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Some Results on the Group of Lower Unitriangular Matrices L(3,$\mathbb{Z}_p$)

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Abstract

The main objective of this paper is to find the order and its exponent, the general form of all conjugacy classes, Artin characters table and Artin exponent for the group of lower unitriangular matrices L(3,$\mathbb{Z}_p$), where p is prime number.

Key Words: Artin character, Artin exponent, cyclic subgroup, group of unitriangular matrices.

Introduction

The group of invertible $n \times n$ matrices over a field F denoted by GL(n,F). Let G be a finite group, all characters of G induced from a principal character of cyclic subgroups of G are called Artin characters of G. Artin induction theorem [1] states that any rational valued character of G is a rational linear combination of the induced principal character of its cyclic subgroups. Lam [5] proved a sharp form of Artin's theorem, he determined the least positive integer A(G) such that A(G)χ is an integral linear combination of Artin character, for any rational valued character χ of G, and he called A(G) the Artin exponent of G and studied it extensively for many groups.

In this paper we consider the group of lower unitriangular matrices L(3,$\mathbb{Z}_p$) and we found that the order of this group is p^3 as in theorem (2.2) and its exponent is p in theorem (2.4). Furthermore we found forms of all conjugace classes in theorem (2.8), the Artin character in theorem (2.12) and finally from the principal character of its cyclic subgroups we found the Artin exponent of this group and denoted by A(L(3,$\mathbb{Z}_p$)) which is equal to p^2 in theorem (2.13).

§1 Preliminaries

In this section, we recall some definitions, theorems and proposition which we needed in the next section.

Definition 1.1 **: [4]**

A rational valued character χ of G is a character whose valued are in Z, that is $\chi(x)\in Z$, for all $x\in G$.

Definition 1.2 **: [2]**

Let H be a subgroup of a group G, and ϕ be a class function of H. Then $\phi\uparrow^G$, the induced class function on G, is given by

$$\phi\uparrow(g)=\frac{1}{|H|}\sum_{x\in G}\phi^{\circ}(xgx^{-1}) \ ,$$

where ϕ° is defined by $\phi^{\circ}(h) = \phi(h)$ if $h \in H$ and $\phi^{\circ}(y) = 0$ if $y \notin H$.
Observe that $\phi\uparrow^G$ is a class function on G and $\phi\uparrow^G(1) = [G:H]\ \phi(1)$.

Another useful formula for computing $\phi\uparrow^G(g)$ explicitly is to choose representatives $x_1, x_2, \dots, x_m$ for the m classes of H contained in the conjugacy class c_g in G which is given by

$$\phi\uparrow^G(g) = |C_G(g)| \sum_{i=1}^{m} \frac{\phi(x_i)}{|C_H(x_i)|}$$

where it is understood that $\varphi\uparrow^G(C_\alpha) = 0$ if $H \cap Cl(g)=\varnothing$. This formula is immediate from the definition of $\phi\uparrow^G$ since as x runs over G, $xgx^{-1} = x_i$ for exactly $|C_G(g)|$ values of x. If H is a cyclic subgroup then

$$\phi\uparrow^G(g) = \frac{|C_G(g)|}{|C_H(g)|} \sum_{i=1}^{m} \phi(x_i) \qquad \dots(1.1)$$

***Definition 1.3*: [5]**

The character induced from the principal character of a cyclic subgroups of G is called Artin character.

***Definition 1.4* : [5]**

Let G be a finite group and let χ be any rational valued character on G. The smallest positive number n such that,

$$n\chi = \sum_{c} a_c \phi_c$$

where $a_c \in Z$ and ϕ_c is Artin character, is called the Artin exponent of G and denoted by A(G).

***Theorem 1.5* : [3]**

Let 1 denote the principal character of G and $d \in Z$, then d is an Artin exponent of G if there exists (uniquely) integers $a_k \in Z$ such that

$$d \cdot 1 = \sum_{k=1}^{q} a_k \mu_k$$

where $\mu_1, \dots, \mu_k$ are the Artin characters.

***Theorem 1.6* : [3]**

For a subgroup H in G, A(H) divides A(G).

***Proposition 1.7* : [5]**

Let G be an arbitrary finite group, and $\{H_1, H_2, \dots, H_q\}$ be a full set of non-gonjugate cyclic subgroups of G, then A(G) is the smallest positive integer m such that:

$$m \cdot 1_G = \sum_{H_k \in H} a_k \cdot 1_{H_k} \uparrow^G \qquad \dots(1.2)$$

With each $a_k \in \mathbb{Z}$.

§2 The Order and its Exponent, The Conjugacy Classes, Artin Character and Artin Exponent of L(3,$\mathbb{Z}_p$)

This section concerns on an important class of groups, the group of lower unitriangular matrices L(3,$\mathbb{Z}_p$). After describing important features of this group and investigating their conjugacy classes we move on to evaluate its Artin Exponent and constructing the table of its induced characters.

***Definition 2.1*: [6]**

Let $L(n,F)=\begin{pmatrix}1 & 0 & \cdots & 0\\ * & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ * & * & \cdots & 1\end{pmatrix}$ be the group of $n\times n$ lower unit triangular matrices with entries in F under matrix multiplication, that is L(n,F) consists of matrices such that L(n,F) is a subgroup of GL(n,F).

***Theorem 2.2*:**

The order of the group $L(3,\mathbb{Z}_p)$ is p^3.

Proof:

$$L(3,\mathbb{Z}_p)=\left\{\begin{bmatrix}1 & 0 & 0\\ x & 1 & 0\\ y & z & 1\end{bmatrix}; x,y,z\in\mathbb{Z}_p\right\}$$

Order of the group $L(3,\mathbb{Z}_p)$ depending on choices number of x, y and z.

Since x, y and z can be chosen arbitrary from $\mathbb{Z}_p$, p choices for x, p choices for y, and p choices for z, thus $|L(3,\mathbb{Z}_p)| = p\cdot p\cdot p = p^3$.

***Theorem 2.3*:**

Every element, excepted identity element e, in the group $G = L(3,\mathbb{Z}_p)$ has order p, that is, $\forall$ g $\in$ G, we have $o(g) = \begin{cases}1 & \text{if } g = e\\ p & \text{if } g\neq e\end{cases}$.

Proof:

If $g = e$, then $o(g) = 1$.

$\forall\, e \neq g \in G$ has the form $g=\begin{bmatrix}1 & 0 & 0\\ g_1 & 1 & 0\\ g_2 & g_3 & 1\end{bmatrix}$ where $g_1, g_2, g_3 \in \mathbb{Z}_p$, and g_1, g_2, g_3 are not all zero.

$$g^2=\begin{bmatrix}1 & 0 & 0\\ 2g_1 & 1 & 0\\ 2g_2+g_1g_3 & 2g_3 & 1\end{bmatrix},\quad g^3=\begin{bmatrix}1 & 0 & 0\\ 3g_1 & 1 & 0\\ 3g_2+3g_1g_3 & 3g_3 & 1\end{bmatrix}$$

In general, $g^r=\begin{bmatrix}1 & 0 & 0\\ rg_1 & 1 & 0\\ r(g_2+\frac{r-1}{2}g_1g_3) & rg_3 & 1\end{bmatrix}$.

Let m be the order of g, then $g^m = e \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ mg_1 & 1 & 0 \\ m(g_2 + \frac{m-1}{2} g_1 g_3) & mg_3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

We get, $mg_1 \equiv 0 \mod p$

$mg_3 \equiv 0 \mod p$

$m(g_2 + \frac{m-1}{2} g_1 g_3) \equiv 0 \mod p$

Since $\mathbb{Z}_p$ is a field and g_1, g_2, g_3 are not all zero, then $m = p$.

Theorem 2.4:

Exponent of the group $G = L(3,\mathbb{Z}_p)$ is p, i.e. $\exp(G) = p$.

Proof:

Let l.c.m(a,b) be the least common multiple of a and b.
By theorem (2.3), $\exp(G) = \text{l.c.m}(1,p) = p$.

Theorem 2.5:

The Center of the group $G = L(3,\mathbb{Z}_p)$ is the cyclic subgroup $Z(G) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & 0 & 1 \end{bmatrix}, r \in \mathbb{Z}_p \right\}$ and $|Z(G)| = p$.

Proof:

Let $g, h \in G$, where $g = \begin{bmatrix} 1 & 0 & 0 \\ g_1 & 1 & 0 \\ g_2 & g_3 & 1 \end{bmatrix}$ and $h = \begin{bmatrix} 1 & 0 & 0 \\ h_1 & 1 & 0 \\ h_2 & h_3 & 1 \end{bmatrix}$.

$$g \cdot h = \begin{bmatrix} 1 & 0 & 0 \\ h_1 + g_1 & 1 & 0 \\ h_2 + g_2 + g_1 h_3 & h_3 + g_3 & 1 \end{bmatrix}, \quad h \cdot g = \begin{bmatrix} 1 & 0 & 0 \\ h_1 + g_1 & 1 & 0 \\ h_2 + g_2 + g_1 h_3 & h_3 + g_3 & 1 \end{bmatrix}$$

If $g_1 = g_3 = 0$, then $\forall h \in G$, we have $g \cdot h = h \cdot g$.

Hence, $g \in Z(G)$ and $Z(G) = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g_2 & 0 & 1 \end{bmatrix}, g_2 \in \mathbb{Z}_p \right\}$.

Since, $\forall g_2 \in \mathbb{Z}_p$, $g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ g_2 & 0 & 1 \end{bmatrix} \in Z(G)$ and $|\mathbb{Z}_p| = p$, then $|Z(G)| = p$.

So, any finite group of prime order is cyclic, then Z(G) is cyclic.

Remark 2.6:

We classify the elements of the group $L(3,\mathbb{Z}_p)$ into three disjoint sets:

(1) Let $W_x = \left\{ x_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & 0 & 1 \end{bmatrix}, i \in \mathbb{Z}_p \right\}$ we called W_x set of all elements of kind x, we note that $W_x = Z(G)$.

(2) Let $W_y = \left\{ y_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & j & 1 \end{bmatrix}, j \neq 0; r, j \in \mathbb{Z}_p \right\}$ we called W_y set of all elements of kind y.

(3) Let $W_z = \left\{ z_{s,t} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ r & t & 1 \end{bmatrix}, s \neq 0; r, s, t \in \mathbb{Z}_p \right\}$ we called W_z set of all elements of kind z.

(4) W_x, W_y and W_z are disjoint sets, i.e., $W_x \cap W_y = \phi$, $W_x \cap W_z = \phi$ and $W_y \cap W_z = \phi$.

***Proposition 2.7*:**

Let $1 \leq m \leq p-1$, then

(1) $\forall\, i = 0,1,\ldots, p-1$; $(x_i)^m$ are elements of kind x, that is $(x_i)^m \in W_x$.

(2) $\forall\, j = 1,2,\ldots, p-1$; $(y_j)^m$ are elements of kind y, that is $(y_j)^m \in W_y$.

(3) $\forall\, s = 1,2,\ldots,p-1$ and $\forall\, t = 0,1,\ldots,p-1$; $(z_{s,t})^m$ are elements of kind z, that is $(z_{s,t})^m \in W_z$.

Proof:

(1) Since, $W_x = Z(G)$ is cyclic, then $\forall\, x_i \in W_x$.

(2) $y_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & j & 1 \end{bmatrix}$ and $(y_j)^m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & j & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ mr & mj & 1 \end{bmatrix}$, where $mr, mj \in \mathbb{Z}_p$.

Since, $m \neq 0$ and $j \neq 0$ then $mj \neq 0$, therefore $(y_j)^m \in W_y$.

(3) $z_{s,t} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ r & t & 1 \end{bmatrix}$ and $(z_{s,t})^m = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ r & t & 1 \end{bmatrix}^m = \begin{bmatrix} 1 & 0 & 0 \\ ms & 1 & 0 \\ m(r + \frac{m-1}{2} st) & mt & 1 \end{bmatrix}$.

Since, $m \neq 0$ and $s \neq 0$ then $ms \neq 0$, therefore $(z_{s,t})^m \in W_z$.

***Theorem 2.8*:**

The group $G = L(3,\mathbb{Z}_p)$ has exactly $p^2 + p - 1$ conjugacy classes:

(1) $\forall\, i = 0,1,\ldots, p-1$; we have classes of the form $C_{x_i} = x_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$ and $|C_{x_i}| = 1$.

(2) $\forall\, j=1,2,\ldots,p-1$; $(y_j)^m$;we have classes of the form $C_{y_i} = \left\{ y_j = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & j & 1 \end{bmatrix}; r = 0,1,\ldots, p-1 \right\}$

and $|C_{y_j}| = p$.

(3) $\forall$ s = 1,2, …, p – 1 and $\forall$ t = 0,1,…, p – 1 ; we have classes of the form $C_{z_{s,t}} = \left\{ z_{s,t} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ r & t & 1 \end{bmatrix}; r = 0,1,\ldots,p-1 \right\}$ and $|C_{z_{s,t}}| = p$.

Proof:

(1) By theorem (2.5), $\forall$ i = 0,1,…, p – 1 ; the elements $x_i \in Z(G)$, then these elements form a conjugacy classes of their own, and $|C_{x_i}| = 1$.

(2) To find a conjugacy classes of y_j, we consider an arbitrary element

$$g = \begin{bmatrix} 1 & 0 & 0 \\ g_1 & 1 & 0 \\ g_2 & g_3 & 1 \end{bmatrix} \in G \text{ and its inverse } g^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -g_1 & 1 & 0 \\ g_1g_3 - g_2 & -g_3 & 1 \end{bmatrix}.$$

Then $gy_jg^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r + jg_1 & j & 1 \end{bmatrix}$.

If $j \neq k$ and y_j is conjugate to y_k, then $gy_jg^{-1} = y_k$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r + jg_1 & j & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & k & 1 \end{bmatrix} \Rightarrow$ j = k; thus $\forall$ j=1,2,…,p – 1, C_{y_j} are all distinct.

In C_{y_j}, r = 0,1,…, p – 1 , $|C_{y_j}| = p$.

(3) To find a conjugacy classes of $z_{s,t}$, we consider an arbitrary element

$$g = \begin{bmatrix} 1 & 0 & 0 \\ g_1 & 1 & 0 \\ g_2 & g_3 & 1 \end{bmatrix} \in G \text{ and its inverse } g^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -g_1 & 1 & 0 \\ g_1g_3 - g_2 & -g_3 & 1 \end{bmatrix}.$$

Then $gz_{s,t}g^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ r + g_1t - g_3s & t & 1 \end{bmatrix}$.

If $s_1 \neq s_2$, $t_1 \neq t_2$; and z_{s_1,t_1} is conjugate to z_{s_2,t_2}, then $g\,z_{s_1,t_1}\,g^{-1} = z_{s_2,t_2}$

$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ s_1 & 1 & 0 \\ r + g_1t_1 - g_3s_1 & t_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ s_2 & 1 & 0 \\ r & t_2 & 1 \end{bmatrix} \Rightarrow$ $s_1 = s_2$ and $t_1 = t_2$.

Thus $\forall$ s = 1,2, …, p – 1 and $\forall$ t = 0,1,…, p – 1 ; $C_{z_{s,t}}$ are all distinct.

In $C_{z_{s,t}}$, r = 0,1,…, p – 1, then $|C_{z_{s,t}}| = p$.

To show that the conjugacy classes C_{x_i}, C_{y_j} and $C_{z_{s,t}}$ are distinct:

We have $C_{x_i} \subseteq W_x$, $C_{y_j} \subseteq W_y$ and $C_{z_{s,t}} \subseteq W_z$, then $C_{x_i} \cap C_{y_j} = \phi$, $C_{x_i} \cap C_{z_{s,t}} = \phi$ and $C_{y_j} \cap C_{z_{s,t}} = \phi$. Hence C_{x_i}, C_{y_j} and $C_{z_{s,t}}$ are distinct.
To find the total number of the conjugacy classes:
Number of conjugacy classes in (1) = p
Number of conjugacy classes in (2) = $p - 1$
Number of conjugacy classes in (3) = $p(p - 1)$
Then the total number of the conjugacy classes is $p + (p - 1) + p(p - 1) = p^2 + p - 1$.

To show that these are all conjugacy classes of the group G = L(3,$\mathbb{Z}_p$), we add up the elements contained in those conjugacy classes, we get:
$P(1) + (p - 1)((p) + [p(p - 1)](p) = p^3 = |G|$.
Thus, this theorem gives all conjugacy classes of the group L(3,$\mathbb{Z}_p$).

***Proposition 2.9*:**

The order of the centralizers, $|C_G(g)|$ of g in the group G = L(3,$\mathbb{Z}_p$) are:
(1) $\forall$ i = 0,1,…, $p - 1$; $|C_G(x_i)| = p^3$.
(2) $\forall$ j=1,2,…,p–1; $(y_j)^m$; $|C_G(y_j)| = p^2$.
(3) $\forall$ s = 1,2, …, $p - 1$ and $\forall$ t = 0,1,…, $p - 1$; $|C_G(z_{s,t})| = p^2$.
Proof:
By theorem (2.10), $|C_G(g)| = \dfrac{|G|}{|C_g|}$ and by theorem (2.2), $|G| = p^3$.

(1) By theorem (2.8), $\forall$ i = 0,1,…, $p - 1$; $|C_{x_i}| = 1$, then $|C_G(x_i)| = \dfrac{|G|}{|C_{x_i}|} = \dfrac{p^3}{1} = p^3$.

(2) By theorem (2.8), $\forall$ j = 1,2,…, $p - 1$; $|C_{y_j}| = p$, then $|C_G(y_j)| = \dfrac{|G|}{|C_{y_j}|} = \dfrac{p^3}{p} = p^2$.

(3) By theorem (2.8), $\forall$ s = 1,2, …, $p - 1$ and $\forall$ t = 0,1,…, $p - 1$; $|C_{z_{s,t}}| = p$, then
$$|C_G(z_{s,t})| = \frac{|G|}{|C_{z_{s,t}}|} = \frac{p^3}{p} = p^2.$$

***Remarks 2.10*:**

(1) Let $x_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ i & 0 & 1 \end{bmatrix}$, where i = 0,1,…, $p - 1$, we note that $x_i \in Z(G)$, by theorem (2.5), Z(G) is cyclic group of order p, and we have
$Z(G) = \{e = x_0, x_1, x_1^2, x_1^3, \ldots, (x_1)^{p-1}\} = \{x_0, x_1, x_2, x_3, \ldots, x_{p-1}\}$.
Since every element excepted e in Z(G) is generator, then
$Z(G) = \langle x_1 \rangle = \langle x_2 \rangle = \ldots = \langle x_{p-1} \rangle$.

(2) $\forall$ j = 1,2,…, p – 1; we take the first element in the conjugacy classes C_{y_j} as a representatives and denoted it by y_i, then $y_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & j & 1 \end{bmatrix}$, so $\forall$ m = 1,2,…, p – 1; $(y_1)^m = y_m$. Then $\langle y_1\rangle = \{e = x_0, y_1, y_1^2, y_1^3, \ldots, (y_1)^{p-1}\} = \{x_0, y_1, y_2, y_3, \ldots, y_{p-1}\}$.

Since $o(y_j) = p$, then every element in $\langle y_1\rangle$ is generator, that is $\langle y_1\rangle = \langle y_2\rangle = \ldots = \langle y_{p-1}\rangle$.

(3) In the same way, we can show that, $\forall$ k = 0, 1,…, p – 1; we have $\langle z_{1,k}\rangle = \langle z_{2,2k}\rangle = \langle z_{3,3k}\rangle = \ldots = \langle z_{p-1,(p-1)k}\rangle$.

Since conjugacy cyclic subgroups give the same Artin characters, then when we construct Artin characters table of $L(3,\mathbb{Z}_p)$ we need only to compute Φ induced from non-conjugate cyclic subgroups and by remark (2.10) they are: $\langle x_0\rangle$, $\langle x_1\rangle$, $\langle y_1\rangle$, $\langle z_{1,0}\rangle$, $\langle z_{1,1}\rangle$, $\langle z_{1,2}\rangle$,…, and $\langle z_{1,p-1}\rangle$.

***Proposition 2.11*:**

Intersection of non-conjugate cyclic subgroups $\langle x_0\rangle$, $\langle x_1\rangle$, $\langle y_1\rangle$, $\langle z_{1,0}\rangle$, $\langle z_{1,1}\rangle$, $\langle z_{1,2}\rangle$,…, and $\langle z_{1,p-1}\rangle$ with all conjugacy classes of the group $L(3,\mathbb{Z}_p)$ are:

Let i, t, k = 0,1,…, p – 1 and j, s = 1,2,…, p – 1; then

(1) (i) $\langle x_0\rangle \cap C_{x_i} = \begin{cases} \{x_0\} & \text{if } i = 0 \\ \phi & \text{if } i > 0 \end{cases}$ **(ii)** $\langle x_0\rangle \cap C_{y_j} = \phi$ **(iii)** $\langle x_0\rangle \cap C_{z_{s,t}} = \phi$

(2) (i) $\langle x_1\rangle \cap C_{x_i} = \{x_i\}$ **(ii)** $\langle x_1\rangle \cap C_{y_j} = \phi$ **(iii)** $\langle x_1\rangle \cap C_{z_{s,t}} = \phi$

(3) (i) $\langle y_1\rangle \cap C_{x_i} = \begin{cases} \{x_0\} & \text{if } i = 0 \\ \phi & \text{if } i > 0 \end{cases}$ **(ii)** $\langle y_1\rangle \cap C_{y_j} = \{y_j\}$ **(iii)** $\langle y_1\rangle \cap C_{z_{s,t}} = \phi$

(4) (i) $\langle z_{1,k}\rangle \cap C_{x_i} = \begin{cases} \{x_0\} & \text{if } i = 0 \\ \phi & \text{if } i > 0 \end{cases}$ **(ii)** $\langle z_{1,k}\rangle \cap C_{y_j} = \phi$

(iii) $\langle z_{1,k}\rangle \cap C_{z_{s,t}} = \begin{cases} \{1 \text{ element}\} & \text{if } t = sk \\ \phi & \text{if } t \neq sk \end{cases}$

Proof:

Let i, t, k = 0, 1,…, p – 1 and j, s = 1, 2, …, p – 1.

(1) (i) Since $\langle x_0\rangle = \{x_0\}$ and $C_{x_i} = \{x_i\}$, then $\langle x_0\rangle \cap C_{x_0} = \{x_0\}$ and $\forall$ i > 0, $\langle x_0\rangle \cap C_{x_i} = \phi$.

(ii) Since $\langle x_0\rangle \subseteq W_x$ and $C_{y_j} \subseteq W_y$, then $\langle x_0\rangle \cap C_{y_j} = \phi$.

(iii) Since $\langle x_0\rangle \subseteq W_x$ and $C_{z_{s,t}} \subseteq W_z$, then $\langle x_0\rangle \cap C_{z_{s,t}} = \phi$.

(2) (i) Since $\langle x_1\rangle = \{x_0, x_1, \ldots, x_{p-1}\} \subseteq W_x$ and $C_{x_i} = \{x_i\}$, then $\langle x_1\rangle \cap C_{x_i} = \{x_i\}$.

(ii) Since $C_{y_j} \subseteq W_y$, then $\langle x_1\rangle \cap C_{y_j} = \phi$.

(iii) Since $C_{z_{s,t}} \subseteq W_z$, then $\langle x_1\rangle \cap C_{z_{s,t}} = \phi$.

(3) (i) Since $\langle y_1\rangle = \{x_0, y_1, y_2, y_3, \ldots, y_{p-1}\}$ then $\langle y_1\rangle \cap C_{x_0} = \{x_0\}$ and $\forall$ i > 0, $\langle y_1\rangle \cap C_{x_i} = \phi$.

(ii) From theorem (2.8), $C_{y_i} = \left\{ y_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & j & 1 \end{bmatrix}; r = 0,1,\ldots,p-1 \right\}$.

Then $y_1 \in C_{y_1}, y_2 \in C_{y_2}, \ldots, y_{p-1} \in C_{y_{p-1}}$ and thus $<y_1> \cap C_{y_j} = \{y_j\}$.

(iii) Since $<y_1>$ have no elements of kind z, then $<y_1> \cap C_{z_{s,t}} = \phi$.

(4) (i) $<z_{1,k}> = \{x_0, z_{1,k}, (z_{1,k})^2, (z_{1,k})^3, \ldots, (z_{1,k})^{p-1}\}$ and by proposition (2.7), $(z_{1,k})^m \in W_z$.

Then $<z_{1,k}> \cap C_{x_0} = \{x_0\}$ and $\forall\, i > 0$, $<z_{1,k}> \cap C_{x_i} = \phi$.

(ii) $<z_{1,k}>$ have no elements of kind y, then $<z_{1,k}> \cap C_{y_j} = \phi$.

(iii) From theorem (2.8), $C_{z_{s,t}} = \left\{ z_{s,t} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ r & t & 1 \end{bmatrix}; r = 0,1,\ldots,p-1 \right\}$.

And from proposition (2.7), $(z_{s,t})^m = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ m(r+\frac{m-1}{2}k) & mk & 1 \end{bmatrix} \in C_{z_{m,mk}}$.

Then $z_{1,k} \in C_{z_{1,k}}, (z_{1,k})^2 \in C_{z_{2,2k}}, (z_{1,k})^3 \in C_{z_{3,3k}}, \ldots, (z_{1,k})^{p-1} \in C_{z_{p-1,(p-1)k}}$.

Therefore, $\forall\, s = 1,2,\ldots, p-1$; $<z_{1,k}> \cap C_{z_{s,sk}} = \{1 \text{element}\}$.

Thus $<z_{1,k}> \cap C_{z_{s,t}} = \begin{cases} \{1 \text{ element}\} & \text{if } t = sk \\ \phi & \text{if } t \neq sk \end{cases}$.

***Theorem 2.12*:**

For any prime number p, Artin characters table of the group $G = L(3,\mathbb{Z}_p)$ is:

$g \in G$	C_{x_0}	C_{x_1}	C_{y_1}	$C_{z_{1,0}}$	$C_{z_{1,1}}$	$C_{z_{1,2}}$	$C_{z_{1,3}}$	…	$C_{z_{1,k}}$	…	$C_{z_{1,p-2}}$	$C_{z_{1,p-1}}$
$\lvert C_g \rvert$	1	1	p	p	p	p	p		p		p	p
$\lvert C_G(g) \rvert$	p^3	p^3	p^2	p^2	p^2	p^2	p^2		p^2		p^2	p^2
Φ_{x_0}	p^3	0	0	0	0	0	0		0		0	0
Φ_{x_1}	p^2	p^2	0	0	0	0	0		0		0	0
Φ_{y_1}	p^2	0	p	0	0	0	0		0		0	0
$\Phi_{z_{1,0}}$	p^2	0	0	p	0	0	0		0		0	0
$\Phi_{z_{1,1}}$	p^2	0	0	0	p	0	0		0		0	0
$\Phi_{z_{1,2}}$	p^2	0	0	0	0	p	0		0		0	0
$\Phi_{z_{1,3}}$	p^2	0	0	0	0	0	p		0		0	0
$\vdots$								$\ddots$				
$\Phi_{z_{1,k}}$	p^2	0	0	0	0	0	0		p		0	0
$\vdots$										$\ddots$		
$\Phi_{z_{1,p-2}}$	p^2	0	0	0	0	0	0		0		p	0
$\Phi_{z_{1,p-1}}$	p^2	0	0	0	0	0	0		0		0	p

Proof:

The order of classes $|C_g|$ and the order of the centralizers $|C_G(g)|$ following from theorem (2.8) and proposition (2.9) respectively.

By proposition (2.11) the intersections of cyclic subgroups of L(3,$\mathbb{Z}_p$) with its conjugacy classes are ({1e.} in the following table means set contain only one element)

$\cap$	C_{x_0}	C_{x_1}	C_{y_1}	$C_{z_{1,0}}$	$C_{z_{1,1}}$	$C_{z_{1,2}}$	$C_{z_{1,3}}$	…	$C_{z_{1,k}}$	…	$C_{z_{1,p-2}}$	$C_{z_{1,p-1}}$
$<x_0>$	{1e.}	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ		ϕ		ϕ	ϕ
$<x_1>$	{1e.}	{1e.}	ϕ	ϕ	ϕ	ϕ	ϕ		ϕ		ϕ	ϕ
$<y_1>$	{1e.}	ϕ	{1e.}	ϕ	ϕ	ϕ	ϕ		ϕ		ϕ	ϕ
$<z_{1,0}>$	{1e.}	ϕ	ϕ	{1e.}	ϕ	ϕ	ϕ		ϕ		ϕ	ϕ
$<z_{1,1}>$	{1e.}	ϕ	ϕ	ϕ	{1e.}	ϕ	ϕ		ϕ		ϕ	ϕ
$<z_{1,2}>$	{1e.}	ϕ	ϕ	ϕ	ϕ	{1e.}	ϕ		ϕ		ϕ	ϕ
$<z_{1,3}>$	{1e.}	ϕ	ϕ	ϕ	ϕ	ϕ	{1e.}		ϕ		ϕ	ϕ
$\vdots$								$\ddots$				
$<z_{1,k}>$	{1e.}	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ		{1e.}		ϕ	ϕ
$\vdots$										$\ddots$		
$<z_{1,p-2}>$	{1e.}	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	{1e.}	ϕ
$<z_{1,p-1}>$	{1e.}	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	{1e.}

Let $V = \{x_0,x_1,y_1,z_{1,0},z_{1,1},z_{1,2},\ldots,z_{1,k},\ldots,z_{1,p-1}\}$, from the intersection table above, we see that $\forall\, h \in V;\ <h> \cap C_{x_0} = \{1 \text{ element}\}$.

Also, $\forall\, h \in V$ and $x_0 \neq g \in V;\ <h> \cap C_g = \begin{cases} \{1 \text{ element}\} & \text{if } h = g \\ \phi & \text{if } h \neq g \end{cases}$.

We using formula (1.1) to compute Artin characters, then we get:

- $h \in V,\ 1_{<h>} \uparrow^G (x_0) = \dfrac{|C_G(x_0)|}{|<h>|} \sum 1 = \begin{cases} \dfrac{p^3}{1}(1) = p^3 & \text{if } h = x_0 \\ \dfrac{p^3}{p}(1) = p^2 & \text{if } h \neq x_0 \end{cases}$.

- $h \in V,\ 1_{<h>} \uparrow^G (x_1) = \dfrac{|C_G(x_1)|}{|<h>|} \sum 1 = \begin{cases} \dfrac{p^3}{p}(1) = p^2 & \text{if } h = x_1 \\ 0 & \text{if } h \neq x_1 \end{cases}$.

- $h \in V,\ 1_{<h>} \uparrow^G (y_1) = \dfrac{|C_G(y_1)|}{|<h>|} \sum 1 = \begin{cases} \dfrac{p^2}{p}(1) = p & \text{if } h = y_1 \\ 0 & \text{if } h \neq y_1 \end{cases}$.

- $\forall\, h \in V$ and $\forall\, k = 0, 1, \ldots, p-1$;

$$1_{<h>} \uparrow^G (z_{1,k}) = \frac{|C_G(z_{1,k})|}{|<h>|} \sum 1 = \begin{cases} \dfrac{p^2}{p}(1) = p & \text{if } h = z_{1,k} \\ 0 & \text{if } h \neq z_{1,k} \end{cases}.$$

***Theorem 2.13*:**

For any prime number p, Artin exponent of the group $G = L(3,\mathbb{Z}_p)$ is $A(L(3,\mathbb{Z}_p)) = p^2$.

Proof:

From Artin characters table of the group $G = L(3,\mathbb{Z}_p)$ in theorem (2.12), we note that

$(-(p+1)/p^2)\,\Phi_{x_0}$	$(-p+1)/p^2)$	0	0	0	0	0	0	…	0	…	0	0
$(1/p^2)\,\Phi_{x_1}$	1	1	0	0	0	0	0		0		0	0
$(1/p)\,\Phi_{y_1}$	p	0	1	0	0	0	0		0		0	0
$(1/p)\,\Phi_{z_{1,0}}$	p	0	0	1	0	0	0		0		0	0
$(1/p)\,\Phi_{z_{1,1}}$	p	0	0	0	1	0	0		0		0	0
$(1/p)\,\Phi_{z_{1,2}}$	p	0	0	0	0	1	0		0		0	0
$(1/p)\,\Phi_{z_{1,3}}$	p	0	0	0	0	0	1		0		0	0
$\vdots$								$\ddots$				
$(1/p)\,\Phi_{z_{1,k}}$	p	0	0	0	0	0	0		1		0	0
$\vdots$										$\ddots$		
$(1/p)\,\Phi_{z_{1,p-2}}$	p	0	0	0	0	0	0		0		1	0
$(1/p)\,\Phi_{z_{1,p-1}}$	p	0	0	0	0	0	0		0		0	1
summation	1	1	1	1	1	1	1	…	1	…	1	1

Then

$$p^2 \cdot 1_G = -(p+1)\Phi_{x_0} + \Phi_{x_1} + p\Phi_{y_1} + p\sum_{k=0}^{p-1}\Phi_{z_{1,k}}$$

And by using (1.2) in proposition (1.7), Artin exponent of $L(3,\mathbb{Z}_p)$ is $A(L(3,\mathbb{Z}_p)) = p^2$.

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