Numerical Solution of Airy Differential Equation by Using Haar Wavelet

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Abstract Haar wavelet is exceedingly simple and optimized completely for computers, so that it can be used for solving ordinary differential equations and partial differential equations without a hassle. In this paper, numerical solutions of Airy differential equations have been obtained by using the Haar Wavelet Method . Comparisons with exact solutions make clear that the Haar Wavelet Method is a powerful candidate for solving the Airy differential equation. Moreover the use of Haar wavelets is found to be accurate, uncomplicated, speedy, adaptable and convenient with very small computation costs and the extra perk of being computationally attractive.
Key Words: Orthogonal Wavelet, Airy Equation, Function Approximation, Operational Matrix

1. Introduction

The significance of the Airy equation has been widely acknowledged by scientists all over the world since it constitutes a classical equation of mathematical physics. Even in this particular field, it has a wide range of applications, including but not restricted to modelling the defraction of light and optics problems. At some times, it also makes possible to transform the differential equation at hand into the well-analyzed and quite popular Airy equation.

The Airy differential equation underlies the form of the intensity near a directional caustic, such as a rainbow. Looking back, this was the problem that led Airy [Airy (1838)] to develop the Airy function [Abramowitz and Stegun (1955)]. The Airy function also happens to be the solution to Schr¨odinger's equation for a particle confined within a triangular potential well and for a particle in a one-dimensional constant force field [Vallee and Soares (2004)]. The solutions to a large number of problems may be expressed in terms of the Airy function. One such problem is the linearized Korteweg–de Vries equation [Vallee and Soares (2004)].

Since the Airy equation is linear in nature, its complete analytical solution is found using a Taylor series expansion at the origin. Fortunately, this Taylor series is convergent for all the points. In the case of a discrete Airy equation, the solution can be found exactly if an equidistant discretizationis allowed [Mickens (2001), Ehrhardt and Mickens (2004)]. Some other numerical and asymptotical methods follow [Grosjean and Meyer (1991) ,Vrahatis et al (1996), Amparo(2001), Lakshmi and Murty (2007)]. Liao in [Liao (1992)] proposed a new analytic method for highly nonlinear problems, namely the Homotopy Analysis Method.

In the last two decades, the approximation of orthogonal functions has been playing an important role in the solution of problem such as parameter identification analysis and optimal control. The main characteristic of this technique is that it converts the differential equation that is being used to describe the problem into a set of algebraic equations. Chen and Hsiao [Chen and Hsiao (1997)] were the first to derive the approximation method via Walsh function. Subsequently, the set of orthogonal functions have been extensively applied to solve the parameter identification of linear lumped time invariant systems [Cheng and Hsu (1982)], bilinear systems [Cheng and Hsu (1982)]and multi-input multi-output systems [Hwang (1997)]. The pioneering work in system analysis via Haar wavelets was led by Chen and Hsiao [Chen and Hsiao (1997)] who first derived a Haar operational matrix for the integrals of the Haar function vector and paved the way for the Haar analysis of the dynamical systems. Later Hsiao [Hsiao (2008)] established the method to find solutions for time varying systems by introducing Kronecker product of matrices for avoiding singularities [Hsiao (1997)] and the timevarying singular bilinear systems [Hsiao and Wang, (2001)]. In this paper, we propose a wavelet method to solve the well known Airy differential equation. The method is based on the Haar wavelet operational matrix defined over the interval [0; 1].

The following strategy has been adopted and applied in the rest of this paper. In section 2 the basis of the Haar Wavelet Method is laid out. Application of the method in the Airy equation is discussed in section 3. Finally, conclusions are drawn in section 4.

2. Mathematical Formation

Among the different wavelet families which are defined by analytical expressions, the most simple in mathematical terms are the Haar wavelets. Due to the simplicity the Haar wavelets are very effective for solving ordinary differential and partial differential equations. In 1910, Alfred Haar[Haar (1910)] introduced the notion of wavelets in the form of a rectangular pulse pair function. His initial theory has been expanded recently into a wide variety of applications, but primarily, it allows for the representation of various functions by a combination of step functions and wavelets over specified interval widths. The Haar wavelet is the only real valued function which is symmetrical, orthogonal and has a compact support[Chui (1992)].

Definition 1. Let $h \in L^2(R)$. For $k \in Z$, let $T_k : L^2(R) \to L^2(R)$ be given by $(T_k h)(t) = h(t-k)$ and $D_h : L^2(R) \to L^2(R)$ be given by $(D^j h)(t) = 2^{\frac{j}{2}} h(2^j t)$ where operators T_k and D^j are called translation and dilation operator.

Definition 2. A function $\phi \in L^2(R)$ is called an orthonormal wavelet for $L^2(R)$ if $\{D^k T_n \phi : k,n \in Z\} = \{2^{k/2}\phi(2^{Kk} - n) : k,n \in Z\}$ is an orthonormal basis for $L^2(R)$.

Definition 3. A set of closed subspace $\{V_j : j \in Z\}$ of $L^2(R)$ is called a Multiresolution Analysis (MRA) if the following properties hold.

- $V_j \subseteq V_{j+1}$, for all $j \in Z$
- $D(V_j) = D(V_{j+1})$, for all $j \in Z$
- $\cup_{j\in Z} V_j = L^2(R)$ and $\cap_{j\in Z} V_j = 0$
- There is a scaling function ϕ for V_o

By scaling function we mean that there exists a function $\phi \in V_o$ such that $\{T_n \phi : n \in Z\}$ is an orthonormal basis for V_o. The first curve $h_o(t)$ also known as scaling function is defined as

$$h_o = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & otherwise \end{cases}$$

and second curve h_1 is obtained after distributing the interval $[0,1]$ in $[0,0.5]$ and $[0.5,1]$.

$$h_1 = \begin{cases} 1 & 0 \le x < 0.5 \\ -1 & 0.5 \le x < 1 \\ 0 & otherwise \end{cases}$$

This is also called mother wavelet. In order to perform wavelet transform, Haar wavelet uses translations and dilations of the function, i.e. the transform make use of $\phi(x) = \phi(2^j x - k)$ which represents shifting and scaling $\phi(x) = \phi(2^j x)$ collectively. All other subsequent curves are generated from $h_1(t)$. $h_2(t)$ is obtained from $h_1(t)$ with dilation [Grossmann and Morlet (1984)].Another way that we can express Haar functions in a more compact form is

$$h_n(x) = h_1\left(2^j x - \frac{k}{2^j}\right) \quad , \qquad n = 2^j + k \quad , \qquad j \ge 0 \quad , \qquad 0 < k \le 2^j$$

Here we observe that $h_1(t)$ is compressed from the whole interval $(0,1)$ to the half interval $(0,1/2)$ to generate $h_2(t)$. $h_3(t)$ is same as $h_2(t)$ but shifted to right by $1/2$. In the same way $h_2(t)$ is compressed from a half interval to generate $h_4(t)$ which is shifted to right by $1/4$, $2/4$, $3/4$ to generate $h_5(t)$, $h_6(t)$, $h_7(t)$ respectively. It can be noticed that all the Haar wavelets are orthogonal to each other

$$\int_0^1 h_i h_l \, dx = 2^{-j}\,\delta_{il} = \begin{cases} 2^{-j} & i = l = 2^{j+k} \\ 0 & i \neq l \end{cases}$$

Moreover, for any square integrable function $u(x)$, approximation can be made using the Haar functions as

$$u(x) = \sum_{i=1}^{2M} a_i h_i(x) \tag{1}$$

where h_m are the Haar functions. Identifying the collocation points as $x_l = \frac{2l-1}{2m}$; $l = 1,2,....,m$ we have $h_m(x) = [h_o(x), h_1(x),.....,h_{m-1}(x)]^T$ and thus we obtain the Haar functions as $h_4(1/8) = [1,1,1,0]$, $h_4(3/8) = [1,1,-1,0]$, $h_4(5/8) = [1,-1,0,1]$, $h_4(5/8) = [1,-1,0,1]$ and so on. So, $H_m = [h_m(1/2m), h_m(3/2m), h_m(1/2m),......, h_m(2m-1/2m)]$. In general, if the interval $[a,b]$ is under consideration, it is partitioned into $2M$ subintervals of equal length $\Delta x = (b-a)/2M$ .Introducing the dilation parameter $j = 0,1,2,....,J$ and translation parameter $k = 0,1,2,....,m-1$ with J as maximal level of resolution. The wavelet number i is identified as $i = m + k + 1$ and the t^{th} haar wavelet is defined as

$$h_i = \begin{cases} 1 & x \in [\xi_1(i), \xi_2(i)] \\ -1 & x \in [\xi_2(i), \xi_3(i)] \\ 0 & otherwise \end{cases}$$

with $\xi_1(i) = a + 2k\mu\Delta x$, $\xi_2(i) = a + (2k+1)\mu\Delta x$ $\xi_3(i) = a + (2k+1)\mu\Delta x$ where $\mu = M/m$. It can be seen that for $i = 1$, scaling function $h_1 = 1$ for $x \in [a,b]$ and $h_1(x) = 0$ otherwise. Taking $\alpha = \frac{k}{m}$, $\beta = \frac{k+0.5}{m}$, $\gamma = \frac{k+1}{m}$, for $m = 2^j$, $j = 0,1,2,....,m-1$, $i = m + k + 1$. While working with the integration of these Haar functions, integrals can be computed piecewise as

$$h_i(x) = \begin{cases} 1 & x \in [\alpha, \beta] \\ -1 & x \in [\beta, \gamma] \\ 0 & otherwise \end{cases}$$

$$p_i(x) = \begin{cases} x - \alpha & x \in [\alpha, \beta] \\ \gamma - x & x \in [\beta, \gamma] \\ 0 & otherwise \end{cases}$$

$$q_i(x) = \begin{cases} \dfrac{(x-\alpha)^2}{2} & x \in [\alpha, \beta] \\ \dfrac{(\alpha-\beta)^2 + (\beta-\gamma)^2 - (\gamma-x)^2}{2} & x \in [\beta, \gamma] \\ \dfrac{(\alpha-\beta)^2 + (\beta-\gamma)^2}{2} & x \in [\gamma, 1] \\ 0 & otherwise \end{cases}$$

In particular when $i=1$, it will give

$$h_1(x)=\begin{cases}1 & x\in[0,1]\\ 0 & otherwise\end{cases}$$

$$p_1(x)=\begin{cases}x & x\in[0,1]\\ 0 & otherwise\end{cases}$$

$$q_1(x)=\begin{cases}\dfrac{x^2}{2} & x\in[0,1]\\ 0 & otherwise\end{cases}$$

and so on. In general, for mth order system, Haar matrix H_m is defined by m Haar functions and we can calculate them as $H_1=(1)$

$$H_2=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

$$H_4=\begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & -1\end{pmatrix}$$

$$H_8=\begin{pmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1\\ 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1\\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1\\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1\end{pmatrix}$$

and the operation matrix P with dimension $m\times m$ is calculated likewise

$$P_{m\times m}=\frac{1}{2m}\begin{pmatrix}2mP_{m/2\times m/2} & -H_{m/2\times m/2}\\ H^{-1}_{m/2\times m/2} & O_{m/2\times m/2}\end{pmatrix}$$

In particular, we get $P_1=[1/2]$,

$$P_2=\frac{1}{4}\begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}$$

$$P_4=\frac{1}{16}\begin{pmatrix}8 & -4 & -2 & -2\\ 4 & 0 & -2 & 2\\ 1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\end{pmatrix}$$

$$P_8=\frac{1}{64}\begin{pmatrix}32 & -16 & -8 & -8 & -4 & -4 & -4 & -4\\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4\\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0\\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0\\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0\\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0\end{pmatrix}$$

and so on. Using these matrices and approximation of the form (1), we can solve the Airy Differential Equation.

3. Numerical Solution of Airy Equation

To illustrate the fundamentals of Haar Wavelet method, let us consider the homogeneous Airy equation, which is sometimes called the Strokes Equation

$$\frac{d^2u}{dt^2} - tu = 0 \tag{2}$$

Equation (2) is accompanied either with boundary condition,

$$u(0) = A_i(0), \quad u(\infty) = 0 \tag{3}$$

or with initial conditions

$$u(0) = B_i(0) \qquad , \qquad u'(0) = B_i(0) \tag{4}$$

Together with these, the solutions of this second order Airy differential equation are called the first and second kind Airy functions $A_i(t)$ and $B_i(t)$ respectively. They play an important role in the theory of the asymptotic expansions of various special functions with the known initial values

$$A_i(0) = \frac{3^{-\frac{2}{3}}}{\Gamma\left(\frac{2}{3}\right)} \quad ; \quad B_i(0) = \sqrt{3}A_i(0) \quad ; \quad B_i'(0) = \frac{3^{\frac{1}{6}}}{\Gamma\left(\frac{1}{3}\right)}$$

From the property of the Haar Wavelet transformation, $y''(x)$ can be approximated by Haar wavelet function as

$$y''(x) = \varphi(x, y(x), y'(x)) = \sum_{1}^{2M} a_i h_i(x) \tag{5}$$

$$y'(x) = y'(0) + \sum_{1}^{2M} \left[\int_0^1 a_i h_i(x) dx - \int_0^x a_i h_i(x)\, dx \right] \tag{6}$$

Substituting the values in equation (2) from eq (3-6) and solving these equation for unknown a_i, the approximate solution $y(x)$ can be found out easily.

In this section, solution obtained from the Haar Wavelet Method has been compared with those obtained from Airy functions. The error between the approximate and numerical solutions has been calculated. It can be seen that Haar wavelet with $m = 64$ shows excellent agreement with the other solution, ergo, approximation can be used to represents the exact solution.

x	Numerical Solution	Solution by Haar Wavelet		Error	
		m=4	m=16	m=4	m=16
0.00	1.0000000	1.0000000	1.0000000	0.0000000	0.0000000
0.05	0.9999792	0.9998427	0.9999782	0.0001365	0.0000010
0.10	0.9998333	0.9993706	0.9998312	0.0004627	0.0000021
0.15	0.9994376	0.9985840	0.9994341	0.0008536	0.0000035
0.20	0.9986670	0.9974826	0.9986615	0.0011844	0.0000055
0.25	0.9973972	0.9960665	0.9973884	0.0013307	0.0000088
0.30	0.9955040	0.9940162	0.9954894	0.0014879	0.0000146
0.35	0.9928644	0.9910118	0.9928393	0.0018526	0.0000251
0.40	0.9893561	0.9870535	0.9893128	0.0023026	0.0000433
0.45	0.9848586	0.9821412	0.9847846	0.0027174	0.0000740
0.50	0.9792533	0.9762749	0.9791295	0.0029784	0.0001238
0.55	0.9724243	0.9691372	0.9722224	0.0032871	0.0002018
0.60	0.9642584	0.9604107	0.9639380	0.0038477	0.0003205
0.65	0.9546466	0.9500953	0.9541510	0.0045512	0.0004955
0.70	0.9434838	0.9381912	0.9427365	0.0052927	0.0007474
0.75	0.9306705	0.9246982	0.9295691	0.0059723	0.0011014
0.80	0.9161127	0.9093049	0.9145238	0.0068078	0.0015889
0.85	0.8997233	0.8916999	0.8974755	0.0080234	0.0022478
0.90	0.8814227	0.8718831	0.8782992	0.0095396	0.0031235
0.95	0.8611397	0.8498547	0.8568700	0.0112850	0.0042697
1.00	0.8388123	0.8256145	0.8330632	0.0131979	0.0057491

Table 1: Comparison between numerical and approximate solution of Airy Equation

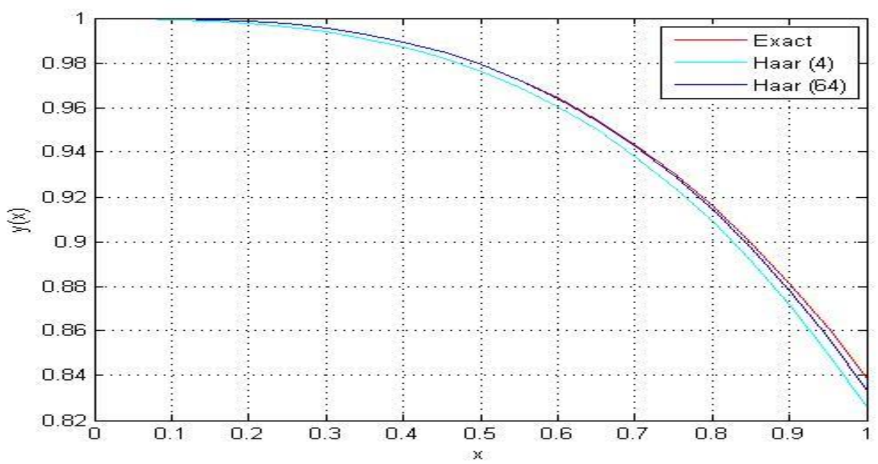

Fig 1: Graph of Approximate solution by Haar wavelet with m=4 and m=16 and numerical solution of Airy differential equation

4. Discussion and Results

The theoretical elegance of the Haar Wavelet approach can be appreciated from the simple mathematical relations; their compact derivations and proofs. It has been well demonstrated that in applying the properties of Haar wavelets, the differential equations can be solved conveniently and accurately by its systematic use. The main goal of this paper is to apply the Haar wavelet method to the well-known Airy differential equation that appears frequently in many scientific applications. In comparison with existing numerical schemes that have been previously used to solve the Airy differential equation, the scheme in this paper is an improvement in terms of accuracy. It is worth mentioning that the Haar Solution provides good results even for small values of m (m =4). As the values of m increase (i.e., m = 64), the accuracy of the results becomes more and more reliable.

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