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Analytic approximate solutions of Volterra's population and some scientific models by power series method

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Abstract.
In this paper, we have implement an analytic approximate method based on power series method (PSM) to obtain a solutions for Volterra's population model of population growth of a species in a closed system. The numerical solution is obtained by combining the PSM and Padé technique. The Padé approximation that often show superior performance over series approximation are effectively used in the analysis to capture essential behavior of the population $u(t)$ of identical individuals. The results demonstrate that the method has many merits such as being derivative-free, overcome the difficulty arising in calculating Adomian polynomials to handle the nonlinear terms in Adomian Decomposition Method (ADM). It does not require to calculate Lagrange multiplier as in Variational Iteration Method (VIM) and no needs to construct a homotopy and solve the corresponding algebraic equations as in Homotopy Perturbation Method (HPM). Moreover, we used this method to solve some scientific models, namely, the hybrid selection model, the Riccati model and the logistic model to provide the analytic solutions. The obtained analytic approximate solutions of applying the PSM is in full agreement with the results obtained with those methods available in the literature. The software used for the calculations in this study was MATHEMATICA® 8.0.

Keywords: Power series method; Volterra's population model; Padé approximation; Hybrid model; Riccati model; Logistic model

1 Introduction

The Volterra integro-differential equations arise from the mathematical modeling of various physical, engineering and biological models for example the population growth of a species in a closed system. These models help us to understand different factors like the behavior of the population evolution over a period of time.

Recently, many attempts have been made to develop analytic and approximate methods to solve the Volterra's population model [1–8]. Although such methods have been successfully applied but some difficulties have appeared, for examples, in calculating Adomian polynomials to handle the nonlinear terms in Adomian Decomposition Method (ADM) and construct a homotopy as in Homotopy Perturbation Method (HPM) and solve the corresponding the algebraic equations and calculate Lagrange multiplier as in Variational Iteration Method (VIM).

In this paper, the applications of the power series method [9, 10] for Volterra's model of population growth will be presented to find the approximate solutions, Moreover, The main attractive features of the current method are being derivative-free, overcome the difficulty in some existing techniques, simple to understand and easy to implement. It is economical in terms of computer power/memory and does not involve tedious calculations such as Adomian polynomials. Also, the proposed method obtains Taylor expansion of the exact solution by using simple computations.

The paper is organized as follows. Section 2 is devoted to the description of the Volterra's population model. In section 3 the basic idea of the power series method (PSM) is illustrated. In section 4 the problem is solved by power series method (PSM). In section 5 the error analysis, comparison with some existing techniques and rate of convergence are discussed. In section 6, some scientific models are solved by power series method (PSM) to obtain the analytic solation and finally in section 7 the conclusion is presented.

2 The Volterra's population model

The model is characterized by the nonlinear Volterra integro-differential equation [1, 2]

$$\kappa \frac{du}{dt} = u - u^2 - u \int_0^t u(x)dx, \quad u(0) = 0.1 \tag{1}$$

where $u = u(t)$ is the scaled population of identical individuals at a time t and κ is a prescribed parameter. The nondimensional parameter $\kappa = c/(ab)$, where $a > 0$ is the birth rate coefficient, $b > 0$ is the crowding coefficient, and $c > 0$ is the toxicity coefficient [1, 2]. The coefficient c indicates the essential behaviour of the population evolution before its level falls to zero in the long run. Volterra introduced this model for a population $u(t)$ of identical individuals which exhibits crowding and sensitivity to the amount of toxins produced. The nonlinear model in Eq.(1) includes the well-known terms of a logistic equation, and in addition it includes an integral term that characterizes the accumulated toxicity [1, 2] produced since time zero.

Wazwaz [1] used ADM to solve the governing problem. Moreover, Mohyud-Din et al. [2] employed the combining of the HPM and Padé technique to obtain the numerical solutions to Volterras population model. Parand [3] used Rational Chebyshev tau method to solve problem. Parand et al. [4] established a method based on collocation approach using Sinc functions and Rational Legendre functions. Also, [5] a numerical method based on hybrid function consist of block-pulse and Lagrange-interpolating polynomials approximations was proposed to solve Volterras population model and in [6] Khana used New Homotopy Perturbation Method (NHPM) which is an improvement of the classical HPM. Al-Khaled [7] implemented the ADM and Sinc-Galerkin method for the solution of some mathematical population growth models. In addition, Ramezani et al. [8] applied the spectral method to solve Volterras population on a semi infinite interval.

3 The basic idea of the Power Series Method (PSM)

Consider the Volterra integral equation [9, 10]

$$u(x) = f(x) + \int_0^x k(x,t)[u(t)]^p dt \tag{2}$$

In Eq.(2), the functions $f(x)$ and $k(x,t)$ are known, and $u(x)$ is the unknown function to be determined, also $p \geq 1$ is a positive integer number. Suppose the solution of Eq.(2) with $e_0 = u(0) = f(0)$ as the initial condition to be as follows,

$$u(x) = e_0 + e_1 x, \tag{3}$$

where, e_1 is a unknown parameter.
If we substitute Eq.(3) into Eq.(2) linear algebraic equation:

$$(a_1 e_1 - b_1)x + Q(x^2) = 0, \tag{4}$$

where, a_1 and b_1 are known constant values and $Q(x^2)$ is a polynomial with the order greater than unity.
By neglecting $Q(x^2)$ in Eq.(4) and solving the system of $a_1 e_1 = b_1$, the unknown parameter e_1 can be determined and therefore the coefficient of x in Eq.(3) obtains.
In the next step, we assume that the solution of Eq.(1) to be,

$$u(x) = e_0 + e_1 x + e_2 x^2, \tag{5}$$

here, e_0 and e_1 both are known and e_2 is unknown parameter.
By substituting Eq.(5) into Eq.(2), we have following system,

$$(a_2 e_2 - b_2)x^2 + Q(x^3) = 0. \tag{6}$$

By neglecting $Q(x^3)$ in Eq.(6) and solving the system of $a_2 e_2 = b_2$, the unknown parameter e_2 will be obtained and therefore the coefficient of x^2 in Eq.(5) obtains.
By repeating the above procedure for m iteration, a power series of the following form derives,

$$u(x) = e_0 + e_1 x + e_2 x^2 + ... + e_m x^m, \tag{7}$$

Eq.(7) is an approximation for the exact solution $u(x)$ of the integral equation (2).

Theorem 3.1 [10]
Let $u = u(x)$ be the exact solution of the following Volterra integral equation

$$u(x) = f(x) + \int_0^x k(x,t)[u(t)]^p dt \tag{8}$$

Then, the proposed method obtains the Taylor expansion of $u(x)$.

Corollary 3.2 [10]
If the exact solution to Eq.(8) be a polynomial, then the proposed method will obtain the real solution.

4 Solution of Volterra's model by power series method (PSM)

In this section we consider the population growth model characterized by nonlinear Volterra integro-differential equation

$$\frac{du}{dt} = 10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx, \quad u(0) = 0.1 \tag{9}$$

where the initial condition $u(0) = 0.1$ and the nondimensional parameter $\kappa = 0.1$ were used by [1] in Eq. (1) for simplicity reasons. By integrating Eq. (9) with respect t from 0 to t, we get following:

$$\int_0^t \frac{du}{dt} = \int_0^t [10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx]dt \tag{10}$$

By using the initial condition $u(0) = 0.1$, then we have the following nonlinear Volterra integral equation:

$$u(t) = 0.1 + \int_0^t [10u(t) - 10u^2(t) - 10u(t)\int_0^t u(x)dx]dt \tag{11}$$

Now apply the power series method to Eq.(11), we suppose the solution of Eq.(11)

with $u_0(t) = u(0) = e_0$ is

$$u_1(t) = e_0 + e_1 t \Longrightarrow u_1(t) = 0.1 + e_1 t \tag{12}$$

Substitute Eq.(12) in Eq.(11), we have,

$$0.1 + e_1 t = 0.1 + \int_0^t [10(0.1 + e_1 t) - 10(0.1 + e_1 t)^2 - 10(0.1 + e_1 t)\int_0^t (0.1 + e_1 x)dx]dt \tag{13}$$

After simplifying, we get, $e_1 = 0.9$ and hence,

$$u_1(t) = 0.1 + 0.9t \tag{14}$$

which is the first approximation for the solution of (11). Now the solution of Eq.(11) can be supposed as:

$$u(t) = 0.1 + 0.9t + e_2 t^2 \tag{15}$$

Substitute Eq.(15) in Eq.(11), we have,

$$\begin{aligned} 0.1 + e_1 t + e_2 t^2 = 0.1 + \int_0^t [10(0.1 + e_1 t + e_2 t^2) - 10(0.1 + e_1 t + e_2 t^2)^2 - \\ 10(0.1 + e_1 t + e_2 t^2)\int_0^t (0.1 + e_1 x + e_2 x^2)dx]dt \end{aligned} \tag{16}$$

After simplifying, we get, $e_2 = 3.55$ and hence,

$$u_2(t) = 0.1 + 0.9t + 3.55t^2 \tag{17}$$

Proceeding in this way the components u_3, u_4, u_5, u_6 and u_7 were also calculated but for brevity not listed and u_8 will be used to obtain,

$$\begin{aligned} u(t) = 0.1 + 0.9t + 3.55t^2 + 6.316667t^3 - 5.5375t^4 - 63.7091667t^5 \\ -156.0804167t^6 - 18.4732341t^7 + 1056.288569t^8 + O(t^9). \end{aligned} \tag{18}$$

The approximation in Eq.(18) is in full agreement with the results obtained by using ADM, HPM and VIM in [1, 2, 11].

5 Error analysis, comparison with other approximate methods and rate of convergence

In this section, we discuss the approximate solution that obtained in Eq.(18) and the relation between increasing the κ and the corresponding absolute error. Moreover, the Comparison between the absolute errors of ADM, HPM, VIM and PSM also presented. In addition, the convergence of the PSM, the main result are proposed in a theorem and the convergent rate are discussed based on the absolute error remainder functions and the maximal error remainder parameters.

5.1 Error analysis and comparison with other approximate methods

The results obtained here can be discussed to get more details about the mathematical structure of $u(t)$. In particular, we seek to study the rapid growth along the logistic curve that will reach a peak, then followed by the slow exponential decay where $u(t) \to 0$ as $t \to \infty$, [1]. In order to study the mathematical structure of $u(t)$ the Padé approximants is used. It widely used in numerical analysis and fluid mechanics, because they are more efficient than polynomials, which have the advantage of manipulating the polynomial approximation into a rational function to gain more information about $u(t)$. It is of interest to note that Padé approximants give results with no greater error bounds than approximation by polynomials [1, 12].

Using the approximation obtained for $u(t)$ in Eq.(18), we find

$$[4/4] = \frac{0.1 + 0.4687931832t + 0.924957398t^2 + 0.9231294892t^3 + 0.4004234788t^4}{1 - 4.312068168t + 12.5581875t^2 - 13.88063927t^3 + 10.8683047t^4}$$

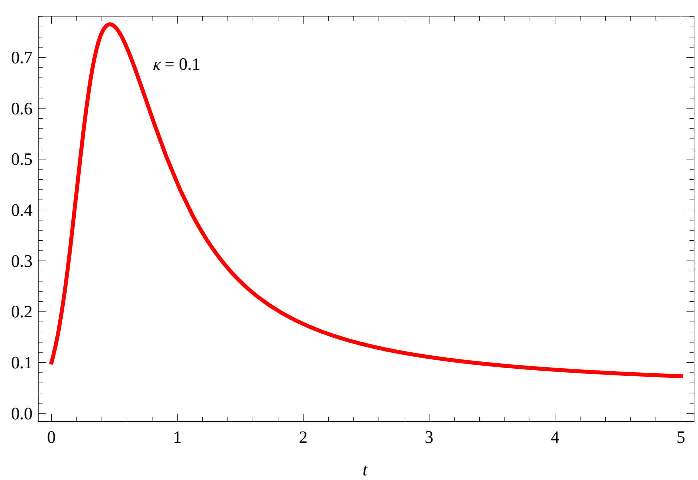

Fig. 1: Relation between Padé approximants [4/4] of $u(t)$ and t for $u(0) = 0.1, \kappa = 0.1$.

Fig.1 shows the relation between the Padé approximants [4/4] of $u(t)$ and t. From Fig.1, we can easily observe that for $u(0) = 0.1$ and $\kappa = 0.1$, we obtain $u_{max} = 0.7651130891$ occurs at $t_{critical} = 0.4644767409$.

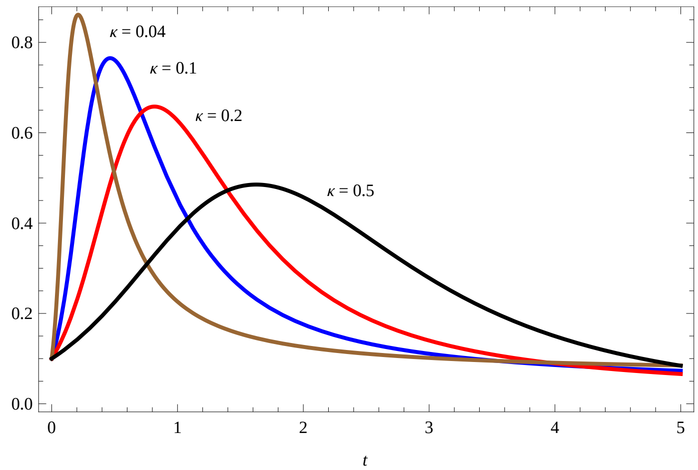

Fig. 2: Relation between Padé approximants $[4/4]$ of $u(t)$ and t for $u(0) = 0.1, \kappa = 0.04, 0.1, 0.2$ and 0.5.

In addition, Fig.1 formally shows the rapid rise along the logistic curve followed by the slow exponential decay after reaching the maximum point. However, Fig.2 shows the Padé approximants $[4/4]$ of $u(t)$ for $u(0) = 0.1$ and for $\kappa = 0.04, 0.1, 0.2$ and 0.5. The key finding of this graph is that when κ increases, the amplitude of $u(t)$ decreases, whereas the exponential decay increases.

Table 1 below summarizes the relation between κ, u_{max} and $t_{critical}$. Moreover, the results of the corresponding absolute errors $\epsilon = |\text{exact solution} - \text{approximate solution}|$ are also presented for the Padé approximants and $[4/4]$. The exact values of u_{max} were evaluated by using

$$u_{max} = 1 + \kappa \ln\left(\frac{\kappa}{1 + \kappa - u_0}\right) \tag{19}$$

obtained by [13]. It can seen clearly from Table 1 that the results obtained by PSM is in a very good agreement with exact solution and the absolute errors are decrease when the values of κ are increases as $u(t)$ becomes smooth.

Tab. 1: Approximation of u_{max} and exact value of u_{max} and absolute errors for $\kappa = 0.04, 0.1, 0.2$ and 0.5 with the Padé approximants $[4/4]$

κ	critical t	Approx.u_{max}	Exact u_{max}	ϵ for $[4/4]$
0.04	0.2102464442	0.8612401810	0.8737199832	1.24798e-02
0.1	0.4644767409	0.7651130891	0.7697414491	4.62836e-03
0.2	0.8168581213	0.657912310	0.6590503816	1.13807e-03
0.5	1.6266622270	0.4852823490	0.4851902914	9.20576e-05

Tab. 2: Comparison between the absolute errors of ADM, HPM, VIM and PSM

κ	ADM [1]	HPM [2]	VIM [11]	PSM
0.04	1.24798e-02	1.24798e-02	1.24798e-02	1.24798e-02
0.1	4.62836e-03	4.62836e-03	4.62836e-03	4.62836e-03
0.2	1.13807e-03	1.13807e-03	1.13807e-03	1.13807e-03
0.5	9.20568e-05	9.20568e-05	9.20568e-05	9.20576e-05

It can be seen from the Table 2 that, the absolute errors of PSM are completely the same as the absolute errors of ADM, HPM and VIM as expected because the approximation in Eq.(18) is the same for all methods. The main advantages of PSM are simple, easy to implement (straightforward) and without required to calculating Adomian polynomials to handle the nonlinear terms as in ADM and construct a homotopy as in HPM and calculating Lagrange multiplier as in VIM.

5.2 Convergence of PSM for Volterra's population model

The following theorem shows the convergence of the PSM for Volterra's population model. The proof of the convergence theorem is a modification to the proof of theorem (2.1) given in [14].

Theorem 5.1.1
Let $u = u(x)$ be the exact solution of the following Volterra integr-differential equation

$$u^{'}(x) = g(x, u(x)) + u(x) \int_{0}^{x} k(x, t, u(t), u^{'}(t))dt, \quad u(0) = a \tag{20}$$

and assume that $u(x)$ has a power series representation. Then, the proposed method obtains it (the Taylor expansion of $u(x)$).

Proof : Assume the approximation solution to Eq.(20) be as follows

$\tilde{u} = e_0 + e_1 x + e_2 x_2 + ...$

Hence, it suffices to prove that $e_m = \frac{u^m(0)}{m!}, m = 1, 2, 3, ...$

If $m = 0$ and with the cooperation of the initial condition, we get

$$a = \tilde{u}(0) = e_0 + e_1 0 + e_2 0 + ... = e_0 \rightarrow e_0 = a$$

If $m = 1$, and since $u(x)$ is the exact solution of Eq.(20), then it satisfies this integro-differential equation. i.e.

$$u^{'}(x) = g(x, u(x)) + u(x) \int_{0}^{x} k(x, t, u(t), u^{'}(t))dt \tag{21}$$

and hence, substituting $x = 0$ in Eq.(21) gives

$$u^{'}(0) = g(0, u(0)) + u(0) \int_{0}^{0} k(x, t, u(t), u^{'}(t))dt = g(0, u(0)) \tag{22}$$

and since $u^{'}(x) = \tilde{u}^{'}(x) = e_1 + 2e_2 x + 3e_3 x^2 + ...$

and therefore, $u^{'}(0) = \tilde{u}^{'}(0)$, then $g(0, u(0)) = u^{'}(0)$

If $m = 2$, then differentiate Eq.(20) with respect to x and substitute $u = u(x)$, we get

$$\begin{aligned} u''(x) &= \tfrac{\partial}{\partial x} g(x, u(x)) + \tfrac{\partial}{\partial u} g(x, u(x))u'(x) + u'(x) \int_{0}^{x} k(x, t, u, u')dt \\ &+u(x)[k(x, t, u, u') + \int_{0}^{x} \tfrac{\partial}{\partial x} k(x, t, u, u')]dt \end{aligned} \tag{23}$$

Evaluate Eq.(23) at $x = 0$, gives

$$u''(0) = \tfrac{\partial}{\partial x} g(0,u(0)) + \tfrac{\partial}{\partial u} g(0,u(0))u'(0) + u'(0)\int_0^0 k(x,t,u,u')dt$$
$$+u(0)[k(x,t,u,u') + \int_0^0 \tfrac{\partial}{\partial x} k(x,t,u,u')]dt \tag{24}$$

$$= \frac{\partial}{\partial x} g(0,u(0)) + \frac{\partial}{\partial u} g(0,u(0))u'(0) + u(0)k(0,u(0),u'(0)) \tag{25}$$

and since

$$u''(x) = \tilde{u}''(x), \text{then } u''(0) = \tilde{u}''(0)$$

Therefore,

$$2e_2 = \frac{\partial}{\partial x} g(0,u(0)) + \frac{\partial}{\partial u} g(0,u(0))u'(0) + u(0)k(0,u(0),u'(0))$$

and hence

$$2e_2 = \frac{\partial}{\partial x} g(0,e_0) + \frac{\partial}{\partial u} g(0,e_0)e_1 + e_0 k(0,e_0,e_1) \tag{26}$$

Comparing Eq.(25) and Eq.(26), we get
$2e_2 = u^{''}(0) \to e_2 = \frac{u^{''}(0)}{2!}$

So by inducting, one can show that

$e_3 = \frac{u^{''}(0)}{3!},\ e_4 = \frac{u^{''}(0)}{4!}$

By mathematical induction and so in general

$e_m = \frac{u^m(0)}{m!},\ \ m = 1, 2, 3, ...$This completes the proof of Theorem 5.1.1. □

Further we can examine the accuracy for the obtained approximate solution by the absolute error remainder functions and the maximal error remainder parameters when there is no explicit exact solution to compare [15]. It is worth to mention here, the exact solution in Eq.(19) is used for calculated the u_{max} only and not useful for comparison with components of the approximate solution in Eq.(18) as it is not contain the parameter t. Therefore, the analytical solution given in [3] and references therein,

$$u(t) = u_0 \exp\left(\frac{1}{\kappa}\int_0^t [1 - u(\tau) - \int_0^\tau u(x)dx]d\tau\right) \tag{27}$$

shows that $u(t) > 0$ for all t if $u_0 > 0$, is used with components of the approximate solution in Eq.(18) for even number n i.e. $u_n,\ n = 2, 4, 6, 8$ on the interval $0 \le t \le 0.1$. The absolute error remainder functions are

$$|ER_n(t)| = \left|u_n(t) - 0.1 * \exp\left(10\int_0^t [1 - u_n(\tau) - \int_0^\tau u_n(x)dx]d\tau\right)\right|,\ n = 2, 4, 6, 8,$$

where $u_0 = \kappa = 0.1$ will be examined, other cases for different values of u_0 and κ can be treated in similar manner, and the maximal error remainder parameters are given in [15]

$$MER_n = \max_{0\le t\le 0.1} |ER_n(t)| \tag{28}$$

The components of $|ER_n(t)|$ have been computed by computer algebra system (MATHEMATICA ® 8.0.), as follows:

$$|ER_2(t)| = |(0.1 + 0.9t + 3.55t^2 - (0.1\exp(10(0.9t - 0.5t^2 - 1.33333t^3 - 0.295833t^4))))|,$$

$$|ER_4(t)| = |[0.1 + 0.9t + 3.55t^2 + 6.31667t^3 - 5.5375t^4 - (0.1\exp(10((0.9t - 0.5t^2 - 1.33333t^3 - \\ -1.875t^4 + 0.791667t^5 + 0.184583t^6))))|,$$

$$|ER_6(t)| = |[0.1 + 0.9t + 3.55t^2 + 6.31667t^3 - 5.5375t^4 - 63.7092t^5 - 156.08t^6 - \\ -(0.1\exp(10((0.9t - 0.5t^2 - 1.33333t^3 - 1.875t^4 + 0.791667t^5 + 10.8028t^6 + 23.8141t^7 + \\ +2.78715t^8))))|,$$

$$|ER_8(t)| = |[0.1 + 0.9t + 3.55t^2 + 6.31667t^3 - 5.5375t^4 - 63.7092t^5 - 156.08t^6 - \\ -18.4732t^7 + 1056.29t^8 - (0.1\exp(10((0.9t - 0.5t^2 - 1.33333t^3 - 1.875t^4 + 0.791667t^5 + \\ +10.8028t^6 + 23.8141t^7 + 5.0963t^8 - 117.109t^9 - 11.7365t^{10}))))|.$$

The logarithmic plots for the maximal error remainder parameters MER_n for $n = 2; 4; 6; 8$ on the interval $0 \le t \le 0.1$ are shown in Fig.3, which demonstrates an approximately exponential rate of convergence.

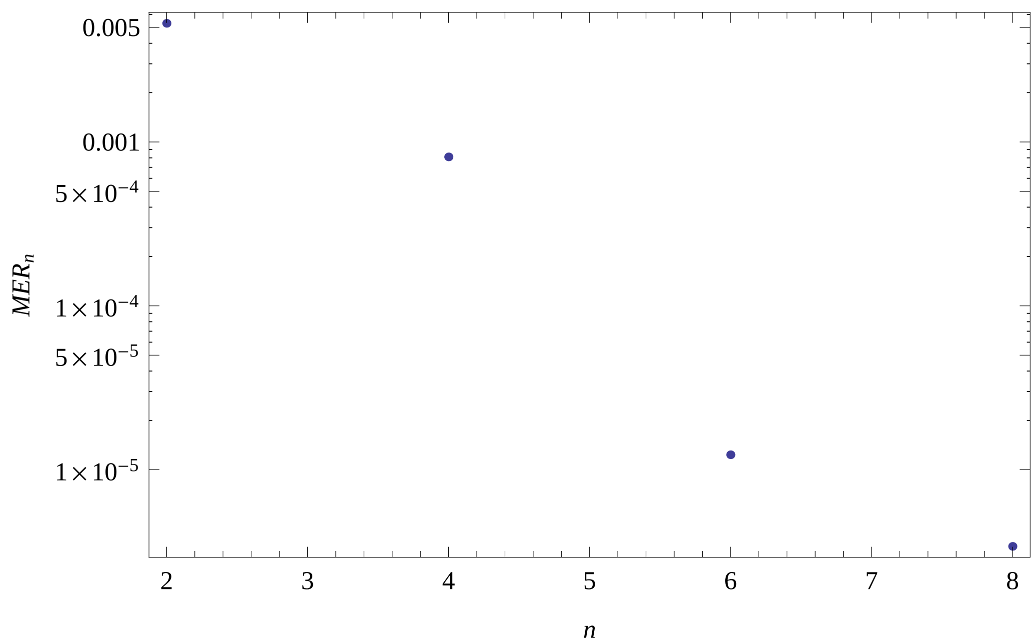

Fig. 3: Logarithmic plots of MER_n for $n = 2$ through 8.

6 Solving some scientific models by power series method (PSM)

In this section we will focus our work on three well-known nonlinear equations, namely the Hybrid selection model, the Riccati model, and the Logistic model.

6.1 The hybrid selection model

We first study the hybrid selection model with constant coefficients that reads [16]

$$u' = ku(1-u)(2-u), \quad u(0) = 0.5 \tag{29}$$

where k is a positive constant that depends on the genetic characteristic. In the hybrid model, $u(t)$ is the portion of population of a certain characteristic, and t is the time measured in generations.

To solve Eq.(29) by PSM, we integrate Eq.(29) with respect to t from 0 to t and apply the initial condition we obtain the following nonlinear Volterra integral equation

$$u(t) = 0.5 + \int_0^t [ku(1-u)(2-u)]dt \tag{30}$$

Now, by applying the power series method to solve Eq.(30), we suppose the solution of Eq.(30) with $u_0(t) = u(0) = e_0$ is

$$u_1(t) = e_0 + e_1 t \Longrightarrow u_1(t) = 0.5 + e_1 t, \tag{31}$$

Substitute Eq.(31) in Eq.(30), we have,

$$0.5 + e_1 t = 0.5 + \int_0^t [k(0.5+e_1 t)(1-(0.5+e_1 t))(2-(0.5+e_1 t))]dt, \tag{32}$$

After simplifying, we get, $e_1 = \frac{3}{8}$ and hence,

$$u_1(t) = 0.5 + \frac{3}{8}kt, \tag{33}$$

which is the first approximation for the solution of Eq.(30).

Now, the solution of Eq.(30) can be supposed as:

$$u(t) = 0.5 + \frac{3}{8}kt + e_2 t^2, \tag{34}$$

Substitute Eq.(34) in Eq.(30), we have,

$$\begin{aligned} &0.5 + \tfrac{3}{8}kt + e_2 t^2 = 0.5 + \int_0^t [k(0.5 + \tfrac{3}{8}kt + e_2 t^2)(1-(0.5 + \tfrac{3}{8}kt + e_2 t^2)) \\ &(2-(0.5 + \tfrac{3}{8}kt + e_2 t^2))]dt, \end{aligned} \tag{35}$$

After simplifying, we get, $e_2 = \frac{3}{64}$ and hence,

$$u_2(t) = 0.5 + \frac{3}{8}kt + \frac{3}{64}(kt)^2, \tag{36}$$

Proceeding in this way, we get,

$$u(t) = 0.5 + \tfrac{3}{8}kt - \tfrac{3}{64}(kt)^2 - \tfrac{17}{256}(kt)^3 - \tfrac{125}{4096}(kt)^4 + \tfrac{721}{81920}(kt)^5 + ... \tag{37}$$

This in turn gives

$$u(t) = \frac{\sqrt{1+3e^{3kt}} - 1}{\sqrt{1+3e^{3kt}}}$$

which is the exact solution of the problem and it is the same results obtained by VIM [16]. Figure 4 shows the solution $u(t)$ which is an increasing function bounded by $u = 1$.

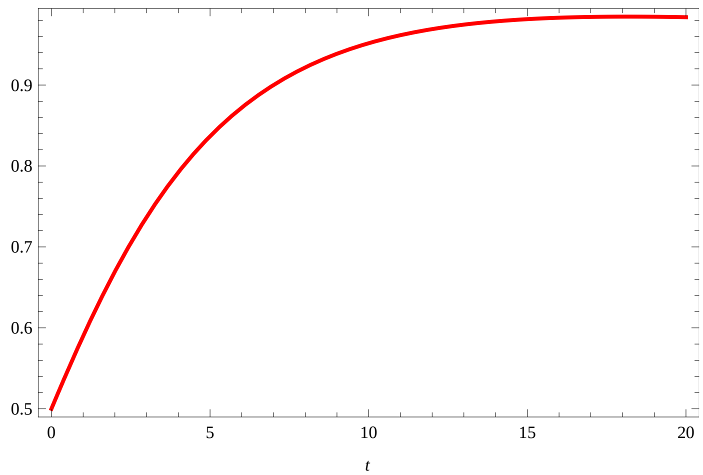

Fig. 4: The solution $u(t)$ for $k = 0.25$, and $0 \leq t \leq 20$

6.2 The Riccati model

The Riccati equation is one of the most interesting nonlinear differential equations of first order. It's written in the form:

$$u' = a(x)u + b(x)u^2 + c(x)$$

where $a(x)$, $b(x)$, $c(x)$ are continuous functions of x and $c(x) \neq 0 \ \ \& \ \ b(x) \neq 0$.
The Riccati equation is used in different areas of mathematics (for example, in algebraic geometry and the theory of conformal mapping and physics). We have the equation [16]:

$$u'(t) = u^2(t) - 2xu(t) + x^2 + 1, \quad u(0) = 0.5, \tag{38}$$

To solve Eq.(38) by PSM, we integrate Eq.(38) with respect to t from 0 to t and apply the initial condition, we obtain the following nonlinear Volterra integral equation

$$u(t) = 0.5 + \int_0^x [u^2 - 2tu + x^2 + 1]dt \tag{39}$$

By applying the power series method to solve Eq.(39), we suppose the solution of Eq.(39) with $u_0(t) = u(0) = e_0$ is

$$u_1(t) = e_0 + e_1 t \Longrightarrow u_1(t) = 0.5 + e_1 t, \tag{40}$$

Substitute Eq.(40) in Eq.(39), we have,

$$0.5 + e_1 t = 0.5 + \int_0^x [(0.5 + e_1 t)^2 - 2t(0.5 + e_1 t) + x^2 + 1]dt \tag{41}$$

After simplifying, we get, $e_1 = \frac{5}{4}$ and hence,

$$u_1(t) = 0.5 + \frac{5}{4}t \tag{42}$$

which is the first approximation for the solution of (39).

The solution of Eq.(39) can be supposed as:

$$u(t) = 0.5 + \frac{5}{4}t + e_2 t^2 \tag{43}$$

Substitute Eq.(43) in Eq.(39), we have,

$$0.5+\frac{5}{4}t+e_2t^2=0.5+\int\limits_0^x[(0.5+\frac{5}{4}t+e_2t^2)^2-2t(0.5+\frac{5}{4}t+e_2t^2)+x^2+1]dt \tag{44}$$

After simplifying, we get, $e_2=\frac{1}{8}$ and hence,

$$u_2(t)=0.5+\frac{5}{4}t+\frac{1}{8}t^2, \tag{45}$$

Proceeding in this way, we get,

$$u(x)=x+\frac{1}{2}(1+\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\frac{1}{16}x^4+\frac{1}{32}x^5+...) \tag{46}$$

That converges to

$$u(x)=x+\frac{1}{2-x},\ |x|<2.$$

which is the exact solution of the problem and it is the same results obtained by VIM [16].

6.3 The logistic model

The logistic function is the solution of the simple first-order non-linear differential equation

$$\frac{d}{dx}u(x)=u(x)(1-u(x)),$$

with initial condition $u(0)=\frac{1}{2}$. The logistic function finds applications in a range of fields, including artificial neural networks, biology, biomathematics, demography, economics, chemistry, mathematical psychology, probability, sociology, political science and statistics.
We first study the logistic nonlinear differential equation [17]:

$$u'=\mu u(1-u),\quad u(0)=\frac{1}{2}, \tag{47}$$

where $\mu>0$ is a positive constant.
To solve Eq.(47) by PSM, we integrate Eq.(47) with respect to t from 0 to t and apply the initial condition we obtain the following nonlinear Volterra integral equation

$$u(x)=0.5+\int\limits_0^x[\mu u(t)(1-u(t)]dt \tag{48}$$

By using the power series method to solve Eq.(48), we suppose the solution of Eq.(48) with $u_0(t)=u(0)=e_0$ is

$$u_1(t)=e_0+e_1t\Longrightarrow u_1(t)=0.5+e_1t, \tag{49}$$

Substitute Eq.(49) in Eq.(48), we have,

$$0.5+e_1t=0.5+\int\limits_0^x[\mu(0.5+e_1t)(1-(0.5+e_1t))]dt, \tag{50}$$

After simplifying, we get, $e_1=\frac{\mu}{4}$ and hence,

$$u_1(t)=0.5+\frac{\mu}{4}t, \tag{51}$$

which is the first approximation for the solution of (48).

The solution of Eq.(48) can be supposed as:

$$u(t) = 0.5 + \frac{\mu}{4}t + e_2 t^2 \tag{52}$$

Substitute Eq.(52) in Eq.(48), we have,

$$0.5 + \frac{\mu}{4}t + e_2 t^2 = 0.5 + \int_0^x [\mu(0.5 + \frac{\mu}{4}t + e_2 t^2)(1 - (0.5 + \frac{\mu}{4}t + e_2 t^2))]dt \tag{53}$$

After simplifying, we get, $e_2 = \frac{\mu^3}{48}$ and hence,

$$u(t) = 0.5 + \frac{\mu}{4}t + \frac{\mu^3}{48}t^2, \tag{54}$$

proceeding in this way, we get,

$$u(x) = \frac{1}{2} + \frac{\mu}{4}x - \frac{\mu^3}{48}x^3 + \frac{\mu^5}{480}x^5 - \frac{17\mu^7}{80640}x^7 - \frac{31\mu^9}{1451520}x^9 + ... \tag{55}$$

This in turn gives

$$u(x) = \frac{e^{\mu x}}{1 + e^{\mu x}}.$$

which is the exact solution of the problem and it is the same results obtained by VIM [17].

7 Conclusion

In this paper, an efficient combined method based on power series and Padé technique namely PSM- Padé is successfully applied to Volterra's population model. This method were used in a direct way without using linearization, perturbation or restrictive assumption. The numerical results show that PSM- Padé technique is an accurate and reliable numerical technique for the solution to Volterra's population model with approximately exponential rate of convergence and can be easily comprehended with only a basic knowledge of Calculus. It is economical in terms of computer power/memory and does not involve tedious calculations such as Adomian polynomials. Also The approximate solution for the Volterra's population model obtained in current paper is the same with the results obtained by using ADM [1], HPM [2] and VIM [11]. Furthermore, the method is used to solve some scientific models, namely, the hybrid selection model, the Riccati model and the logistic model to provide the analytic solutions and the results showed the PSM provided the same results obtained by VIM [17]. It is worth to mention here the power series method is straightforward without required to calculating Adomian polynomials to handle the nonlinear terms as in ADM and construct a homotopy as in HPM and calculate the Lagrange multiplier as in VIM.

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