Explicit upper bound for the function of sum of divisors σ(n)

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Abstract: we developed the result proved by A.Evi$\hat{c}$ [6]where he proved the following theorem :

Theorem:[6] let $n \geq 7$,then :

$$\sigma(n) < (2.59) n \log\log n, \ldots\ldots\ldots\ldots(1)$$

this proving required several Lemmas which are given to be :

Lemma1:[6] if $n \geq 39$,then:

$$\log p_n < \frac{7}{5}\log n$$

Lemma2:[6] if $n \geq 31$,then:

$$\prod_{p|n}\left(1+\frac{1}{p}\right) < \frac{28}{15}\log\log n$$

Lemma3:[6] if $n \geq 31$, $n \neq 42, n \neq 210$,then:

$$\prod_{p|n}\frac{p}{p-1} < 2.59 \log\log n.$$

We a dope, the procedure of Evi$\hat{c}$ by improving the lemmas and get the following results:

Lemma 2:If $n \geq 82$, then:

$$\log p_n < \frac{59}{43}\log n$$

Lemma3; If $n \geq 31$, then:

$$\prod_{p|n}\left(1+\frac{1}{p}\right) < \frac{236}{129} \log\log n$$

Lemma4: For $n \geq 82$, $n \neq 42$, $n \neq 210$,

$$\prod_{p|n}\frac{p}{p-1} < 2.509038354 \log \log n$$

$\sigma(n)$ which given in the Where our lemma give above leads us to gut the new upper bound of the function following theorem:

Theorem1: If $n \geq 82$,then:

$$\sigma(n) < 2.509038354 \; n \log\log n$$

Key words: Explicit upper bound for the function of sum of divisor $\sigma(n)$, $n \geq 1$.

Introduction:

In this paper, we discuss the upper bound of the multiplicative number theoretic function $\sigma(n)$, where $\sigma(n)$ represent the sum of all positive divisors of n, $n \geq 1$.this function is written to be,

$$\sum_{d|n} d \, ,$$

In Hardy and whright [4] the bound given to be explicit which is

$$\sigma(n) = 0(n \log\log n), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Also, R.L. Duncan [3] showed that ,

$$\sigma(n) < \frac{n}{6}\,[7\omega(n) + 10] \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Where $n = p_1^{a1} \, . \, p_2^{a2} \ldots\ldots\ldots p_r^{ar}$ and $\omega(n) = r$ and this is an explicit bound interns of $\omega(n)$. this bound can not be determined for large n , in which $\omega(n)$ can not be counted.

The best explicit upper bound proved by Aleksander Ivi$\hat{c}$ [6]when he proved .

$$\sigma(n) < (2.59) n \log\log n, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

where $n \geq 7$.

Our work dealing with Aleksander Ivi$\hat{c}$ results in (3) and we improve this bound by increasing n to be $n \geq 82$,and proved the following result .

$$\sigma(n) < (2.5090383) n \log\log n, \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Definition ω(n): Let $n = p_1^{a1} \cdots p_r^{ar}$.Then $\omega(n) = r$, represent the number of primes of n, if $n = 2.3^2.5$, then w(n)=3.

Definition π(x) :the number of all primes less them or equal to x denoted by π(x),

from Definition ,we can write:

$$\pi(x) = \sum 1$$

For if, x=8.5, π(8.5)=4,

x=2 , π(2)=1,

x=12 , π(12)=5.

Theorem1:[4]The number of primes not exceeding x is asymptotic $\frac{x}{\log x}$ i.e

$$(x) \sim \frac{x}{\log x}$$

Theorem2:[6]there exist a constants c_1 and c_2 such that,

$$, \frac{c_2 x}{\log x} < \pi(x) < \frac{c_1 x}{\log x}$$

We shall improve the bound given in [6]for the function $\sigma(n)$.**Evic′** proved the following theorem.

Theorem3:[4] If n≥7, then

$$\sigma(n) < (2.59)\, n \log\log n\,.$$

Theorem4:[3]If n≥1, then

Our improvment will required Some estimates given by Rosser and schcoefeld in[5].

$$\sigma(n) < \frac{n}{6}\,(7\, w(n) + 10)$$

From J.Rosser and L.Schoefeld [5],we have the following estimates:

Theorem5: [3] If n≥ 1,then:

$$\sigma(n)=0(\log\log n)$$

Theorem6: [5] If $e^{3/2} < x$ and $x \geq 67$, then:

and:

$$\frac{x}{\log x - \frac{1}{2}} < \pi(x)$$

$$\pi(x) < \frac{x}{\log x - \frac{3}{2}}$$

Corollary 1:[5] If $x \geq 17$, then:

$$\frac{x}{\log x} < \pi(x)$$

Corollary 2:[5] If x>1, then:

$$\pi(x) < \frac{(1.25556)x}{\log x}$$

Theorem7: [5] If x>1, and B=0.261497 ………, then:

$$\operatorname{Log}\log x + B - \frac{1}{2\log^2} < \sum_{p\leq x} \frac{1}{p}$$

and for x≥280,then:

$$\sum_{p\leq x} \frac{1}{p} < \log\log x + B + \frac{1}{2\log^2 x}$$

Corollary1: **[5]**If x>1, then:

$$\log\log x < \sum_{p\leq x} \frac{1}{p}$$

Theorem8: **[5]**If x>1,then:

and when $x \leq 280$, then;

$$e^{\gamma}(\log x)\left(1 - \frac{1}{2\log^2 x}\right) < \prod_{p\leq x} \frac{p}{p-1}$$

and when $x \leq 280$, then:

$$\prod_{p\leq x} \frac{p}{p-1} < e^{\gamma}(\log x)\left(1 + \frac{1}{2\log^2 x}\right)$$

Corollary1: [5] If x>1,then:

$$\prod_{p \leq x} \frac{p}{p-1} < e^{\gamma}(\log x)\left(1+\frac{1}{\log^2 x}\right)$$

Where γ=0.5772157 is Euler's constant .

We need the following definition.

Definition: Let $n \geq 1$,then p_n denoted the n th prime , where:

$$n=1 \;,\; p_1 = 2,$$

$$n=2 \;,\; p_2 = 3,$$

$$n=3 \;,\; p_3 = 5,$$

$$n=4 \;,\; p_4 = 7,$$

Now, we shall give an improvement of **Evic**' result by proving the following theorem.

Theorem(1) : For $n \geq 82$, then

$$\sigma(n) < 2.509038354 \text{ n log log}$$

Now, in order to prove theorem(4),we need the following Lemmas:

Lemma(1): [5]Let $n \geq 2$, then:

$$p_n \sim n \text{ Iog } n$$

as $n \to \infty$. this Lemme,we can written in the from:

$$p_n < (1+\epsilon) n \text{ Iog } n, \ldots \ldots .. (1)$$

for some $\epsilon, \forall \epsilon > 0$ and from, we can write

$$\log p_n < (1+\epsilon) \text{ n Iog n}$$

where $n \geq n_0(\epsilon)$. some fixed number n_0.

Now, we shall give an improvement of **Evic**' result by proving the following theorem.

Lemma 2:If $n \geq 82$, then:

$$\log p_n < \frac{59}{43} \log n$$

Proof: we shall the bound given the lemma (1) to prove lemma(2)by

using the inequality given in by G.Rosser and L.Schoenfeld[5]which given in the

corollary (1)of theorem(6)

$$\pi(x) > x/\log x \qquad \text{for } x > 17$$

with $x = p_n$ and $\pi(x) = n$, then:

$$n > \frac{p_n}{\log p_n} \qquad \text{for } n \geq 82$$

$$p_n < n \log p_n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and so:

$$p_n < n \sqrt[3]{p_n} \qquad \text{for } n \geq 82$$

$$p_n^{2/3} < n,$$

$$\log p_n^{2/3} < \log n,$$

$$\frac{2}{3} \log p_n < \log n,$$

and this gives us that;

$$n \log p_n < \frac{3}{2} n \log n$$

there fore,

$$p_n < n \log p_n < \frac{3}{2} n \log n,$$

$$n \log p_n < \frac{3}{2} n \log n\,,$$

$$\therefore \qquad p_n < n^{3/2}\,, \qquad \text{for } n \geq 82$$

$$\frac{3}{2} n \log n \leq n^{7/5}\,, \qquad \text{for } n \geq 161$$

There fore,

$$p_n < n^{7/5}\,, \qquad \text{for } n \geq 161$$

$$\log p_n < \frac{7}{5} \log n\,, \qquad \text{for } 82 \leq n \leq 161$$

$$\therefore \qquad \log p_n < \frac{59}{43} \log n\,, \qquad \text{for } n \geq 82 \qquad ■$$

Lemma3; If $n \geq 31$, then:

$$\prod_{p|n} \left(1 + \frac{1}{p}\right) < \frac{236}{129} \log \log n$$

Proof: for $x \geq 286$, and B=0.261497……., we shall have that:

$$\log x > \text{Iog } 286 \ \rightarrow \text{Iog} x > 5.6559$$

$$\text{Iog}^2 > 31.98920481$$

$$\log^2 > 2 \times 31.98920481$$

$$\frac{1}{2\log^2 x} < \frac{1}{63.97840962} = 0.015630 < 0.016$$

$$\frac{1}{(2\log^2 x)} = \frac{1}{(2\log^2 286)} < 0.016$$

$$\sum_{p\le x}\frac{1}{p} < \log\log x + B + 1/(2\log^2 x) < \log\log x + \log\frac{4}{3}$$

$$\therefore \quad \sum_{p\le x}\frac{1}{p} < \log\log x + 0.261497 + 0.016$$

$$\therefore \quad \sum_{p\le x}\frac{1}{p} < \log\log x + 0.277497$$

$$\therefore \quad \sum_{p\le x}\frac{1}{p} < \log\left(\frac{4}{3}\log x\right)$$

There fore,

$$\log\prod_{p|n}\left(1+\frac{1}{p}\right) = \log[(1+1/p_1)\ \ldots\ldots\ldots\ldots\ldots\ (1+1/p_r)]$$

$$\log\prod_{p|n}\left(1+\frac{1}{p}\right) = [\log(1+1/p_1)\ \ldots\ldots \log(1+1/p_r)]$$

$$= \sum_{p|n}\log(1+1/p)$$

Also since, $p\geq 2$ then:

$$\frac{1}{p}\leq\frac{1}{2} \ \rightarrow\ 1+\frac{1}{p}\leq\frac{3}{2}$$

$$\log\left(1+\frac{1}{p}\right)\leq \log\ \frac{3}{2}\leq\frac{1}{p}$$

$$\therefore \quad \sum_{p|n}\log\left(1+\frac{1}{p}\right)\ \leq\ \sum_{p|n}\frac{1}{p}\leq\sum_{p\le p_k}\frac{1}{p}\ <\ (\log\frac{4}{3})\log p_k$$

By using Lemma(1) and $p_k \geq 286$, where k=ω(n), then we have the following cases:

Case (1):

(i) If ω(n)> 62,

$$\log\prod_{p|n}\left(1+\frac{1}{p}\right) < \log[\frac{4}{3}\ \log p_k]$$

$$< \log[\frac{4}{3}\times\frac{59}{43}\ \log k]$$

$$< \log[\frac{236}{129}\ \log\ k]$$

$$\therefore \quad \log\prod_{p|n}\left(1+\frac{1}{p}\right) < \log[\frac{236}{129}\ \log k]$$

$$\therefore \quad \log\prod_{p|n}\left(1+\frac{1}{p}\right) < \log[\frac{236}{129}\ \log\log n]$$

(ii) when ω(n)≤ 61, and $p_n < n\log p_n$,where n≥ 82, then by calculation, we get:

$$\prod_{p|n}\left(1+\frac{1}{p}\right) \leq \prod_{p\le p_n}\left(1+\frac{1}{p}\right) = 10.22575004$$

$$\therefore \qquad \prod_{p|n}\left(1+\frac{1}{p}\right) < 10.22575004 < \frac{236}{129}\log\log n$$

Case(2):

(i) If ω(n)>12 ,then:

$$\prod_{p\le p_{12}} p > 7.42\times 10^{12},$$

(ii) If $\omega(n)\le 11$ then, by calculation, we get :

$$\prod_{p|n}\left(1+\frac{1}{p}\right) \le \prod_{p\le p_{11}}\left(1+\frac{1}{p}\right) < 6.542269709$$

$$\therefore \qquad \prod_{p|n}\left(1+\frac{1}{p}\right) < 6.542269709 < \frac{236}{129}\log\log n$$

Case(3) ;

If n >16000,and $n\le 16000$,with $\omega(n)\le 5$,then by calculation we get:

$$\prod_{p|n}\left(1+\frac{1}{p}\right) \le \frac{1152}{385} < \frac{236}{129}\log\log n$$

Case(4);

If n>200, and n≤200,with $\omega(n)\le 3$,then by calculation we get:

$$\therefore \qquad \prod_{p|n}\left(1+\frac{1}{p}\right) \le \frac{3}{2}\cdot\frac{4}{3}\cdot\frac{6}{5} \le \frac{12}{5} < \frac{239}{129}\log\log n$$

Lemma4: For $n\ge 82,\ n\ne 42,\ n\ne 210,$

$$\prod_{p|n}\frac{p}{p-1} < 2.509038354\ \log\ \log n$$

Proof: take x>455, then:

$$\log x > \log 455$$

$$\log^2 x > \log 455$$

$$\log^2 x > 37.4580405$$

$$\log^{-2} x < \frac{1}{37.4580405}$$

$$< 0.026696537$$

$$1+\log^{-2} x < 1.026696537 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

since $e^{\gamma} < 1.7810727$ and

$$\prod_{p\le x}\frac{p}{p-1} < e^{\gamma}\log x\,(1+\log^{-2} x)$$

$$\prod_{p\leq x} \frac{p}{p-1} < (1.7810727)(1.026696537) \log x$$

$$<1.828621173 \log x ,$$

Using Lemma (2) and k=ω(n),then if n> 82 with ω(n)<log n ,

$$\prod_{p|n} \frac{p}{p-1} \leq \prod_{p\leq p_k} \frac{p}{p-1} < 1.828621173 \log p_k.$$

$$\leq 1.828621173 \times \frac{59}{43} \log k \leq 2.509038354 \log k$$

$$\therefore \quad \prod_{p|n} \frac{p}{p-1} \leq 2.509038354 \log k \leq 2.509038354 \log\log n$$

$$\therefore \quad \prod_{p|n} \frac{p}{p-1} \leq 2.509038354 \log \log n$$

Case(1):

By taking k=ω(n)≥ 82 and $p_k \geq p_{82} = 421$,then

$$\prod_{p|n} \frac{p}{p-1} \leq 2.509038354 \log\log n$$

So, if ω(n)≤ 82,then:

$$\prod_{p|n} \frac{p}{p-1} \leq \prod_{p\leq p_{82}} \frac{p}{p-1} < 10.84851192$$

$$\therefore \quad \prod_{p|n} \frac{p}{p-1} \leq \prod_{p\leq p_{82}} \frac{p}{p-1} < 10.84851192 < 2.509038354 \log\log n$$

Case(2):

If $n \geq 5 \times 10^{17}$,then since $\prod_{p\leq p_{17}} p > 10^{18}$, $\omega(n) \leq 16$ and n≤ 5.10^{17} in this case we get:

$$\therefore \quad \prod_{p|n} \frac{p}{p-1} \leq \prod_{p\leq p_{16}} \frac{p}{p-1} < 7.348238613 < 2.509038354 \log\log n$$

Case(3):

If n≥ 10^8, then since $\prod_{p\leq p_8} p > 2 \times 10^8$, $\omega(n) \leq 8$ and $n \leq 10^8$ in this case we get:

$$\prod_{p|n} \frac{p}{p-1} \leq \prod_{p\leq p_8} \frac{p}{p-1} < 5.8471318$$

$$\therefore \quad \prod_{p|n} \frac{p}{p-1} \leq \prod_{p\leq p_8} \frac{p}{p-1} < 5.8471318 < 2.5090383 \log\log n$$

Case(4):

if $n > 30000$, then since $\prod_{p\leq p_k} p = 30030$, $\omega(n) \leq 5$ and n≤ 30000 in this case we get:

$$\prod_{p|n} \frac{p}{p-1} \leq 2 \cdot \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} = \frac{77}{16}$$

$$\therefore \qquad \prod_{p|n} \frac{p}{p-1} \leq \frac{77}{16} < 2.5090383 \log\log n$$

Case(5):

if n$< 300, then\ \omega(n) \leq 3$, except for n=210 =2·3· 5 ·7 and for n=210 then:

$$\prod_{p|n} \frac{p}{p-1} < 2.5090383 \log\log n$$

which is not true, If ω(n)≤ 3,

$$\prod_{p|n} \frac{p}{p-1} \leq 2 \cdot \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{4} = 3.75$$

$$\prod_{p|n} \frac{p}{p-1} \leq \frac{15}{4} = 3.75 < 2.5090383 \log\log n$$

Theorem1: If n≥ 82,then:

$$\sigma(n) < 2.509038354\ n \log\log n$$

Proof: If n = $p_1^{a1} . p_2^{a2} \cdots\cdots p_r^{ar}$,then since σ(n) is multiplicative function then:

$$\sigma(n) = \prod_{p_i|n} (1 + p + \cdots\cdots p^{ai}), \qquad (1 \leq i \leq r)$$

$$= \prod_{p_i|n} \left(\frac{p^{ai+1} - 1}{p_i - 1}\right)$$

By lemma(4):

$$\left[\text{If } n \geq 82, \quad n \neq 42, \quad n \neq 210 \text{ then: } \prod_{p|n} \frac{p}{p-1} < 2.509038354 \log\log n \right]$$

$$\sigma(n) = \prod \frac{p_i^{ai+1}}{p_i}$$

$$= \prod \frac{p^{ai+1}\left(1 - 1/p_i^{ai+1}\right)}{p_i - 1}$$

$$= \prod \frac{p^{ai} . p\left(1 - 1/p_i^{ai+1}\right)}{p_i - 1}$$

$$= n \prod \frac{p_i - p_i^{-ai}}{p_i - 1}$$

To show:

$$\frac{p - p^{-a}}{p - 1} \leq \frac{p}{p - 1}$$

Take L.H.S, than since:

$$1\text{-}1/p^{a+1} \leq 1,$$

$$\frac{p(1-1/p^{a+1})}{p-1} \le \frac{p}{p-1}$$

$$\frac{p-p^{-a}}{p-1} \le \frac{p}{p-1}$$

$$\sigma(n)=\prod \frac{p_i - p_i^{-ai}}{p_i - 1}$$

$$\sigma(n) \le n \prod_{p|n} \frac{p}{p-1}$$

$$\sigma(n) < (2.5090383)\log\log n.n$$

for $n \ge 31, n \ne 42, n \ne 210$. for n=42 and n=210 is not true in

$$[\sigma(n) < 2.5090383 \log\log n.n\ ]$$ ■

Our result . declare for several values of $n \ge 82$ which shows the validity of our bound which given in the table (1).

Table(1)

n	$\sigma(n)$	2.5090383 n log log n
82	126	305.1409562
83	84	309.4342252
84	129	313.7327872
85	108	318.0365693
86	132	322.3455001
87	120	326.6595105
88	135	330.9785327
89	90	335.3025008
90	138	339.6313504
91	112	343.9650188
92	141	348.3034445
93	128	352.6465679
94	144	356.9943303
95	120	361.3466748
96	147	365.7035454
97 . .	98	370.0648877

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