

A STUDY OF DERIVATIONS ON LATTICES

Funmilola Balogun

Department of Mathematical Sciences and Information Technology, Federal University Dutsinma, Katsina State, Nigeria

fbalogun@fudutsinma.edu.ng

Abstract

In this paper, we introduce the notion $f(x \wedge y) = x \wedge fy$, where f is a derivation on a lattice L and $x, y \in L$, using this notion, equivalence relations were established on L. Secondly, we extend some results of isotone derivations on a distributive lattice. Finally, it is shown that $\langle F(L), \vee, \wedge \rangle$ is a modular lattice, where L is a modular lattice and $F(L)$ is the set of all isotone derivations on L.

Keywords: Isotone derivation, distributive lattice, modular lattice.

1. Introduction

Several authors [2,3,6,10,11,13,20,22] have studied analytical and algebraic properties of lattices. In the eighteenth century, George Boole [7] initiated the study of lattices. In this context Richard Dedekind in a series of paper around 1900 laid foundation for lattice theory. The distributive lattices were introduced by Gratzer [12]. These lattices have provided the motivation for many results in general lattice theory.

Lattice theory has quite a number of applications in many research areas such as information retrieval, and information access controls (see [8] and [20]). Sandhu [20] showed that lattice based mandatory access controls can be enforced by appropriate configuration of Role Based Access Control (RBAC) components. In [11], the author solved several problems in cryptanalysis using tools from the geometry of numbers. The probability density under a general hypergraphical model was expressed using co-information lattice in [3].

Derivations in rings and near rings have been widely researched [4, 5, 15, 18, 19]. The concept of derivation in a lattice was introduced in [22], Xin et al characterized modular lattices and distributive lattices by isotone derivations and gave conditions under which a derivation is isotone for bounded lattices, modular lattices and distributive lattices. Several other authors [1, 9, 21, 23] also studied derivations on lattices. In [21], the author using fixed sets of isotone derivations established characterizations of a chain, a distributive lattice, a modular lattice and a relatively pseudo-complimented lattice. The application of the notion of derivation in ring and near-ring theory to BCI-algebras was given by Jun and Xin [16], see also [24]. Section 2, is devoted to some basic definitions and results. In Section 3, we define $f(x \wedge y) = x \wedge fy$ and establish equivalence relations using isotone derivations on L. Section 4 gives an extension of isotone derivations on distributive lattices, which was studied in [22]. Also, by defining a partial order on the set of all isotone derivations on a modular lattice, we prove that this set of isotone derivations together with the operations of meet $' \wedge '$ and join $' \vee '$ form a modular lattice.

2. Preliminaries

The following are basic definitions and results on lattices and derivations on lattices

Definition 2.1[6]: Let L be a nonempty set endowed with operations $\wedge$ and $\vee$. Then $(L, \wedge, \vee)$ is called a lattice if it satisfies the following conditions for all $x, y, z \in L$:

i. $x \wedge x = x, \ x \vee x = x$

ii. $x \wedge y = y \wedge x, \;\; x \vee y = y \vee x$

iii. $(x \wedge y) \wedge z = x \wedge (y \wedge z), \;\; (x \vee y) \vee z = x \vee (y \vee z)$

iv. $(x \wedge y) \vee x = x, \;\; (x \vee y) \wedge x = x.$

Definition 2.2 [6]: A lattice $(L,\wedge,\vee)$ is called a distributive lattice if it satisfies any of the following conditions for all $x, y, z \in L$:

v. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

vi. $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

In any lattice, conditions v and vi are equivalent.

Definition 2.3 [2]: A lattice $(L,\wedge,\vee)$ is called a modular lattice if it satisfies the following conditions for all $x, y, z \in L$:

vii. If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$

Condition vii is called the modular identity.

Example 2.4: A distributive lattice of fundamental importance is the two-element chain$(2,\wedge,\vee)$**.** This lattice features prominently in logic as the lattice of truth values.

In [14] the lattice L is called n-distributive if, $x \wedge (\vee_{i=0}^{n} y_i) = \vee_{i=0}^{n} \left(x \wedge \left(\vee_{j(\neq i)=0}^{n} y_j\right)\right)$

Figure 1 represents a modular lattice.

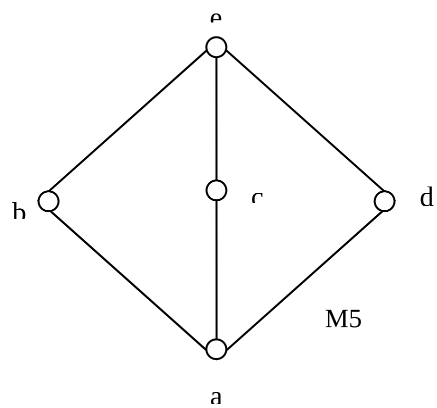

Figure 1 A modular lattice

Definition 2.5 [6]: Let $(L,\wedge,\vee)$ be a lattice. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

We need the following result:

Lemma 2.6 [6]: Let $(L,\wedge,\vee)$ be a lattice. Define the binary relation $\leq$ as in definition 2.5. Then $(L, \leq)$ is a partially ordered set **(**poset) and for any $x, y \in L, x \wedge y$ is the g.l.b of $\{x, y\}$, and $x \vee y$ is the l.u.b of $\{x, y\}$.

Definition 2.7 [6]: An ideal I of the lattice $(L,\wedge,\vee)$ is a nonempty subset I of L with the properties:

viii. $x \leq y, \;\; y \in I \Longrightarrow x \in I$

ix. $x, y \in I \Longrightarrow x \vee y \in I$

I_1, I_2 are ideals of L, implies that $I_1 \wedge I_2$ is an ideal of L.

Definition 2.8 [6]: Let $\theta: L \longrightarrow M$ be a function from a lattice L to a lattice M. Then θ is a lattice homomorphism if $\theta(x \wedge y) = \theta(x) \wedge \theta(y)$ and

$\theta(x \vee y) = \theta(x) \vee \theta(y)$, for all $x, y \in L$

A homomorphism is called an isomorphism if it is bijective, an epimorphism if it is onto and a monomorphism if it is one-to-one.

The following is an analogous form of the Leibniz's formula for derivations on a ring.

Definition 2.9 [22]: Let L be a lattice and $f: L \to L$ a function. We call f a derivation on L if it satisfies the condition:

$$f(x \wedge y) = (f(x) \wedge y) \vee (x \wedge f(y))$$

Proposition 2.10 [22]: Let L be a lattice and f a derivation on L. Then the following hold:

1. $fx \leq x$;
2. $fx \wedge fy \leq f(x \wedge y) \leq fx \vee fy$;
3. If I is an ideal of L, then $fI \subseteq I$, where $fI = \{fx: x \in I\}$;
4. If L has a least element 0, $d0 = 0$.

From proposition 2.10, it is obvious that derivations in lattices are contraction mappings i.e. $fx \leq x$.

Definition 2.11 [22]: Let L be a lattice and f a derivation on L. If $x \leq y$ implies $fx \leq fy$, then f is called an isotone derivation.

Proposition 2.12 [22]: Let L be a lattice and f a derivation on L. If $y \leq x$ and $fx = x$, then $fy = y$.

Theorem 2.13 [22]: Let L be a distributive lattice and f a derivation on L. The following conditions are equivalent:

1. f is an isotone derivation
2. $f(x \wedge y) = fx \wedge fy$
3. $f(x \vee y) = fx \vee fy$

3. Derivations on Lattices.

The following results were established in [21], we include the proof for the sake of convenience for the reader.

Proposition 3.1: Let L be a lattice and $a \in L$. Define a function f_a by $f_a(x) = x \wedge a$ for all $x \in L$ then f_a is a derivation on L. Such derivations are called principal derivations.

$f_a(x \wedge y) = (f_a x \wedge y) \vee (x \wedge f_a y) = \big((x \wedge a) \wedge y\big) \vee \big(x \wedge (y \wedge a)\big) = \big((x \wedge y) \wedge a\big) \vee \big((x \wedge y) \wedge a\big) = (x \wedge y) \wedge (a \vee a) = (x \wedge y) \wedge a$

Proposition 3.2: Every principal derivation of a lattice L is an isotone derivation of L.

Proof

Let f_a be a principal derivation of a lattice L. Since for any $x, y \in L$, we have $f_a x = x \wedge a \leq y \wedge a = f_a y$ and hence f_a is isotone.

Next we prove the following results:

Theorem 3.3: Let L be a lattice and $f: L \to L$ be a derivation. The following conditions are equivalent:

1. f is an isotone derivation
2. $f(x \wedge y) = x \wedge fy$

Proof

$1 \Longrightarrow 2$

Suppose f is an isotone derivation,

we have, $x \leq y \Longrightarrow fx \leq fy, \forall x, y \in L$

Then $f(x \wedge y) = (fx \wedge y) \vee (x \wedge fy) \geq x \wedge fy \ldots .(i)$

Also, $x \wedge y \leq x$ and $x \wedge y \leq y$

This implies that, $f(x \wedge y) \leq fx$ and $f(x \wedge y) \leq fy$

We have, $f(x \wedge y) \leq fx \wedge fy \leq x \wedge fy \ldots .(ii)$

From (i) and (ii) we have $f(x \wedge y) = x \wedge fy$

$2 \Longrightarrow 1$,

Suppose $f(x \wedge y) = x \wedge fy \ \forall x, y \in L$

We have; $f(x \wedge y) = (x \wedge fy) \vee (fx \wedge y) = x \wedge fy$

If $x \leq y, since\ f(x \wedge y) = x \wedge fy \Longrightarrow fx = x \wedge fy$

We have $fx \vee fy = (x \wedge fy) \vee fy = fy$

Therefore, $fx \leq fy$, hence f is isotone. □

Theorem 3.4: Let L be a lattice and $f: L \longrightarrow L$ be a derivation. The following conditions are equivalent:

1. $f(x \wedge y) = x \wedge fy$
2. $f(x \wedge y) = fx \wedge fy$

Proof

(1)$\Longrightarrow$ (2)

Suppose $f(x \wedge y) = x \wedge fy$, clearly, $fx \wedge fy \leq fx \wedge y$

Since $x \wedge fy = f(x \wedge y)$ also, $fy \wedge x \leq fy$ and $fx \wedge y \leq fx$

This implies that $fy \wedge x = fx \wedge y \leq fx \wedge fy \Longrightarrow x \wedge fy = y \wedge fx \leq fx \wedge fy$

Therefore, $f(x \wedge y) = fx \wedge fy$.

(2)$\Longrightarrow$ (1)

Suppose $f(x \wedge y) = fx \wedge fy \ \forall x, y \in L$

If $x \leq y$ then $fx = f(x \wedge y) = fx \wedge fy \Longrightarrow fx \leq fy$, hence f is an isotone derivation. Since f is an isotone derivation, by Theorem 3.3, we have $f(x \wedge y) = x \wedge fy$ □

From Theorems 3.3 and 3.4 the following result can be established:

Theorem 3.5: Let L be a lattice and $f: L \rightarrow L$ be a derivation. The following conditions are equivalent:

1. f is an isotone derivation
2. $f(x \wedge y) = x \wedge fy$
3. $f(x \wedge y) = fx \wedge fy$

4. Derivations on Distributive and Modular Lattices

Distributive lattices have provided the motivation for many results in general lattice theory. Many conditions on lattices are weakend forms of distributivity. Hence derivations on distributive lattices have stronger properties.

Theorem 4.1 [21]: Let L be a distributive lattice and $f_1\ and\ f_2$ be two isotone derivations on L. Define

$$(f_1 \wedge f_2)(x) = f_1 x \wedge f_2 x,$$
$$(f_1 \vee f_2)(x) = f_1 x \vee f_2 x.$$

Then $f_1 \wedge f_2$ and $f_1 \vee f_2$ are also isotone derivation on L.

In this sequel we establish the following result:

Theorem 4.2: Let f_1, f_2 and f_3 be isotone derivations on a distributive lattice L, defined by

$$\big((f_1 \wedge f_2) \wedge f_3\big)x = (f_1x \wedge f_2x) \wedge f_3x$$

$$\big((f_1 \vee f_2) \vee f_3\big)x = (f_1x \vee f_2x) \vee f_3x$$

Then $(f_1 \wedge f_2) \wedge f_3$ and $(f_1 \vee f_2) \vee f_3$ are also isotone derivations on L.

Proof

Now,

$$\big((f_1 \vee f_2) \vee f_3\big)(x \wedge y) = f_1(x \wedge y) \vee f_2(x \wedge y) \vee f_3(x \wedge y)$$
$$= (x \wedge f_1y) \vee (x \wedge f_2y) \vee (x \wedge f_3y) = x \wedge \big((f_1y \vee f_2y) \vee f_3y\big)$$
$$= x \wedge \big((f_1 \vee f_2) \vee f_3\big)y \;....(i)$$

Similarly,

$$\big((f_1 \vee f_2) \vee f_3\big)(x \wedge y) = \big((f_1 \vee f_2) \vee f_3\big)x \wedge y \;...(ii)$$

Combining (i) and (ii) we have,

$$\big((f_1 \vee f_2) \vee f_3\big)(x \wedge y) = \Big(\big((f_1 \vee f_2) \vee f_3\big)x \wedge y\Big) \vee \Big(\big((f_1 \vee f_2) \vee f_3\big)y \wedge x\Big)$$

Therefore $(f_1 \vee f_2) \vee f_3$ is a derivation on L.

Also, by Theorem 2.13 $(f_1 \vee f_2) \vee f_3$ is isotone since

$$\big((f_1 \vee f_2) \vee f_3\big)(x \vee y) = \big((f_1 \vee f_2)(x \vee y)\big) \vee f_3(x \vee y) = f_1(x \vee y) \vee f_2(x \vee y) \vee f_3(x \vee y)$$
$$= (f_1x \vee f_1y) \vee (f_2x \vee f_2y) \vee (f_3x \vee f_3y) = \big((f_1x \vee f_2x) \vee f_3x\big) \vee \big((f_1y \vee f_2y) \vee f_3y\big)$$
$$= \big((f_1 \vee f_2)x \vee f_3x\big) \vee \big((f_1 \vee f_2)y \vee f_3y\big) = \big((f_1 \vee f_2) \vee f_3\big)x \vee \big((f_1 \vee f_2) \vee f_3\big)y$$

Similarly, $(f_1 \wedge f_2) \wedge f_3$ is an isotone derivation on L. □

The following result is due to [21]

Theorem 4.3: Let L be a distributive lattice and $D(L)$ be a set of all isotone derivations on L. Then, $\langle D(L), \vee, \wedge \rangle$ is a distributive lattice.

In this sequel, we establish the following result for modular lattices:

Theorem 4.4: Let L be a modular lattice and $F(L)$ be a set of all isotone derivations on L.

Then, $\langle F(L), \vee, \wedge \rangle$ is a modular lattice.

Proof

From Theorem 4.2, we know that $\wedge$ *and* $\vee$ are binary operators on $F(L)$. Define a partial order $\leq$ on $F(L)$ by $f_1 \leq f_2$ if and only if $f_1 \wedge f_2 = f_1$ *and* $f_1 \vee f_2 = f_2$.

The g.l.b $\{f_1, f_2\} = f_1 \wedge f_2$, and l.u.b $\{f_1, f_2\} = f_1 \vee f_2$,

hence $\langle F(L), \vee, \wedge \rangle$ is a lattice.

Furthermore, for any $f_1, f_2, f_3 \in F(L)$ and for all $x \in L$ we have,

$$\big(f_1 \vee (f_2 \wedge f_3)\big)x = f_1x \vee (f_2x \wedge f_3x) = (f_1x \vee f_2x) \wedge (f_1x \vee f_3x) = \big((f_1 \vee f_2)x\big) \wedge \big((f_1 \vee f_3)x\big)$$
$$= f_2x \wedge f_3x = (f_2 \wedge f_3)x = f_2x$$

also,

$$\big((f_1 \vee f_2) \wedge f_3\big)x = (f_1x \vee f_2x) \wedge f_3x = (f_1x \wedge f_3x)x \vee (f_2x \wedge f_3x) = (f_1 \wedge f_3)x \vee (f_2 \wedge f_3)x = f_1x \vee f_2x$$
$$= (f_1 \vee f_2)x = f_2x$$

Therefore, $\big(f_1 \vee (f_2 \wedge f_3)\big)x = \big((f_1 \vee f_2) \wedge f_3\big)x$ □

References

1. Alshehri, N. O. (2010). Generalized Derivations of Lattices, Int. J. Contemp. Math. Sciences **5**, 629-640.
2. Balbes, R. and Dwinger, P. (1974). Distributive Lattices, University of Missouri Press, Columbia, USA.
3. Bell, A.J. (2003). The co-information lattice, In: 4th International Symposium on Independent Component Analysis and Blind Signal Seperation (ICA2003), Nara, pp. 921-926.
4. Bell, H.E. and Kappe, L.C. (1989). Rings in which derivations satisfy certain algebraic conditions. Acta Math. Hng. **53**(3-4),339-346.
5. Bell, H. E. and Mason, G. (1987). On derivations in near-rings and near-fields. North-Holl. Math.Std.**137**, 31-35.
6. Birkhoff, G. (1940). Lattice Theory, American Mathematical Society, New York.
7. Boole, G. (1854). An investigation of the Laws of Thought, Cambridge University Press.
8. Carpineto, C. and Romano, G. (1996). Information retrieval through hybrid navigation of lattice representations. Int. J. Human-Comput. Stud. **45**,553-578.
9. Chaudhry, M. A. and Ullah, Z. (2011). On Generalized α,β-Derivations on Lattices, Quaestiones Mathematicae **34**, 417-424.
10. Degang, C., Wenxiu, Z., Yeung, D. and Tsang, E.C.C. (2006). Rough approximations on a complete distributive lattice with applications to generalized rough sets, Inf. Sci. **176**, 1829-1848.
11. Durfee, G. (2002). Cryptanalysis of RSA using algebraic and lattice methods. A dissertation submitted to the Department of Computer Sciences and the committee on graduate studies, Stanford University, California, USA.
12. Gratzer, G. (2008). *Lattice Theory*. First concepts and Distributive Lattices. W. H. Freeman and Co., San Fransisco, calif., 1971. xv+212pp, softcover edition, Dover publications.
13. Honda, A. and Grabisch, M. (2006). Entropy of capacities on lattices and set systems. Inf. Sci. **176**, 3472-3489.
14. Huhn, A. (1975). n-distribtivity and some questions of the equational theory of lattices, Colloquia Math. Soc. J. Bolyai, 17. Contributions to niversal algebra, Szeged, 1975, North-Holland, Amsterdam, 1977, 167-178.
15. Hvala, B. (1998). Generalized derivations in prime rings. Commun. Algebra **26**, 1147-1166.
16. Jun, Y. B. and Xin, X. L. (2004). On derivations of BCI-algebras. Inf. Sci. **159**, 167-176.
17. Karacal, F. (2006). On the direct decomposability of strong negations and S-implication operators on product lattices. Inf. Sci. **176**, 3011-3025.
18. Kaya, K. (1987-1988). Prime rings with α derivations. Bull. Natural Sci. Eng.**16-17**, 63-71.
19. Posner, E. (1957). Derivations in prime rings. Proc. Amer. Math. Soc. **8**,1093-1100.
20. Sandhu, R.S. (1996). Role hierarchies and constraints for lattice-based access controls. In: Proceedings of the 4th European Symposium on Research in Computer Security, Rome, pp. 65-79.
21. Xin, X. L. (2012). The fixed set of a derivation in lattices, Fixed Point Theory and Applications(a Springer Open Journal)
22. Xin, X. L., Li, T. Y. and Lu, J.H. (2008). On derivations of lattices. Inf. Sci. **176**, 307-316.
23. Yilmaz, C. and Ozturk, M. A. (2008). On f-derivations of lattices, Bull. Korean Math. Soc. 1675-1684.
24. Zhan, J. and Liu, T. Y. (2005). On f-derivations of BCI-algebras, Int. J. Math. Sci. **11**, 1675-1684.