Bayesian One- Way Repeated Measurements Model as a Mixed Model

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Abstract

In the Bayesian approach to inference, all unknown quantities contained in a probability model for the observed data are treated as random variables. Specifically, the fixed but unknown parameters are viewed as random variables under the Bayesian approach. In this paper, Bayesian approach is employed to making inferences on the one- way repeated measurements model as mixed model , and we prove some theorems about posterior.

Keywords: Mixed models, One- way repeated measurements model , Bayesian inference, Prior density, Posterior density.

1.Introduction

Mixed models are an extension of regression models that allow for the incorporation of random effects. A more contemporary application of mixed models is the analysis of longitudinal data, clustered data repeated measurements and spatially correlated data. Often random effects are used to describe the correlation structure in this type of this data. [4],[9],[10],[12].

Repeated measurements is a term used to describe data in which the response variable for each experimental units is observed on multiple occasions and possible under different experimental conditions . Repeated measures data is a common form of multivariate data, and linear models with correlated error which are widely used in modeling repeated measures data. Repeated measures is a common data structure with multiple measurements on a single unit repeated over time. Multivariate linear models with correlated errors have been accepted as one of the primary modeling methods for repeated measures data. We can represent repeated measurements model as a mixed model[1],[2],[4],[10],[12] .

In the Bayesian approach to inference, all unknown quantities contained in a probability model for the observed data are treated as random variables. Specifically, the fixed but unknown parameters are viewed as random variables under the Bayesian approach. Bayesian techniques based on Markov chain Monte Carlo provide what we believe to be the most satisfactory approach to fitting complex models as well as the direction that model is most likely to take in the future [3],[5],[6],[7],[8],[9],[11],[13],[14] .

In this paper, a simple Bayesian approach is employed to the linear one- way repeated measurements model to make inferences on the resulting mixed model coefficients. We investigate the posterior density and identify the analytic form of the Bayes factor. To illustrate the effectiveness of the our methodology. We have choosing the data set which A study was conducted at date palm research center laboratories, university of Basra, during 2007-2008 season

2. One- Way Repeated Measurements Model

Consider the model

$$y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk} \qquad (1)$$

Where

i=1,…., n is an index for experimental unit within group j ,

j=1,…,q is an index for levels of the between-units factor (Group) , k=1,…,p is an index for levels of the within-units factor (Time) ,

y_{ijk} is the response measurement at time k for unit i within group j,

μ is the overall mean ,

τ_j is the added effect for treatment group j ,

$\delta_{i(j)}$ is the random effect for due to experimental unit i within treatment group j ,

γ_k is the added effect for time k ,

$(\tau\gamma)_{jk}$ is the added effect for the group j × time k interaction ,

e_{ijk} is the random error on time k for unit i within group j ,

For the parameterization to be of full rank, we imposed the following set of conditions

$\sum_{j=1}^{q} \tau_{j=0}$, $\sum_{k=1}^{p} \gamma_{k=0}$, $\sum_{j=1}^{q} (\tau\gamma)_{jk=0}$ for each k=1,…,p

$\sum_{k=1}^{p} (\tau\gamma)_{jk=0}$ for each j=1,…,q

And we assumed that the e_{ijk} and $\delta_{i(j)}$ are indepndent with

$e_{ijk} \sim i.i.d\ N(0,\sigma_\epsilon^2)$, $\delta_{ij} \sim i.i.d\ N(0,\sigma_\delta^2)$.

Sum of squares due to groups, subjects(group), time, group*time and residuals are then defined respectively as follows:

$SS_G = np\sum_{j=1}^{q}\left(\overline{y}_{.j.} - \overline{y}_{...}\right)^2$, $SS_{U(G)} = p\sum_{i=1}^{n}\sum_{j=1}^{q}(\overline{y}_{ij.} - \overline{y}_{.j.})^2$

$SS_{time} = nq\sum_{k=1}^{p}\left(\overline{y}_{..k} - \overline{y}_{...}\right)^2$, $SS_{G\times time} = n\sum_{j=1}^{q}\sum_{k=1}^{p}(\overline{y}_{.jk} - \overline{y}_{.j.} - \overline{y}_{..k} + \overline{y}_{...})^2$

$$SS_E = \sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \overline{y}_{.jk} - \overline{y}_{ij.} + \overline{y}_{.j.})^2$$

Where

$\overline{y}_{...} = \dfrac{\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p} y_{ijk}}{nqp}$ *is the overall mean.*

$\overline{y}_{.j.} = \dfrac{\sum_{i=1}^{n}\sum_{k=1}^{p} y_{ijk}}{np}$ *is the mean for group j.*

$\overline{y}_{ij.} = \dfrac{\sum_{k=1}^{p} y_{ijk}}{p}$ *is the mean for the* i^{th} *subject in group j.*

$\overline{y}_{..k} = \dfrac{\sum_{i=1}^{n}\sum_{j=1}^{q} y_{ijk}}{nq}$ *is the mean for time k.*

$\overline{y}_{.jk} = \dfrac{\sum_{i=1}^{n} y_{ijk}}{n}$ *is the mean for group j at time k.*

Table(1): ANOVA table for one-way Repeated measures model

Source of variation	d.f	SS	MS	E(MS)
Group	$q-1$	SS_G	$\frac{SS_G}{q-1}$	$\frac{np}{(q-1)}\sum_{j=1}^{q}\tau_j^2 + p\sigma_\delta^2 + \sigma_e^2$
Unit (Group)	$q(n-1)$	$SS_{U(G)}$	$\frac{SS_{U(G)}}{q(n-1)}$	$p\sigma_\delta^2 + \sigma_e^2$
Time	$p-1$	SS_{time}	$\frac{SS_{time}}{p-1}$	$\frac{nq}{(p-1)}\sum_{k=1}^{p}\gamma_k^2 + \sigma_e^2$
Group*Time	$(q-1)(p-1)$	$SS_{G\times time}$	$\frac{SS_{G\times time}}{(q-1)(p-1)}$	$\frac{n}{(p-1)(q-1)}\sum_{j=1}^{q}\sum_{k=1}^{p}(\tau\gamma)_{jk}^2 + \sigma_e^2$
Residual	$q(p-1)(n-1)$	SS_E	$\frac{SS_E}{q(p-1)(n-1)}$	σ_e^2

The model (1) is rewritten as follows

$$Y = X\beta + Zb + \epsilon \qquad (2)$$

Where

$$Y = \begin{bmatrix} y_{111} \\ y_{112} \\ \vdots \\ y_{nqp} \end{bmatrix}_{nqp\times 1} \quad , \quad \beta = \begin{bmatrix} \mu \\ \tau \\ \gamma \\ (\tau\gamma) \end{bmatrix}_{(qp+q+p+1)\times 1} \quad , \quad Z = \begin{pmatrix} 1_{P\times 1} & \cdots & 0_{P\times 1} \\ \vdots & \ddots & \vdots \\ 0_{P\times 1} & \cdots & 1_{P\times 1} \end{pmatrix}_{nqp\times nq} \quad ,$$

$$b = \begin{bmatrix} \delta_{1(1)} \\ \delta_{1(2)} \\ \vdots \\ \delta_{n(q)} \end{bmatrix}_{nq\times 1} \quad , \quad \epsilon = \begin{bmatrix} e_{111} \\ e_{112} \\ \vdots \\ e_{nqp} \end{bmatrix}_{nqp\times 1} \quad ,$$

and design matrix X is an $\big(nqp * (qp+q+p+1)\big)$ matrix.

We assume that $H_0: Y = X\beta + \epsilon$ versus $H_1: Y = X\beta + Zb + \epsilon$ (3)

As for the prior of β ,and ϵ under H_1, we assume β and b are independent and

$$\beta \sim N(0, \Sigma_0) \quad , \quad \Sigma_0 = \sigma_\beta^2 I_{1+q+p+qp} \quad , \quad b \sim N(0, \Sigma_1) \quad , \quad \Sigma_1 = \sigma_b^2 I_{nq} \qquad (4)$$

Also we assume that

$$\sigma_b^2 \sim Invers\ Gamma(\alpha_b, \beta_b) \quad , \quad \sigma_\epsilon^2 \sim Invers\ Gamma\ (\alpha_\epsilon, \beta_\epsilon) \qquad (5)$$

3. Posterior calculations

We have

$$Y|\theta, \sigma_\epsilon^2, \sigma_b^2 \sim N\big(C\theta, \sigma_\epsilon^2 I_{nqp}\big), \qquad (6)$$

$where\ C = [X\ \ Z]\ and\ \theta = [\beta\ b]^T$

Then the likelihood function $L(Y|\theta, \sigma_\epsilon^2, \sigma_b^2)$ can be expressed as

$$L(Y|\theta, \sigma_\epsilon^2, \sigma_b^2) \propto \left|\sigma_\epsilon^2 I_{nqp}\right|^{-1/2} exp\left\{\frac{-1}{2}(Y - C\theta)^T\big(\sigma_\epsilon^2 I_{nqp}\big)^{-1}(Y - C\theta)\right\}. \qquad (7)$$

Then the joint posterior density to coefficients θ and the error variances σ_ϵ^2 and σ_b^2 given by the expression

$\pi_1(\theta|Y, \sigma_\epsilon^2, \sigma_b^2) \propto L(Y|\theta, \sigma_\epsilon^2, \sigma_b^2)\pi_0(\theta),$ (8)

$\rightarrow \pi_1(\theta|Y, \sigma_\epsilon^2, \sigma_b^2) \propto exp\left\{\frac{-1}{2}(Y - C\theta)^T\left(\sigma_\epsilon^2 I_{nqp}\right)^{-1}(Y - C\theta)\right\}\pi_0(\theta),$ (9)

and

$\pi_1(\sigma_\epsilon^2|Y, \theta, \sigma_b^2) \propto \left|\sigma_\epsilon^2 I_{nqp}\right|^{-1/2} exp\left\{\frac{-1}{2}(Y - C\theta)^T\left(\sigma_\epsilon^2 I_{nqp}\right)^{-1}(Y - C\theta)\right\}\pi_0(\sigma_\epsilon^2),$ (10)

$\pi_1(\sigma_b^2|Y, \theta, \sigma_\epsilon^2) \propto \left|\sigma_\epsilon^2 I_{nqp}\right|^{-1/2} exp\left\{\frac{-1}{2}(Y - C\theta)^T(\sigma_\epsilon^2 I_n)^{-1}(Y - C\theta)\right\}\pi_0(\sigma_b^2).$ (11)

Therefore, it follows that

$$\theta|Y, \sigma_b^2, \sigma_\epsilon^2 \sim N(A_1 Y, A_2) \quad (12)$$

Where

$$A_1 = \{\Sigma D\}\{\sigma_\epsilon^2 I_N + [\Sigma D C^T C]\}^{-1} C^T \quad (13)$$

$$A_2 = \Sigma D - \Sigma^2 D\, C^T\{\sigma_\epsilon^2 I_N + [\Sigma D C^T C]\}^{-1}\{CD\}. \quad (14)$$

Where

$\Sigma = \begin{bmatrix} \sigma_\beta^2 & 0 \\ 0 & \sigma_b^2 \end{bmatrix}$ and $D = \begin{bmatrix} I_{1+q+p+qp} & 0 \\ 0 & I_{nq} \end{bmatrix}$, then $\Sigma D = \begin{bmatrix} \sigma_\beta^2 I_{1+q+p+qp} & 0 \\ 0 & \sigma_b^2 I_{nq} \end{bmatrix}$, and $N = 1 + q + p + qp + nq$.

Now we employ spectral decomposition to obtain $DC^T C = P\,\Lambda\,P^T$,[4] where $\Lambda = diag(\Lambda_1, \dots, \Lambda_N)$ is the matrix of eigenvalues and P is the orthogonal matrix of eigenvectors. Thus,

$$\sigma_\epsilon^2 I_N + [\Sigma D C^T C] = \sigma_\epsilon^2 I_N + P\Sigma\Lambda P^T = P(\sigma_\epsilon^2 I_N + \Sigma\Lambda)P^T = P\sigma_\epsilon^2(I_N + \frac{\Sigma}{\sigma_\epsilon^2}\Lambda)P^T$$

$$= \sigma_\epsilon^2 P(I_N + \begin{bmatrix} uI_{1+q+p+qp} & 0 \\ 0 & vI_{nq} \end{bmatrix}\Lambda)p^T$$

Where $v = \sigma_b^2/\sigma_\epsilon^2$ and $u = \sigma_\beta^2/\sigma_\epsilon^2$ .Then, the conditional density of Y given σ_ϵ^2, u and v can be written as

$$m(Y|\sigma_\epsilon^2, u, v) = \frac{1}{(2\pi\sigma_\epsilon^2)^{nqp/2}} \frac{1}{det\left[I_N + \begin{bmatrix} uI_{1+q+p+qp} & 0 \\ 0 & vI_{nq} \end{bmatrix}\Lambda\right]^{1/2}} exp\{-\frac{1}{2\sigma_\epsilon^2} Y^T P\left(I_N + \begin{bmatrix} uI_{1+q+p+qp} & 0 \\ 0 & vI_{nq} \end{bmatrix}\Lambda\right)^{-1} P^T Y\}$$

$$= \frac{1}{(2\pi\sigma_\epsilon^2)^{nqp/2}} \frac{1}{[\prod_{i=1}^{1+q+p+qp}[1 + ud_i]]^{1/2}[\prod_{i=2+q+p+qp}^{nq}[1 + vd_i]]^{1/2}}$$

$$exp\{-\frac{1}{2\sigma_\epsilon^2}(\sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i})\}. \quad (15)$$

Where $s = (s_1, \dots, s_N)^T = B^T Y$. We choose the prior on σ_ϵ^2, $v = \sigma_b^2/\sigma_\epsilon^2$ and $u = \sigma_\beta^2/\sigma_\epsilon^2$, qualitatively similar to the used in [3]. Specifically, we take $\pi_1(\sigma_\epsilon^2, u, v)$ to be proportional to the product of an inverse gamma density $\{\beta_\epsilon{}^{\alpha_\epsilon}/\Gamma(\alpha_\epsilon)\}\, exp(-\beta_\epsilon/\sigma_\epsilon^2)(\sigma_\epsilon^2)^{-(\alpha_\epsilon+1)}$ for σ_ϵ^2 and the gamma density for u and the density of a $F(b,a)$distribution for v (for suitable choice of β_ϵ, α_ϵ, b and a).The posterior density of u, v given Y, the posterior mean and covariance matrix of θ as in the following theorems.

Theorem1: the posterior density of u, v given Y is:

$\pi_1(u, v|Y) \propto$

$$\frac{v^{(b/2)-1}u^{\alpha\epsilon-1}\,exp\frac{Cu}{\beta_\epsilon}}{(a+bv)^{-(a+b)/2}}\left(\prod_{i=1}^{1+q+p+qp}(1 + ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1 + vd_i)\right)^{-1/2}\left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)^{-(nqp+2\alpha_\epsilon+2)/2} \quad (16)$$

Proof:

$$\pi_1(u,v|Y) = \int m(Y|\sigma_\epsilon^2,u,v)\, f(u,\alpha_\epsilon,\beta_\epsilon)\, f(v,b,a)\, f(\sigma_\epsilon^2,\alpha_\epsilon,\beta_\epsilon) d\sigma_\epsilon^2$$

$$= \int \frac{1}{(2\pi\sigma_\epsilon^2)^{nqp/2}} \left(\prod_{i=1}^{1+q+p+qp}(1+ud_i)\right)^{-1/2} \left(\prod_{i=2+q+p+qp}^{nq}(1+vd_i)\right)^{-1/2}$$

$$\frac{c}{\beta_\epsilon \Gamma(\alpha_\epsilon)}\left(\frac{cu}{\beta_\epsilon}\right)^{\alpha_\epsilon - 1} exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(\sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right\}$$

$$exp\frac{cu}{\beta_\epsilon}\frac{b^{b/2}a^{a/2}}{B(b,a)}\frac{v^{(b/2)-1}}{(a+bv)^{-(a+b)/2}}\frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)}(\sigma_\epsilon^2)^{-(\alpha_\epsilon+1)} exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right) d\sigma_\epsilon^2$$

Where $c = \sigma_\beta^2$

$$= (2\pi)^{-nqp/2}\frac{(cu)^{\alpha_\epsilon-1} exp\frac{cu}{\beta_\epsilon}}{(\Gamma(\alpha_\epsilon))^2}\frac{b^{b/2}a^{a/2}}{B(b,a)}\frac{v^{(b/2)-1}}{(a+bv)^{-(a+b)/2}}\int\left(\prod_{i=1}^{1+q+p+qp}(1+ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1+vd_i)\right)^{-1/2} exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right\}(\sigma_\epsilon^2)^{-(nqp+2\alpha_\epsilon+2)/2} d\sigma_\epsilon^2$$

$$= (2\pi)^{-\frac{nqp}{2}}\frac{(cu)^{\alpha_\epsilon-1} exp\frac{cu}{\beta_\epsilon}}{(\Gamma(\alpha_\epsilon))^2}\frac{b^{b/2}a^{a/2}}{B(b,a)}\frac{v^{(b/2)-1}}{(a+bv)^{-(a+b)/2}}(2)^{(nqp+2\alpha_\epsilon+2)/2}$$

$$\int\left(\prod_{i=1}^{1+q+p+qp}(1+ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1+vd_i)\right)^{-1/2} exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right\}\left(\frac{2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}}{2\sigma_\epsilon^2}\right)^{(nqp+2\alpha_\epsilon+2)/2}\left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)^{-(nqp+2\alpha_\epsilon+2)/2} d\sigma_\epsilon^2$$

$$\propto \frac{v^{(b/2)-1}u^{\alpha_\epsilon-1} exp\frac{cu}{\beta_\epsilon}}{(a+b\delta)^{-(a+b)/2}}\int\left(\prod_{i=1}^{1+q+p+qp}(1+ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1+vd_i)\right)^{-1/2} exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right\}\left(\frac{2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}}{2\sigma_\epsilon^2}\right)^{[(nqp+2\alpha_\epsilon+4)/2]-1}\left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)^{-(nqp+2\alpha_\epsilon+2)/2} d\sigma_\epsilon^2$$

$$\propto \frac{v^{(b/2)-1}u^{\alpha_\epsilon-1} exp\frac{cu}{\beta_\epsilon}}{(a+bv)^{-(a+b)/2}}\Gamma((n+2\alpha_\epsilon+4)/2)\left(\prod_{i=1}^{1+q+p+qp}(1+ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1+vd_i)\right)^{-1/2}\left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)^{-(nqp+2\alpha_\epsilon+2)/2}$$

$$\therefore\ \pi_1(u,v|Y) \propto \frac{v^{(b/2)-1}u^{\alpha_\epsilon-1} exp\frac{cu}{\beta_\epsilon}}{(a+bv)^{-(a+b)/2}}\left(\prod_{i=1}^{1+q+p+qp}(1+ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1+vd_i)\right)^{-1/2}\left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)^{-(nqp+2\alpha_\epsilon+2)/2}$$

Theorem2: The posterior mean and covariance matrix of θ are:

$$E(\theta/Y) = DPE\left\{\left(I_N + \begin{bmatrix} uI_{1+q+p+qp} & 0 \\ 0 & vI_{nq}\end{bmatrix}\Lambda\right)^{-1}/Y\right\}C^T s \qquad (17)$$

And

$$var(\theta|Y) = \frac{1}{nqp+2\alpha_\epsilon+2} E\left[\frac{\left(2\beta_\epsilon+\left(\sum_{i=1}^{1++q+p+qp}\frac{s_i^2}{1+ud_i}+\sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right)}{Y}\right]D - \frac{1}{nqp+2\alpha_\epsilon+2}DC^TPE\left[\left(2\beta_\epsilon + \right.\right.$$

$$\left(\sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i}+\sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\Big)\left[I_N+\begin{bmatrix}uI_{1+q+p+qp} & 0\\ 0 & vI_{nq}\end{bmatrix}\Lambda\right]^{-1}/Y\Big]P^TCD +$$

$$E[R(u,v)R(u,v)^T/Y], \quad (18)$$

Where $R(u,v) = DP(I_N+\begin{bmatrix}uI_{1+q+p+qp} & 0\\ 0 & vI_{nq}\end{bmatrix}\Lambda)^{-1}C^Ts$

Proof:

From (11): $E(\theta|Y) = A_1Y$

$$= \{\Sigma D\}\{\sigma_\epsilon^2 I_N + [\Sigma DC^TC]\}^{-1}C^TY$$

$$= \Sigma D\left\{\sigma_\epsilon^2 P(I_N+\begin{bmatrix}uI_{1+q+p+qp} & 0\\ 0 & vI_{nq}\end{bmatrix}\Lambda)P^T\right\}^{-1}C^TY$$

$$= \frac{\Sigma}{\sigma_\epsilon^2}D\,P^{T^{-1}}\left(I_N+\begin{bmatrix}uI_{1+q+p+qp} & 0\\ 0 & vI_{nq}\end{bmatrix}\Lambda\right)^{-1}P^{-1}C^TY$$

$\because P$ is the orthogonal matrix of eigenvectors, then $P^{-1} = P^T$ and $P^{T^{-1}} = P$

Therefore

$$E(\theta|Y) = D\,P\begin{bmatrix}uI_{1+q+p+qp} & 0\\ 0 & vI_{nq}\end{bmatrix}\left(I_N+\begin{bmatrix}uI_{1+q+p+qp} & 0\\ 0 & vI_{nq}\end{bmatrix}\Lambda\right)^{-1}p^TC^TY)$$

$$= D\,P\,E\left(\left(I_N+\begin{bmatrix}uI_{1+q+p+qp} & 0\\ 0 & vI_{nq}\end{bmatrix}\Lambda\right)^{-1}\Bigg|y\right)C^Ts$$

Where the expectation $E\left(\left(I_N+\begin{bmatrix}uI_{1+q+p+qp} & 0\\ 0 & vI_{nq}\end{bmatrix}\Lambda\right)^{-1}\Bigg|y\right)$ is taken with respect to $\pi_1(u,v|Y)$(see theorem 1 above). And by same way can prove the variance of θ given Y.

4. Model checking and Bayes factors

We would like to choose between a Bayesian mixed repeated measurements model and its fixed counterpart by the criterion of the Bayes factor for two hypotheses :

$$H_o : Y = X\beta+\epsilon \;\; versus \;\; H_1 : Y = X\beta+Zb+\epsilon. \quad (19)$$

We compute the Bayes factor, B_{01}, of H_o relative to H_1 for testing problem (19) as following

$$B_{01}(Y) = \frac{m(Y|H_o)}{m(Y|H_1)} \quad (20)$$

where $m(Y|H_i)$ is the predictive density of Y under model $H_i, i = 0,1$.

We have

$$m(Y|H_o) = \int f(Y|\beta,\sigma_\epsilon^2)\pi_0(\sigma_\epsilon^2)\,d\sigma_\epsilon^2,$$

where under H_o , $\pi_0(\sigma_\epsilon^2)$ induced by $\pi_0(\theta,\sigma_b^2,\sigma_\epsilon^2)$ is the only part needed , and

$$m(Y|H_1) = \int f(Y|\theta,\sigma_b^2,\sigma_\epsilon^2)\pi_0(\theta,\sigma_b^2,\sigma_\epsilon^2)\,d\theta d\sigma_b^2 d\sigma_\epsilon^2,$$

where $\pi_0(\sigma_b^2,\sigma_\epsilon^2)$ will be constant in θ. Therefore,

$$m(Y|H_o) = \int f(Y|\beta,\sigma_\epsilon^2)\pi_0(\sigma_\epsilon^2)\,d\sigma_\epsilon^2$$

$$= (2\pi)^{-nqp/2}\frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)}\int(\sigma_\epsilon^2)^{-nqp/2}\,exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right)(\sigma_\epsilon^2)^{-(\alpha_\epsilon+1)}\,exp(-\frac{(Y-X\beta)^2}{2\sigma_\epsilon^2})d\sigma_\epsilon^2$$

$$= (2\pi)^{-nqp/2}\frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)}\int(\sigma_\epsilon^2)^{-(\frac{nqp}{2}+\alpha_\epsilon+1)}exp(-\frac{\beta_\epsilon+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-(\mu+\tau_j+\gamma_k+(\tau\gamma)_{jk}))^2}{\sigma_\epsilon^2})d\sigma_\epsilon^2$$

$$= (2\pi)^{-\frac{nqp}{2}}\frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)}\int(\sigma_\epsilon^2)^{-(\frac{nqp}{2}+\alpha_\epsilon+1)}(\beta_\epsilon +$$

$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - (\mu + \tau_j + \gamma_k + (\tau\gamma)_{jk}))^2)^{-\left(\frac{nqp}{2}+\alpha_\epsilon+1\right)} \left(\beta_\epsilon + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - (\mu + \tau_j + \gamma_k + (\tau\gamma)_{jk}))^2\right)^{\left(\frac{nqp}{2}+\alpha_\epsilon+1\right)}$

$$exp\left(-\frac{\beta_\epsilon+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-(\mu+\tau_j+\gamma_k+(\tau\gamma)_{jk}))^2}{\sigma_\epsilon^2}\right) d\sigma_\epsilon^2$$

$$= (2\pi)^{-\frac{nqp}{2}} \frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)} \int \frac{(\beta_\epsilon+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-(\mu+\tau_j+\gamma_k+(\tau\gamma)_{jk}))^2)^{(\frac{nqp}{2}+\alpha_\epsilon+1)}}{(\sigma_\epsilon^2)^{\left(\frac{nqp}{2}+\alpha_\epsilon+1\right)}}$$
$$exp\left(-\frac{\beta_\epsilon+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-(\mu+\tau_j+\gamma_k+(\tau\gamma)_{jk}))^2}{\sigma_\epsilon^2}\right)(\beta_\epsilon + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - (\mu + \tau_j + \gamma_k + (\tau\gamma)_{jk}))^2)^{-(\frac{nqp}{2}+\alpha_\epsilon+1)} d\sigma_\epsilon^2$$

$$= (2\pi)^{-\frac{nqp}{2}} \frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)} \int \left(\frac{(\beta_\epsilon+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-(\mu+\tau_j+\gamma_k+(\tau\gamma)_{jk}))^2)}{\sigma_\epsilon^2}\right)^{\left(\frac{nqp}{2}+\alpha_\epsilon+1\right)}$$
$$exp\left(-\frac{\beta_\epsilon+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk}-(\mu+\tau_j+\gamma_k+(\tau\gamma)_{jk})^2}{\sigma_\epsilon^2}\right)(\beta_\epsilon + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - (\mu + \tau_j + \gamma_k + (\tau\gamma)_{jk}))^2)^{-(\frac{nqp}{2}+\alpha_\epsilon+1)} d\sigma_\epsilon^2$$

$\therefore m(Y|H_o) =$

$$(2\pi)^{-\frac{nqp}{2}} \frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)} \Gamma(\frac{nqp}{2} + \alpha_\epsilon + 1)(\beta_\epsilon + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \mu + \tau_j + \gamma_k + (\tau\gamma)_{jk})^2)^{-(\frac{nqp}{2}+\alpha_\epsilon+1)} \quad (21)$$

Since

$$M(Y|H_1, \sigma_\epsilon^2, u, v) = (2\pi\sigma_\epsilon^2)^{-\frac{nqp}{2}}\left(\prod_{i=1}^{1+q+p+qp}(1 + ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1 + vd_i)\right)^{-1/2}$$
$$exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(\sum_{i=1}^{1+q=p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right\}. \quad (22)$$

Therefore,

$$m(Y|H_1) = \int m(Y|M_1, \sigma_\epsilon^2, u, v)\, \pi_0(\sigma_\epsilon^2, u, v)\, d\sigma_\epsilon^2\, du\, dv$$
$$= \int \frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)}(\sigma_\epsilon^2)^{-(\alpha_\epsilon+1)} exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right)(2\pi\sigma_\epsilon^2)^{-nqp/2}\left(\prod_{i=1}^{1+q+p+qp}(1 + ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1 + vd_i)\right)^{-1/2} exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(\sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right\}\pi_0(u, v)\, d\sigma_\epsilon^2\, du\, dv$$

$$= \frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)}(2\pi)^{-nqp/2}\int\left(\prod_{i=1}^{1+q+p+qp}(1 + ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1 + vd_i)\right)^{-1/2}\pi_0(u, v)$$
$$\left\{\int exp\left\{-\frac{1}{\sigma_\epsilon^2}\left(\beta_\epsilon + \frac{1}{2}\sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right\} d\sigma_\epsilon^2\right\} du\, dv$$

$\therefore$

$$m(Y|H_1) = \frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)}(2\pi)^{-nqp/2}\Gamma(nqp/2 + \alpha_\epsilon)\int\left(\prod_{i=1}^{1+q+p+qp}(1 + ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1 + vd_i)\right)^{-1/2}\pi_0(u, v)(\beta_\epsilon + \frac{1}{2}\sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{n}\frac{s_i^2}{1+vd_i})^{-(nqp/2+\alpha_\epsilon-1)} du\, dv \quad (23)$$

5.Example (The storage experiment)

In this section, we illustrate the effectiveness of the our methodology. We have choosing the data set which a study was conducted at date palm research center laboratories, university of Basra, during 2007-2008 season. The objective of the study is to improve storage ability of date palm fruits cv. Barhi at khalal stage. Fruits of both cultivars were soaked in calcium solution at (0,1,2)% concentration for 5 minutes and stored at $0C^{\circ}$, $3C^{\circ}$ and room temperature. the design of the experiment was done according to the model (1). Table (2) below show the results for the analysis of variance for model, from this table we can see that the calculated F-values is greater than the tabulated F-values at 0.05 level significant that is means there is significant effect for calcium chloride on storage capability for date palm fruits under different temperatures. The values of parameters $(\mu, \tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2)$ for the model (1) based on ANOVA table shown in table (3).

Table 2 : ANOVA table for one-way Repeated measures model

Source of variation	*d.f*	*S.S*	*M.S*	*E(M.S)*	*F-Test*
Group	*2*	*2956.3*	*1478.15*	*41765.659*	$F_c = \frac{MS_G}{MS_{U(G)}} = 13.448^*$ $F_t(2,6,0.05)=5.14$
Unit (Group)	*6*	*659.5*	*109.917*	*10724.508*	
Time	*6*	*42468.7*	*7078.117*	*63745.491*	$F_c = \frac{MS_T}{MS_E} = 166.787^*$ $F_t(6,21,0.05)=2.57$
Group*Time	*12*	*22868.9*	*1905.742*	*5759.664*	$F_c = \frac{MS_{G\times T}}{MS_E} = 44.906^*$ $F_t(12,21,0.05)=2.25$
Residual	*21*	*891.2*	*42.4381*	*42.4381*	
Total	*47*	*69844.6*			

Table(3) estimation values for parameters $(\mu, \tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2)$ by ANOVA table

$\hat{\mu}$	$\hat{\tau}_j$	$\widehat{\delta_{i(j)}}$	$\widehat{\gamma_k}$	$\widehat{(\tau\gamma)_{jk}}$	$\widehat{\sigma_\delta^2}$	$\widehat{\sigma_e^2}$
63.584	*141.929*	*411.037*	*413.784*	*398.039*	*55.352*	*42.4381*

We next applied our methodology (Bayesian method) to the storage experiment data. Figure(1) represent the posterior density of coefficients for the model (1). Figure (2) shows the number for iterations of the Gibbs sampler which used in this study, which was 10000 iterations for this data, while figure (3) shows density estimates based on 10000 iterations of σ_e^2 and σ_δ^2 .Table(4) presents the values of the parameters$(\mu, \tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2)$ based on Bayesian method. From table(3) and table(4), we can see that the values of parameters obtained in both ANOVA and Gibbs sampling are nearly alike and encouraging.

Table(4)estimation values for parameters$(\mu, \tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}, \sigma_\delta^2, \sigma_e^2)$ by Bayesian method

$\hat{\mu}$	$\hat{\tau}_j$	$\widehat{\delta_{i(j)}}$	$\widehat{\gamma_k}$	$\widehat{(\tau\gamma)_{jk}}$	$\widehat{\sigma_\delta^2}$	$\widehat{\sigma_e^2}$
63.55	*145.3*	*435.8*	*381.5*	*381.2*	*50.814*	*43.403*

The model checking approach based on Bayes factor which its value was

$$B_{01}(Y) = 2.0022\times 10^{-8} ,$$

that is mean the Bayes factor favors H_1 with strong evidence for the storage experiment data.

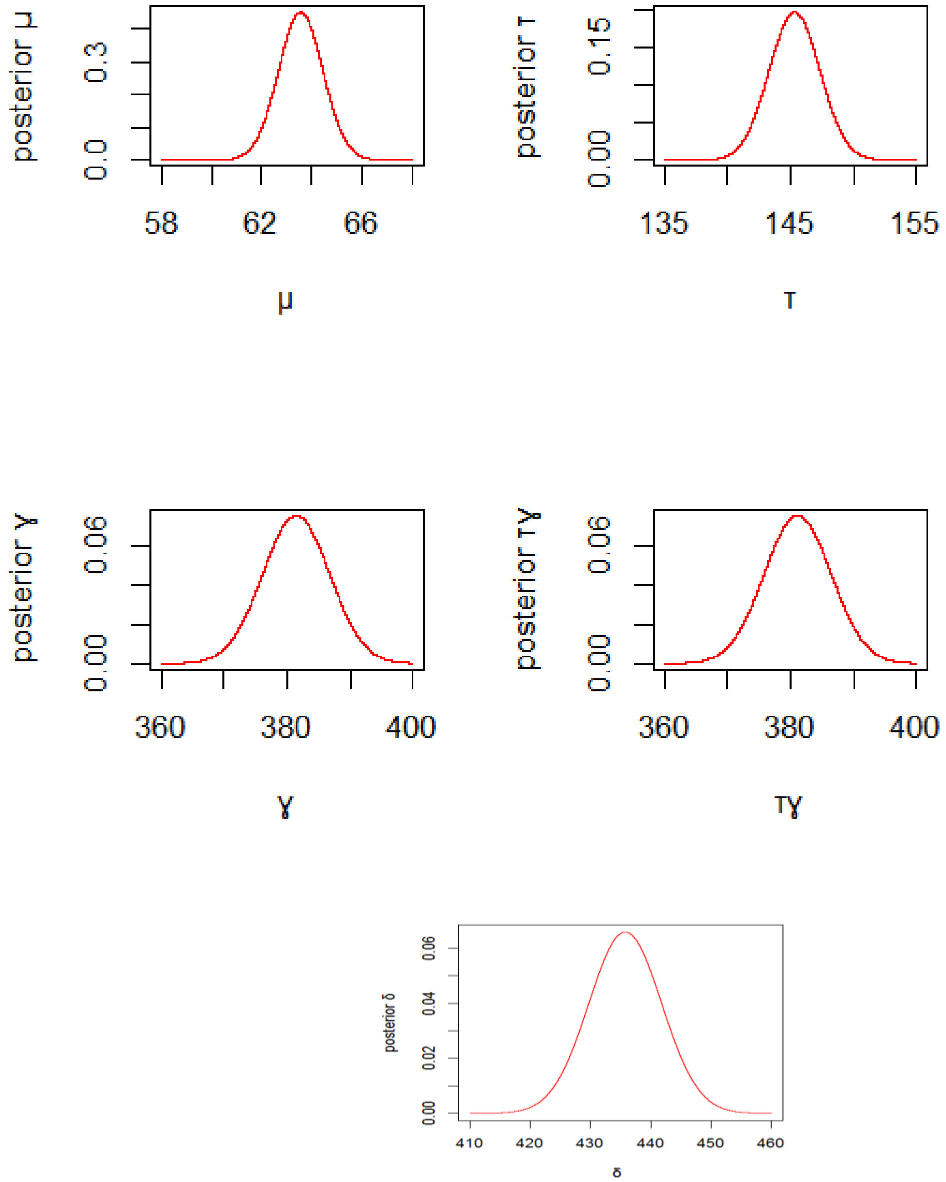


Figure (1) the posterior density of one-way repeated measurements model coefficients($\mu, \tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}$)

Figure (2) shows 10000 iterations of the based Gibbs sampler for the this data

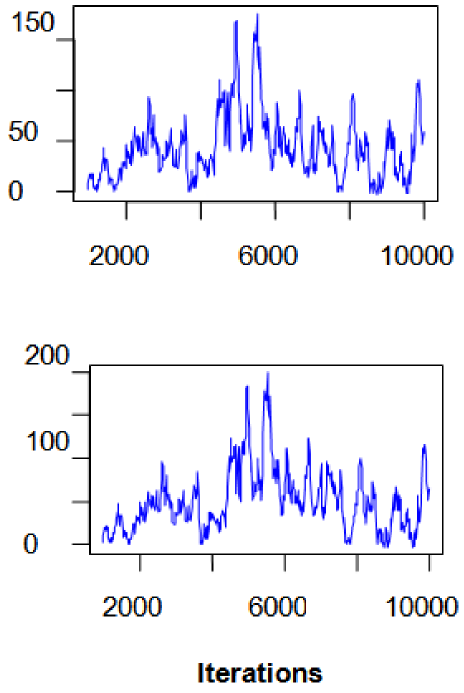
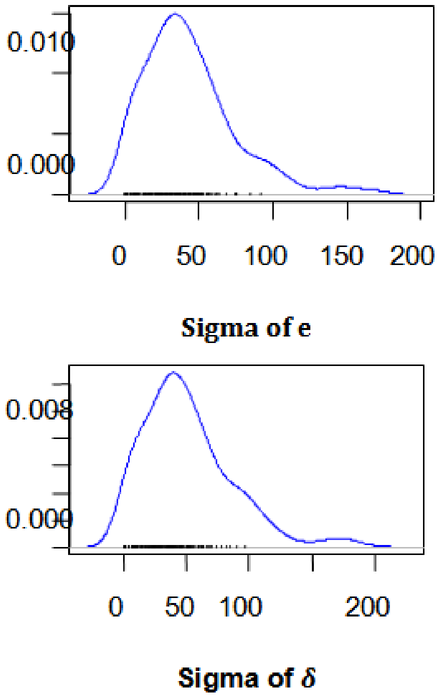


Figure (3) shows density estimates on 10000 iterations of σ_e^2 and σ_δ^2



6.Conclusions

1.We show that the posterior density of u, v given Y in Bayesian one- way repeated measurements model is:

$$\pi_{22}(u,v|Y) \propto \frac{v^{(b/2)-1}u^{\alpha_\epsilon -1}\, exp\frac{Cu}{\beta_\epsilon}}{(a+bv)^{-(a+b)/2}} \left(\prod_{i=1}^{1+q+p+qp}(1+ud_i)\right)^{-1/2} \left(\prod_{i=2+q+p+qp}^{nq}(1+vd_i)\right)^{-1/2} \left(2\beta_\epsilon + \sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)^{-(nqp+2\alpha_\epsilon+2)/2}$$

2.The posterior mean of θ given Y in Bayesian one- way repeated measurements is:

$$E(\theta/Y) = DPE\{\left(I_N + \begin{bmatrix} uI_{1+q+p+qp} & 0 \\ 0 & vI_{nq} \end{bmatrix}\Lambda\right)^{-1} /Y\}C^T s$$

3.The posterior covariance matrix of θ given Y in Bayesian one- way repeated measurements is:

$$Var(\theta|Y) = \frac{1}{nqp+2\alpha_\epsilon+2} E\left[\frac{\left(2\beta_\epsilon+\left(\sum_{i=1}^{1++q+p+qp}\frac{s_i^2}{1+ud_i}+\sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right)}{Y}\right]D - \frac{1}{nqp+2\alpha_\epsilon+2}DC^T PE\left[\left(2\beta_\epsilon + \left(\sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{nq}\frac{s_i^2}{1+vd_i}\right)\right)\left[I_N + \begin{bmatrix} uI_{1+q+p+qp} & 0 \\ 0 & vI_{nq} \end{bmatrix}\Lambda\right]^{-1} /Y\right]P^T CD + E[R(u,v)R(u,v)^T/Y],$$

$$Where\ R(u,v) = DP(I_N + \begin{bmatrix} uI_{1+q+p+qp} & 0 \\ 0 & vI_{nq} \end{bmatrix}\Lambda)^{-1}C^T s$$

4.Bayes factor in Bayesian one- way repeated measurements for testing the two models

$H_o : Y = X\beta + \epsilon$ *versus* $H_1 : Y = X\beta + Zb + \epsilon$ *is:*

$$B_{01}(Y) = \frac{m(Y|H_o)}{m(Y|H_1)}$$

where $m(Y|H_i)$ is the predictive density of Y under model $H_i, i = 0, 1$. We have

$$m(Y|H_o) = (2\pi)^{-\frac{nqp}{2}} \frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)} \Gamma(\frac{nqp}{2} + \alpha_\epsilon + 1)(\beta_\epsilon + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{q}\sum_{k=1}^{p}(y_{ijk} - \mu + \tau_j + \gamma_k + (\tau\gamma)_{jk})^2)^{-(\frac{nqp}{2}+\alpha_\epsilon+1)},$$

$$and\ m(Y|H_1) = \frac{\beta_\epsilon^{\alpha_\epsilon}}{\Gamma(\alpha_\epsilon)}(2\pi)^{-nqp/2}\Gamma(nqp/2 + \alpha_\epsilon)\int\left(\prod_{i=1}^{1+q+p+qp}(1 + ud_i)\right)^{-1/2}\left(\prod_{i=2+q+p+qp}^{nq}(1 + vd_i)\right)^{-1/2}\pi_0(u, v)(\beta_\epsilon + \frac{1}{2}\sum_{i=1}^{1+q+p+qp}\frac{s_i^2}{1+ud_i} + \sum_{i=2+q+p+qp}^{n}\frac{s_i^2}{1+vd_i})^{-(nqp/2+\alpha_\epsilon-1)}du\, dv.$$

5.There is significant effect for calcium chloride on storage capability for date palm fruits cv. Barhi under different temperatures.

6.The values of parameters obtained in both ANOVA and Bayesian method are nearly alike and encouraging.

7.The Bayes factor favors H_1 with strong evidence for the storage experiment data that is mean the correct model is mixed one-way repeated measurements model .

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