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Asymptotic Properties of Bayes Factor in One- Way Repeated Measurements Model

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Abstract

In this paper, we consider the linear one- way repeated measurements model which has only one within units factor and one between units factor incorporating univariate random effects as well as the experimental error term. In this model, we investigate the consistency property of Bayes factor for testing the fixed effects linear one- way repeated measurements model against the mixed one- way repeated measurements alternative model. Under some conditions on the prior and design matrix, we identify the analytic form of the Bayes factor and show that the Bayes factor is consistent.

Keywords: One- Way Repeated Measurements Model, ANOVA, Mixed model, Prior Distribution, Posterior Distribution, covariance matrix, Bayes Factor, Consistent.

1. Introduction

Repeated measurements occur frequently in observational studies which are longitudinal in nature, and in experimental studies incorporating repeated measures designs. For longitudinal studies, the underlying metameter for the occasions at which measurements are taken is usually time. Here, interest often centers around modeling the response as linear or nonlinear function of time, Repeated measures designs, on the other hand, entail one or more response variables being measured repeatedly on each individual over arrange of conditions. Here the metameter may be time or it may be a set of experimental conditions (e.g. ,dose levels of a drug). Repeated measurements analysis is widely used in many fields, for example , in the health and life science, epidemiology, biomedical, agricultural, industrial, psychological, educational researches and so on.[2],[3],[17]

A balanced repeated measurements design means that the p occasions of measurements are the same for all of the experimental units. A complete repeated measurements design means that measurements are available each time point for each experimental unit. A typical repeated measurements design consist of experimental units or subjects randomized to a treatment (a between-units factor) and remaining on the assigned treatment throughout the course of the experiment. [2],[3],[17]

A one –way repeated measurements analysis of variance refers to the situation with only one within-units factor and a multi-way repeated measurements analysis of variance to the situation with more than one within units factor. This is because the number of within factors, and not the number or between factor, dictates the complexity of repeated measurements analysis of variance.[2],[3],[4],[12],[17]

In this paper, we consider the linear one- way repeated measurements model which has only one within units factor and one between units factor incorporating univariate random effects as well as the experimental error term, we can represent repeated measurements model as a mixed model.

The asymptotic properties of the Bayes factor have been studied mainly in nonparametric density estimation problems related to goodness of fit testing [1],[5],[7],[8],[9],[15],[16]. Choi, Taeryon and et al in (2009) [6] studied the semiparametric additive regression models as the encompassing model with algebraic smoothing and obtained the closed form of the Bayes factor and studied the asymptotic behavior of the Bayes factor based on the closed form. Mohaisen, A. J. and Abdulhussain , A. M. in (2013)[11] investigated large sample properties of the Bayes factor for testing the pure polynomial component of spline null model whose mean function consists of only the polynomial component against the fully spline semiparametric alternative model whose mean function comprises both the pure polynomial and the component spline basis functions. In this paper, we investigate the consistency property of Bayes factor for testing the fixed effects linear one- way repeated measurements model against the mixed one- way repeated measurements alternative model. Under some conditions on the prior and design matrix, we identify the analytic form of the Bayes factor and show that the Bayes factor is consistent.

2. One- Way Repeated Measurements Model

Consider the model

$$y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk} \quad (1)$$

Where

i=1,….,n is an index for experimental unit within group j ,

j=1,…,q is an index for levels of the between-units factor (Group) ,

k=1,…,p is an index for levels of the within-units factor (Time) ,

y_{ijk} is the response measurement at time k for unit i within group j,

μ is the overall mean ,

τ_j is the added effect for treatment group j ,

$\delta_{i(j)}$ is the random effect for due to experimental unit i within treatment group j ,

γ_k is the added effect for time k ,

$(\tau\gamma)_{jk}$ is the added effect for the group j × time k interaction ,

e_{ijk} is the random error on time k for unit i within group j ,

For the parameterization to be of full rank, we imposed the following set of conditions

$\sum_{j=1}^{q} \tau_{j=0}$, $\sum_{k=1}^{p} \gamma_{k=0}$, $\sum_{j=1}^{q} (\tau\gamma)_{jk=0}$ for each k=1,…,p

$\sum_{k=1}^{p} (\tau\gamma)_{jk=0}$ for each j=1,…,q .

And we assumed that the e_{ijk} and $\delta_{i(j)}$ are indepnndent with

$$e_{ijk} \sim \text{i.i.d } N(0,\sigma_e^2) \quad , \quad \delta_{ij} \sim \text{i.i.d } N(0,\sigma_\delta^2) \quad (2)$$

The model (1) is rewritten as follows

$$Y = X\beta + Zb + \epsilon \quad (3)$$

Where

$$Y = \begin{bmatrix} y_{111} \\ y_{112} \\ \vdots \\ y_{nqp} \end{bmatrix}_{nqp\times 1} , \quad \beta = \begin{bmatrix} \mu_{1\times 1} \\ \tau_{q\times 1} \\ \gamma_{p\times 1} \\ (\tau\gamma)_{qp\times 1} \end{bmatrix}_{(qp+q+p+1)\times 1} , \quad Z = \begin{pmatrix} 1_{P\times 1} & \cdots & 0_{P\times 1} \\ \vdots & \ddots & \vdots \\ 0_{P\times 1} & \cdots & 1_{P\times 1} \end{pmatrix}_{nqp\times nq} ,$$

$$b = \delta = \begin{bmatrix} \delta_{1(1)} \\ \delta_{1(2)} \\ \vdots \\ \delta_{n(q)} \end{bmatrix}_{nq\times 1} , \quad \epsilon = \begin{bmatrix} e_{111} \\ e_{112} \\ \vdots \\ e_{nqp} \end{bmatrix}_{nqp\times 1}$$ and $X = (X_1^T, X_2^T, X_3^T, X_4^T)^T$ is an

$(nqp \times (1 + q + p + qp))$ design matrix of fixed effects.

We assume $nqp < 1 + q + p + qp + nq, nqp + s = 1 + q + p + qp + nq$, ($s$ is integer number and greater than 1), and we assume that the design matrix as the following way:

$$M_{nqp}^T M_{nqp} = nqpI_{nqp} \qquad \forall nqp \geq 1 \tag{4}$$

where:

$M_{nqp} = [X\ Z_r]$ and Z_r be the $nqp \times [nqp - (1 + q + p + qp)]$ matrix, and let $Z = [Z_r\ Z_s]$.

We would like to choose between a Bayesian mixed one- way repeated measurements model and it's one- way repeated measurements model (the model without random effects) by the criterion of the Bayes factor for two hypotheses,

$H_0: Y = X\beta + \epsilon$ i . e H_0: $y_{ijk} = \mu + \tau_j + \gamma_k + (\tau\gamma)_{jk} + e_{ijk}$ versus

$H_1: Y = X\beta + Zb + \epsilon$ i . e H_1: $y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk}$. (5)

We assume β and b are independent and the prior distribution for the parameters are

$\beta \sim N(0, \Sigma_0)$, where

$$\Sigma_0 = \sigma_\beta^2 I_{1+q+p+qp} = \begin{bmatrix} \sigma_\mu^2 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} \\ 0_{q\times 1} & \sigma_\tau^2 I_{q\times q} & 0_{q\times p} & 0_{q\times qp} \\ 0_{p\times 1} & 0_{q\times p} & \sigma_\gamma^2 I_{p\times p} & 0_{p\times qp} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & \sigma_{(\tau\gamma)}^2 I_{qp\times qp} \end{bmatrix}_{(1+q+p+qp)\times(1+q+p+qp)}, \tag{6}$$

and $b \sim N(0, \Sigma_1)$, where

$$\Sigma_1 = \sigma_b^2 I_{nq} = \sigma_\delta^2 I_{nq} = \begin{bmatrix} \sigma_{\delta_{11}}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{\delta_{12}}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{\delta_{nq}}^2 \end{bmatrix}_{nq\times nq}. \tag{7}$$

2.Posterior distribution

From the model (1), we have the following:

$$Y_{nqp}|\xi_{nqp} \sim N_{nqp}(\xi_{nqp}, \sigma_\epsilon^2 I_{nqp}), \ \xi_{nqp} \sim N_{nqp}(0_{nqp}, X\Sigma_0 X^T + Z\Sigma_1 Z^T) \tag{8}$$

where

$\xi_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk}$,for, $i = 1,2, \dots, n, j = 1, \dots, q$ and $k = 1, \dots, p$.

Then, the posterior distribution $\pi_1(\xi_{nqp}|Y_{nqp})$ is the multivariate normal with

$E(\xi_{nqp}|Y_{nqp}, \sigma_\epsilon^2) = \Sigma_2(\Sigma_2 + \sigma_\epsilon^2 I_N)^{-1} Y_{nqp}$, where N=1+q+p+qp+nq and

$Var(\xi_{nqp}|Y_{nqp}, \sigma_\epsilon^2) = (\Sigma_2^{-1} + (\sigma_\epsilon^2 I_N)^{-1})^{-1} = \Sigma_2(\Sigma_2 + \sigma_\epsilon^2 I_N)^{-1}\sigma_\epsilon^2 I_N$

where

$$\Sigma_2 = X\Sigma_0 X^T + Z\Sigma_1 Z^T = X \begin{bmatrix} \sigma_\mu^2 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} \\ 0_{q\times 1} & \sigma_\tau^2 I_{q\times q} & 0_{q\times p} & 0_{q\times qp} \\ 0_{p\times 1} & 0_{p\times q} & \sigma_\gamma^2 I_{p\times p} & 0_{p\times qp} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & \sigma_{(\tau\gamma)}^2 I_{qp\times qp} \end{bmatrix} X^T + Z$$

$$\begin{bmatrix} \sigma_{\delta_{11}}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{\delta_{12}}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{\delta_{nq}}^2 \end{bmatrix} Z^T$$

Then, $Y_{nqp} \sim MVN(0, \sigma_\epsilon^2 I_{nqp} + \Sigma_2)$. (9)

Hence, the distribution of Y_{nqp} under H_0 and H_1 are respectively $N_{nqp}(0, \sigma_\epsilon^2 I_{nqp} + X\Sigma_0 X^T)$ and $N_{nqp}(0, \sigma_\epsilon^2 I_{nqp} + X\Sigma_0 X^T + Z\Sigma_1 Z^T)$. (10)

We investigate the consistency of the Bayes factor, under the one- way repeated measurements model with design matrix of the random effects (Z) to the first nqp terms. We get

$y_{ijk} = \mu + \tau_j + \delta_{K^*} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk}$, $K^* = 1, \dots, (nqp - (1 + q + p + qp))$.The covariance matrix of the distribution of Y_{nqp} under H_1 as:

$$\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r} Z_r^T = \sigma_\epsilon^2 I_{nqp} + M_{nqp} D_{nqp} M_{nqp}^T \quad (11)$$

where $\sum_{1,r=}\sigma_b^2 I_{nqp-(1+q+p+qp)} = \begin{bmatrix} \sigma_{\delta_{11}}^2 & 0 & & \cdots & 0 \\ 0 & \sigma_{\delta_{12}}^2 & & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & \cdots & \sigma_{\delta_{nqp-(1+q+p+qp)}}^2 & \end{bmatrix}_{nqp-(1+q+p+qp)\times nqp-(1+q+p+qp)}$

$= \sigma_\delta^2 I_{nqp-(1+q+p+qp)}$ and $D_{nqp} = \begin{bmatrix} \sum_0 & 0_{r1} \\ 0_{1r} & \sum_{1,r} \end{bmatrix}$, where 0_{r1}, 0_{1r} are the matrices for zeros element with size $(1+q+p+qp)\times(nqp-(1+q+p+qp))$ and $(nqp-(1+q+p+qp))\times(1+q+p+qp)$, respectively.

Then, we can rewrite (11) by using D_{nqp} as:

$$M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right] M_{nqp}^T. \quad (12)$$

Then, the covariance matrix of the distribution of Y_{nqp} under H_0, can be represented as:

$\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r 0_{rr} Z_r^T = \sigma_\epsilon^2 I_{nqp} + M_{nqp} C_{nqp} M_{nqp}^T$, where:

$C_{nqp} = \begin{bmatrix} \sum_0 & 0_{r1} \\ 0_{1r} & 0_{rr} \end{bmatrix}$

and 0_{rr} is the matrix for zeros element with size $(nqp-(1+q+p+qp))\times(nqp-(1+q+p+qp))$, then we can rewrite the covariance matrix of the distribution of Y_{nqp} under H_0, the following:

$$M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + C_{nqp}\right] M_{nqp}^T \quad (13)$$

and the covariance matrix of the distribution of Y_{nqp} under H_1, can be represented as:

$\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T = \sigma_\epsilon^2 I_{nqp} + [X\ Z] G_{nqp} [X\ Z]^T$, where:

$G_{nqp} = \begin{bmatrix} \sum_0 & 0_{r1} \\ 0_{1r} & \sum_1 \end{bmatrix}$

then we can rewrite the covariance matrix of the distribution of Y_{nqp} under H_1, the following:

$$[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right][X\ Z]^T \quad (14)$$

Lemma(1): Let the covariance matrices of the distribution of Y_{nqp} under H_O and H_1 be defined as in (12) and (13) respectively, and suppose M_{nqp} satisfy (4). Then, the following hold.

$1 - M_{nqp} M_{nqp}^T = M_{nqp}^T M_{nqp} = nqp I_{nqp}$

$2 - \det\left(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T\right) = nqp^{1+q+p+qp}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\mu^2\right)\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\tau^2\right)^q\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\gamma^2\right)^p\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_{(\tau\gamma)}^2\right)^{qp}$ $(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}$

$3 - \det\left(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r} Z_r^T\right) =$

$(nqp)^{nqp}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\mu^2\right)\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\tau^2\right)^q\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\gamma^2\right)^p\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_{(\tau\gamma)}^2\right)^{qp}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\delta^2\right)^{nqp-(1+q+p+qp)}$

$4 - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0 X^T)^{-1} = \frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + C_{nqp}\right]^{-1} M_{nqp}^T$

$5 - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r} Z_r^T)^{-1} = \frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T$

Proof:-

$1 -$ By multiplying M_{nqp} and M_{nqp}^T to equation (4) from right and left side, we have:-

$$\begin{aligned} M_{nqp} M_{nqp}^T M_{nqp} M_{nqp}^T &= M_{nqp}\, nqp I_{nqp} M_{nqp}^T \\ &= M_{nqp}\, nqp M_{nqp}^T \\ &= nqp M_{nqp} M_{nqp}^T \end{aligned}$$

Multiplying $(M_{nqp} M_{nqp}^T)^{-1}$ to the above equation from left side, we get:-

$M_{nqp} M_{nqp}^T M_{nqp} M_{nqp}^T (M_{nqp} M_{nqp}^T)^{-1} = nqp M_{nqp} M_{nqp}^T (M_{nqp} M_{nqp}^T)^{-1}$

$M_{nqp} M_{nqp}^T = nqp I_{nqp}$

$2 - \sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T = M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + C_{nqp}\right] M_{nqp}^T$, and by using (4), we get

$$= [X\ ,Z_r]\begin{bmatrix} (\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2) & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & (\frac{\sigma_\epsilon^2 I_q}{nqp}+\sigma_\tau^2 I_q) & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & (\frac{\sigma_\epsilon^2 I_p}{nqp}+\sigma_\gamma^2 I_p) & 0_{p\times qp} & 0_{p\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & (\frac{\sigma_\epsilon^2 I_{qp}}{nqp}+\sigma_{(\tau\gamma)}^2 I_{qp}) & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \frac{\sigma_\epsilon^2}{nqp} I_{nqp-(1+q+p+qp)} \end{bmatrix}[X\ ,Z_r]^T$$

$$= \begin{bmatrix} nqp(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2) & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & nqp(\frac{\sigma_\epsilon^2 I_q}{nqp}+\sigma_\tau^2 I_q) & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & nqp(\frac{\sigma_\epsilon^2 I_p}{nqp}+\sigma_\gamma^2 I_p) & 0_{p\times qp} & 0_{p\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & nqp(\frac{\sigma_\epsilon^2 I_{qp}}{nqp}+\sigma_{(\tau\gamma)}^2 I_{qp}) & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \sigma_\epsilon^2 I_{nqp-(1+q+p+qp)} \end{bmatrix}$$

Then

$$\det\left(\sigma_\epsilon^2 I_{nqp} + X\textstyle\sum_0 X^T\right) = nqp^{1+q+p+qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp} (\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}$$

$$3 - \sigma_\epsilon^2 I_{nqp} + X\textstyle\sum_0 X^T + Z_r\sum_{1,r}Z_r^T = M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + D_{nqp}\right]M_{nqp}^T$$

$$= [X\ \ Z_r]\begin{bmatrix} (\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2) & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & (\frac{\sigma_\epsilon^2 I_q}{nqp}+\sigma_\tau^2 I_q) & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & (\frac{\sigma_\epsilon^2 I_p}{nqp}+\sigma_\gamma^2 I_p) & 0_{p\times qp} & 0_{p\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & (\frac{\sigma_\epsilon^2 I_{qp}}{nqp}+\sigma_{(\tau\gamma)}^2 I_{qp}) & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & (\frac{\sigma_\epsilon^2}{nqp} I_{nqp-(1+q+p+qp)} + \sigma_\delta^2 I_{nqp-(1+q+p+qp)}) \end{bmatrix}[X\ \ Z_r]^T$$

$$= \begin{bmatrix} nqp(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2) & 0_{1\times q} & 0_{1\times p} & 0_{1\times q} & 0_{1\times q} \\ 0_{q\times 1} & nqp(\frac{\sigma_\epsilon^2 I_q}{nqp}+\sigma_\tau^2 I_q) & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & nqp(\frac{\sigma_\epsilon^2 I_p}{nqp}+\sigma_\gamma^2 I_p) & 0_{p\times qp} & 0_{p\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & nqp(\frac{\sigma_\epsilon^2 I_{qp}}{nqp}+\sigma_{(\tau\gamma)}^2 I_{qp}) & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & nqp(\frac{\sigma_\epsilon^2}{nqp} I_{nqp-(1+q+p+qp)} + \sigma_\delta^2 I_{nqp-(1+q+p+qp)}) \end{bmatrix}.$$

Then

$$\det\left(\sigma_\epsilon^2 I_{nqp} + X\textstyle\sum_0 X^T + Z_r\sum_{1,r}Z_r^T\right) = (nqp)^{nqp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2)^{nqp-(1+q+p+qp)}$$

$$4 - (\sigma_\epsilon^2 I_{nqp} + X\textstyle\sum_0 X^T)^{-1} = \{M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]M_{nqp}^T\}^{-1}$$

$$= {M_{nqp}^T}^{-1}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1}{M_{nqp}}^{-1}$$

$$= \frac{I_{nqp}^{-1}}{nqp}nqpI_{nqp}{M_{nqp}^T}^{-1}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1}{M_{nqp}}^{-1}nqpI_{nqp}\frac{I_{nqp}^{-1}}{nqp}$$

$$= \frac{1}{(nqp)^2}M_{nqp}M_{nqp}^T{M_{nqp}^T}^{-1}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1}{M_{nqp}}^{-1}M_{nqp}M_{nqp}^T$$

$$= \frac{1}{(nqp)^2}M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1}M_{nqp}^T \text{ (by lemma 1)}$$

$$5 - (\sigma_\epsilon^2 I_{nqp} + X\textstyle\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} = \{M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + D_{nqp}\right]M_{nqp}^T\}^{-1}$$

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$$= {M_{nqp}^{T}}^{-1}\left[\frac{\sigma_{\epsilon}^{2}}{nqp}I_{nqp} + D_{nqp}\right]^{-1} {M_{nqp}}^{-1}$$

$$= \frac{I_{nqp}^{-1}}{nqp} nqpI_{nqp}{M_{nqp}^{T}}^{-1}\left[\frac{\sigma_{\epsilon}^{2}}{nqp}I_{nqp} + D_{nqp}\right]^{-1} {M_{nqp}}^{-1} nqpI_{nqp}\frac{I_{nqp}^{-1}}{nqp}$$

$$= \frac{1}{(nqp)^2} M_{nqp}M_{nqp}^{T}{M_{nqp}^{T}}^{-1}\left[\frac{\sigma_{\epsilon}^{2}}{nqp}I_{nqp} + D_{nqp}\right]^{-1} {M_{nqp}}^{-1}M_{nqp}M_{nqp}^{T}$$

$$= \frac{1}{(nqp)^2} M_{nqp}\left[\frac{\sigma_{\epsilon}^{2}}{nqp}I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^{T}. \blacksquare$$

3- The Bayes factor

The Bayes factor for testing problem (5) is given by:

$$B_{01}(\mathrm{y_{ijk}}) = \frac{\mathrm{m}(\mathrm{y_{ijk}}|H_o)}{\mathrm{m}(\mathrm{y_{ijk}}|H_1)}$$

$$= \frac{\frac{1}{\sqrt{2\pi nqp^{1+q+p+qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+q)}}}\exp\left\{-\frac{1}{2}Y_{nqp}^T(\frac{1}{nqp^2}M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp}+C_{nqp}\right]^{-1}M_{nqp}^T)Y_{nqp}\right\}}{\frac{1}{\sqrt{2\pi(nqp)^{nqp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2\right)\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2\right)^q\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2\right)^p\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2\right)^{qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2)^{nq}}}\exp\left\{-\frac{1}{2}Y_{nqp}^T(\frac{1}{nqp^2}[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp}+G_{nqp}\right]^{-1}[X\ Z]^T)Y_{nqp}\right\}}$$

$$= \frac{\sqrt{(nqp)^{nqp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2\right)\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2\right)^q\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2\right)^p\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2\right)^{qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2)^{nq}}}{\sqrt{nqp^{1+q+p+qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}}}\exp\{-\frac{1}{2nqp^2}Y_{nqp}^T$$

$$(M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1} M_{nqp}^T-[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + G_{nqp}\right]^{-1}[X\ Z]^T)Y_{nqp}\}. \qquad (15)$$

Now let the approximation of B_{01} with design matrix of the random effects (Z) to the first nqp terms as following

$$\tilde{B}_{01} = \frac{\sqrt{(nqp)^{nqp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2)^{nqp-(1+q+p+qp)}}}{\sqrt{nqp^{1+q+p+qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}}}\exp\{-\frac{1}{2nqp^2}Y_{nqp}^T$$

$$(M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1} M_{nqp}^T-M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T)Y_{nqp}\}. \qquad (16)$$

Then,

$$\log\tilde{B}_{01} = \frac{1}{2}\log\frac{(nqp)^{nqp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2\right)^{nqp-(1+q+p+qp)}}{nqp^{1+q+p+qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}} - \frac{1}{2nqp^2}Y_{nqp}^T$$

$$(M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1} M_{nqp}^T - M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T)Y_{nqp}$$

$$2\log\tilde{B}_{01} = \log\frac{(nqp)^{nqp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2\right)^{nqp-(1+q+p+qp)}}{nqp^{1+q+p+qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}} - \frac{1}{nqp^2}Y_{nqp}^T\ (M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1} M_{nqp}^T$$

$$-M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T)Y_{nqp}$$

$$\rightarrow 2\log\tilde{B}_{01} = \log\frac{(nqp)^{nqp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2)^{nqp-(1+q+p+qp)}}{nqp^{1+q+p+qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}} - \frac{1}{(nqp)^2}\left\{Y_{nqp}^TM_{nqp}\left\{\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + C_{nqp}\right]^{-1} - \left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} + \right.\right.\right.$$

$$\left.\left.\left. D_{nqp}\right]^{-1}\right\}M_{nqp}^TY_{nqp}\right\}$$

$$= \log\frac{(nqp)^{nqp-(1+q+p+qp)}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2\right)^{nqp-(1+q+p+qp)}}{(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}} - \frac{1}{(nqp)^2}\{Y_{nqp}^TM_{nqp}$$

$$\begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}\right) I_{nqp-(1+q+p+qp)} \end{bmatrix}_{nqp\times nqp} M_{nqp}^T Y_{nqp}\}$$

$$= \log\left(\frac{\sigma_\epsilon^2+nqp\sigma_\delta^2}{\sigma_\epsilon^2}\right)^{nqp-(1+q+p+qp)} - \{Y_{nqp}^T Q_{nqp} Y_{nqp}\}$$

where

$$Q_{nqp} = \frac{1}{(nqp)^2} M_{nqp} \begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}\right) I_{nqp-(1+q+p+qp)} \end{bmatrix}_{nqp\times nqp} M_{nqp}^T .$$

The Bayes factor for testing the model in (5) satisfy is consistent if [6]

$$lim_{nqp\to\infty} B_{01} = \infty \quad , \quad almost\ surely\ in\ probablity\ P_0^\infty \text{ under } H_0, \tag{17}$$

$$lim_{nqp\to\infty} B_{01} = 0 \quad , \quad almost\ surely\ in\ probablity\ P_1^\infty \text{ under } H_1, \tag{18}$$

Lemma(2): let $Y_{nqp} = (y_{111}, \dots, y_{nqp})^T$ where $y_{ijk}'s$ are independent normal random variable with mean $\mu + \tau_j + \gamma_k + (\tau\gamma)_{jk}$ and variance $\sigma_\epsilon^2 = 1$, then there exist a positive constant C_1 such that $\frac{1}{S_{nqp,1}}\left[\sum_{i=1}^{nqp-(1+q+p+qp)} \log(1+nqp\sigma_\delta^2) - Y_{nqp}^T Q_{nqp} Y_{nqp}\right] > C_1$, with probability tending to 1, i.e. $\log\tilde{B}_{01} \xrightarrow{nqp\to\infty} \infty$, in probability, where $S_{nqp,1} = \sum_{i=1}^{nqp} \frac{nqp\sigma_\delta^2}{(1+nqp\sigma_\delta^2)}$.

Proof:

Let $X_{nqp} = (X_{111}, X_{121}, \dots, X_{nqp})$ are independent standard normal random variables, Note that $Y_{nqp} \overset{d}{=} X_{nqp} + X\beta$ and

$$Y_{nqp}^T Q_{nqp} Y_{nqp} = (X_{nqp} + X\beta)^T Q_{nqp} (X_{nqp} + X\beta)$$

$$= X_{nqp}^T Q_{nqp} X_{nqp} + (X\beta)^T Q_{nqp} (X\beta) + X_{nqp}^T Q_{nqp} (X\beta) + (X\beta)^T Q_{nqp} X_{nqp}.$$

By the property of the design matrix (4) get

$$(X\beta)^T Q_{nqp} = (X\beta)^T \frac{1}{(nqp)^2} M_{nqp} \begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}\right) I_{nqp-(1+q+p+qp)} \end{bmatrix}$$

$M_{nqp}^T =$

$$\frac{1}{(nqp)^2}[\mu,\tau,\gamma,(\tau\gamma)]nqp\left(I_{1+q+p+qp}, 0_{nqp-(1+q+p+qp)}\right) \begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}\right) I_{nqp-(1+q+p+qp)} \end{bmatrix}$$

$$nqp\left(I_{1+q+p+qp}, 0_{nq-(1+q+p+qp)}\right)[\mu,\tau,\gamma,(\tau\gamma)]^T = 0.$$

Then $Y_{nqp}^T Q_{nqp} Y_{nqp} = X_{nqp}^T Q_{nqp} X_{nqp}$.

Now by using the expectation formula of the quadratic form[4],[10],[12],[14], get

$$0 < E\left(X_{nqp}^T Q_{nqp} X_{nqp}\right) = tr\left(Q_{nqp}\right) = \sum_{i=1}^{nqp} \frac{nqp\sigma_\delta^2}{(1+nqp\sigma_\delta^2)} = S_{nqp,1}$$

Note that

$$Q_{nqp}^2 = \frac{1}{(nqp)^4} M_{nqp} \begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & (\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}) I_{nqp-(1+q+p+qp)} \end{bmatrix} M_{nqp}^T M_{nqp}$$

$$\begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & (\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}) I_{nqp-(1+q+p+qp)} \end{bmatrix} M_{nqp}^T$$

$$= \frac{1}{(nqp)^3} M_{nqp} \begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & (\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2})^2 I_{nqp-(1+q+p+qp)} \end{bmatrix} M_{nqp}^T$$

and

$$tr(Q_{nqp}^2) = \frac{1}{(nqp)^2}\sum_{i=1}^{nqp}(\frac{(nqp)^2\sigma_\delta^2}{1+nqp\sigma_\delta^2})^2 = \sum_{i=1}^{nqp}(\frac{nqp\sigma_\delta^2}{1+nqp\sigma_\delta^2})^2 = nqp^3(\frac{\sigma_\delta^2}{1+nqp\sigma_\delta^2})^2 = S_{nqp,2}.$$

Using the variance formula of the quadratic form[4],[10],[12],[14], get

$$var(X_{nqp}^T Q_{nqp} X_{nqp}) = 2tr(Q_{nqp}^2) = 2S_{nqp,2} = 2\sum_{i=1}^{nqp}(\frac{nqp\sigma_\delta^2}{1+nqp\sigma_\delta^2})^2.$$

Let $c_{nqp} = \frac{S_{nqp,3}-S_{nqp,1}}{2S_{nqp,1}}$, where $S_{nqp,3} = \sum_{i=1}^{nqp}\log(1+nqp\sigma_\delta^2) = nqp\log(1+nqp\sigma_\delta^2)$. Then,

$$\Pr\{\frac{1}{S_{nqp,1}}[\sum_{i=1}^{nqp-(1+q+p+qp)}\log(1+nqp\sigma_\delta^2)^2 - Y_{nqp}^T Q_{nqp} Y_{nqp}] \le c_{nqp}\} = \Pr[\frac{\{Y_{nqp}^T Q_{nqp} Y_{nqp}\}}{S_{nqp,1}} \ge \frac{S_{nqp,3}}{S_{nqp,1}} - c_{nqp}]$$

$$= \Pr[\frac{\{Y_{nqp}^T Q_{nqp} Y_{nqp}\}}{S_{nqp,1}} - \frac{E(Y_{nqp}^T Q_{nqp} Y_{nqp})}{S_{nqp,1}} \ge \frac{S_{nqp,3}}{S_{nqp,1}} - c_{nqp} - \frac{E(Y_{nqp}^T Q_{nqp} Y_{nqp})}{S_{nqp,1}}]$$

$$= \Pr[\frac{1}{S_{nqp,1}}\{(Y_{nqp}^T Q_{nqp} Y_{nqp}) - E(Y_{nqp}^T Q_{nqp} Y_{nqp})\} \ge \frac{S_{nqp,3}-E(Y_{nqp}^T Q_{nqp} Y_{nqp})}{S_{nqp,1}} - c_{nqp}]$$

$$\le \frac{1}{S_{nqp,1}^2}\frac{var(Y_{nqp}^T Q_{nqp} Y_{nqp})}{(\frac{S_{nqp,2}-S_{nqp,1}}{2S_{nqp,1}})^2} = \frac{8S_{nqp,2}}{(S_{nqp,3}-S_{nqp,1})^2} \xrightarrow{nqp\to\infty} 0,$$ from the Chebyshev inequality[13] .

Since $S_{nqp,3} - S_{nqp,1} \ge c_1 S_{nqp,1}$ for some $c_1 > 0$ and let $c_{nqp} \ge c_1/2$.

Then, there exists a positive constant $C_1 = c_1/2$ such as

$\frac{1}{S_{nqp,1}}[\sum_{i=1}^{nqp-(1+q+p+qp)}\log(1+nqp\sigma_\delta^2)^2 - Y_{nqp}^T Q_{nqp} Y_{nqp}] > C_1$, with probability tending to 1.

Then,

$2\log\tilde{B}_{01} = \sum_{i=1}^{nqp-(1+q+p+qp)}\log(1+nqp\sigma_\delta^2)^2 - Y_{nqp}^T Q_{nqp} Y_{nqp} \xrightarrow{nqp\to\infty} \infty$ in probability under H_0 i.e. $\log\tilde{B}_{01} \xrightarrow{nqp\to\infty} \infty$

Lemma 3: Suppose we have the one- way repeated measurements model, then

$$\frac{1}{S_{nqp,1}} tr\left(\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T - [X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1} [X\ Z]^T\right) \xrightarrow{nqp\to\infty} 0$$

Proof:

Since $[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right][X\ Z]^T - M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right] M_{nqp}^T$ is the covariance matrix of $\xi^* = \delta_{K^*} = Z_s b$, $K^* = nqp - (1+q+p+qp), \dots, nqp+s$, it is a positive definite matrix. Thus $\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T - \frac{1}{nqp^2}[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1}[X\ Z]^T$ is also a positive definite matrix.

Thus, $tr\left(\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T\right) \ge tr(\frac{1}{nqp^2}[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1}[X\ Z]^T)$, also

$tr\left(Z_s^T \frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T\right) \le \sqrt{tr(Z_s^T Z_s) tr\left(\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T\right)}$

$= \|Z_s\| \left\|\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T\right\|$ by the Cauchy-Schwarz inequality, where

$\left\|\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T\right\| = \{ tr\left((\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right] M_{nqp}^T)^{-2}\right)\}^{\frac{1}{2}}$

$= [\frac{1}{nqp^2}\{\left(\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\mu^2\right) + \sum_{i=1}^{q}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\tau^2\right) + \sum_{i=1}^{p}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\gamma^2\right) + \sum_{i=1}^{qp}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_{(\tau\gamma)}^2\right)\right)^{-2} +$

$\sum_{i=1}^{nqp-(1+q=p+qp)}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\delta^2\right)^{-2}\}]^{\frac{1}{2}}$.

Since

$\left(\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\mu^2\right) + \sum_{i=1}^{q}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\tau^2\right) + \sum_{i=1}^{p}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\gamma^2\right) + \sum_{i=1}^{qp}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_{(\tau\gamma)}^2\right)\right)^{-2} = \frac{1}{\left(\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2\right)+q\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2\right)+p\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2\right)+qp\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2\right)\right)^2} =$

$\frac{nqp^2}{\left((\sigma_\epsilon^2+nqp\sigma_\mu^2)+q(\sigma_\epsilon^2+nqp\sigma_\tau^2)+p(\sigma_\epsilon^2+nqp\sigma_\gamma^2)+qp\left(\sigma_\epsilon^2+nqp\sigma_{(\tau\gamma)}^2\right)\right)^2} \le \frac{nqp^2}{\left((nqp\sigma_\mu^2)+q(nqp\sigma_\tau^2)+p(nqp\sigma_\gamma^2)+qp\left(nqp\sigma_{(\tau\gamma)}^2\right)\right)^2}$

$= \frac{1}{\left((\sigma_\mu^2)+q(\sigma_\tau^2)+p(\sigma_\gamma^2)+qp\left(\sigma_{(\tau\gamma)}^2\right)\right)^2} = O(1)$, and

$\sum_{i=1}^{nqp-(1+q+p+qp)}\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\delta^2\right)^{-2} = \frac{nqp-(1+q+p+qp)}{\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2\right)^2} \le \frac{nqp}{\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2\right)^2} = \frac{(nqp)^3}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)^2} \le \frac{(nqp)^3}{(nqp\sigma_\delta^2)^2} = O(nqp).$

Then, $\left\|\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T\right\| = O\{(nqp)^{\frac{1}{2}}\}$.

Since $sup\|Z_s\| = 1$ from the distances between the elements of design matrix Z, then $\|Z_s\|^2 \le nqp$.

Thus , $\frac{1}{S_{nqp,1}} tr\left(\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T - \frac{1}{nqp^2}[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1}[X\ Z]^T\right) =$

$\frac{1}{S_{nqp,1}} tr\left(\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T \left\{\frac{1}{nqp^2}[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right][X\ Z]^T - \frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right] M_{nqp}^T\right\}[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1}[X\ Z]^T\right)$

$= \frac{1}{S_{nqp,1}} \sum_{K^*=nqp-(1+q+p+qp)}^{nqp+s} \sigma_\delta^2 (Z_{K^*}^T \frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T (Z_{K^*}^T \frac{1}{nqp^2}[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1}[X\ Z]^T)^T) \le \frac{\sigma_\delta^2}{S_{nqp,1}}\left\|\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T\right\|^2 \sum_{K^*=nqp-(1+q+p+qp)}^{nqp+s}\|z_{K^*}\|^2$

$\le \frac{nqp\sigma_\delta^2}{nqpS_{nqp,1}} \sum_{K^*=nqp-(1+q+p+qp)}^{nqp+s} 1 \le \frac{K^*\sigma_\delta^2}{nqp\ S_{nqp,1}} = \frac{K^*\sigma_\delta^2}{\frac{(nqp)^2\sigma_\delta^2}{(1+nqp\sigma_\delta^2)}} = \frac{K^*(1+nqp\sigma_\delta^2)}{(nqp)^2}$

Thus

$\frac{1}{S_{nqp,1}} tr\left(\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T - \frac{1}{nqp^2}[X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1}[X\ Z]^T\right) \le \frac{K^*(1+napσ_\delta^2)}{(nqp)^2} \xrightarrow{nqp\to\infty} 0$

Theorem(1): Suppose that the true model is the one- way repeated measurements model with fixed effect only (the model under H_0 in 5), and let P_0^{nqp} denote the true distribution of the data, with probability density function $P_0(y_{ijk}|\mu,\tau_j,\gamma_k,(\tau\gamma)_{jk}) = \Phi\{(y_{ijk} - \mu - \tau_j - \gamma_k - (\tau\gamma)_{jk})/\sigma_\epsilon\}$, where $\Phi(\cdot)$ is the standard normal density, then, the Bayes factor is consistent under the null hypothesis H_0 i.e

$\lim_{nqp\to\infty} B_{01} = \infty$ in probability P_0^{nqp}.

Proof:

$\log B_{01} = \frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0 X^T+Z\Sigma_1 Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0 X^T)} - \frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0 X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0 X^T + Z\Sigma_1 Z^T)^{-1})Y_{nqp}$

$= \frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0 X^T+Z\Sigma_1 Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0 X^T+Z_r\Sigma_{1,r} Z_r^T)} + \frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0 X^T+Z_r\Sigma_{1,r} Z_r^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0 X^T)} - \frac{1}{2}Y_{nqp}^T(\sigma_\epsilon^2 I_{nqp} + X\Sigma_0 X^T)^{-1}$

$-(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1})Y_{nqp} \ -\frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1})Y_{nqp}$

$= \log\tilde{B}_{01} + \frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z\sum_1 Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z_r\sum_{1,r}Z_r^T)} - \frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1})Y_{nqp}$

Since $(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T) - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)$ is a positive definite matrix, $det\ (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T) \ \geq\ det(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)$, thus

$\frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z\sum_1 Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z_r\sum_{1,r}Z_r^T)} \geq 0$

$E(\ Y_{nqp}^T\big((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1})Y_{nqp}\big) = \ \sigma_\epsilon^2\ tr\big((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1}\big) + (X\beta)^T\big((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1}\big)(X\beta)$,

since $(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1}$, $(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1}$ and $(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1}$ are all nonnegative definite matrices, it follows that

$0 \leq\ (X\beta)^T\big((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + (X\sum_0 X^T + Z\sum_1 Z^T)^{-1}\big)(X\beta) \leq (X\beta)^T (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1}\ (X\beta).$

By the property of design matrix (4), we get

$$X^T M_{nqp} =\ M_{nqp}^T X = nqp\ \big(I_{1+q+p+qp}\ , 0_{nqp-(1+q+p+qp)}\big).$$

Now as the proof of lemma (3), we get:

$(X\beta)^T(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1}(X\beta) = \frac{1}{(nqp)^2}(X\beta)^T M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp}I_{nqp} +\ D_{nqp}\right]^{-1} M_{nqp}^T(X\beta)$

$= \frac{1}{(nqp)^2}[\mu,\tau,\gamma,(\tau\gamma)]nqp\ \big(I_{1+q+p+qp}\ , 0_{nq-(1+q+p+qp)}\big)$

$$\begin{bmatrix} (\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2) & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & (\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)I_q & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & (\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)I_p & 0_{p\times qp} & 0_{p\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & (\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)I_{qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & (\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2)I_{nqp-(1+q+p+qp)} \end{bmatrix}^{-1} nqp$$

$\big(I_{1+q+p+qp}\ , 0_{nq-(1+q+p+qp)}\big)\ [\mu,\tau,\gamma,(\tau\gamma)]^T$

$= \ \mu^2\left[\frac{\sigma_\epsilon^2}{nqp}+\ \sigma_\mu^2\right]^{-1} + \sum_{i=1}^{q}\tau_i^2\left[\frac{\sigma_\epsilon^2}{nqp}+\ \sigma_\tau^2\right]^{-1} + \sum_{i=1}^{p}\gamma_i^2\left[\frac{\sigma_\epsilon^2}{nqp}+\ \sigma_\gamma^2\right]^{-1} + \sum_{i=1}^{qp}(\tau\gamma)_i^2\left[\frac{\sigma_\epsilon^2}{nqp}+\ \sigma_{(\tau\gamma)}^2\right]^{-1}$

$= O(1).$

Since $pr\left\{\frac{1}{S_{nqp,1}}Y_{nqp}^T\big((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1})Y_{nqp} >\ \varepsilon\right\} \leq \frac{1}{S_{nqp,1}}\left\{\frac{E(\ Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z_r\sum_{1,r}Z_r^T)^{-1}-(\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z\sum_1 Z^T)^{-1})Y_{nqp})}{\varepsilon}\right\}$

$\leq\ \frac{1}{S_{nqp,1}}\frac{tr((\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z_r\sum_{1,r}Z_r^T)^{-1}-(\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z\sum_1 Z^T)^{-1}}{\varepsilon} + \frac{1}{S_{nqp,1}}$

$\frac{(X\beta)^T(\sigma_\epsilon^2 I_{nqp}+X\sum_0 X^T+Z_r\sum_{1,r}Z_r^T)^{-1}\ (X\beta)}{\varepsilon} \xrightarrow{nqp\to\infty} 0$ (by the Markov inequality[13], where $\varepsilon > 0$) .

Since $(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} -\ (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1}$ is nonnegative definite $\rightarrow$ $Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1})Y_{nqp}$ is nonnegative .

Therefore,

$$-\frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1})Y_{nqp} =\ o\big(S_{nqp,1}\big)$$

$$\Rightarrow \frac{-\frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)^{-1})Y_{nqp}}{S_{nqp,1}} = o(1)$$

Then, as in lemma (2) there exists a constant C_1 such that

$\frac{1}{S_{nqp,1}} \log B_{01} \geq \frac{1}{S_{nqp,1}} \log \tilde{B}_{01} + o(1) > C_1 + o(1).$

That is,

$B_{01} \geq \exp(S_{nqp,1}\{ C_1 + o(1)\}) \xrightarrow{nqp\to\infty} \infty$ in probability P_0^{nqp}.■

Lemma(4):

Assuming that the true model is the mixed one-way repeated measurements model (the model under H_1 in 5). Let $Y_{nqp} = (y_1, \dots, y_{nqp})$ where $y_{ijk}'s$ are independent normal random variables with mean $\mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk}$ and variance $\sigma_\epsilon^2 = 1$ and let $w_{nqp} = E(Y_{nqp})^T Q_{nqp} E(Y_{nqp})$.

Then, $\lim_{nqp\to\infty} \frac{1}{w_{nqp}} Y_{nqp}^T Q_{nqp} Y_{nqp} = 1$, in probability P_1^∞.

Proof:

From the moment formula of the quadratic form of normal random variables, the expectation and variance of quadratic form $Y_{nqp}^T Q_{nqp} Y_{nqp}$ are given by:

$E(Y_{nqp}^T Q_{nqp} Y_{nqp}) = tr\,(Q_{nqp}) + E(Y_{nqp})^T Q_{nqp} E(Y_{nqp})$

$= \sum_{i=1}^{nqp-(1+q+p+qp)} \frac{nqp\sigma_\delta^2}{(1+nqp\sigma_\delta^2)} + E(Y_{nqp})^T Q_{nqp} E(Y_{nqp})$

$= S_{nqp,1} + w_{nqp}$, and

$var(Y_{nqp}^T Q_{nqp} Y_{nqp}) = 2tr\,(Q_{nqp}{}^2) + 4E(Y_{nqp})^T Q_{nqp}{}^2 E(Y_{nqp})$.

Note

$w_{nqp} = E(Y_{nqp})^T Q_{nqp} E(Y_{nqp})$

$$= (\mu, \tau^T, \gamma^T, (\tau\gamma)^T, \delta_{nqp-(1+q+p+qp)}^T) M_{nqp}^T \frac{1}{nqp^2} M_{nqp}$$

$$\begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{p\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}\right) I_{nqp-(1+q+p+qp)} \end{bmatrix} M_{nqp}^T M_{nqp} (\mu, \tau^T, \gamma^T, (\tau\gamma)^T, \delta_{nqp-(1+q+p+qp)}^T)^T$$

$$= (\mu, \tau^T, \gamma^T, (\tau\gamma)^T, \delta_{nqp-(1+q+p+qp)}^T) \begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}\right) I_{nqp-(1+q+p+qp)} \end{bmatrix}$$

$$(\mu, \tau^T, \gamma^T, (\tau\gamma)^T, \delta_{nqp-(1+q+p+qp)}^T)^T$$

$$= \delta_{nqp-(1+q+p+qp)}^T \left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}\right) I_{nqp-(1+q+p+qp)} \delta_{nqp-(1+q+p+qp)} = \sum_{i=1}^{nqp-(1+q+p+qp)} \frac{(nqp)^2\sigma_\delta^2\delta_i^2}{1+nqp\sigma_\delta^2},$$

since $sup_i\, \delta_i < \infty$, we have

$w_{nqp} \leq sup_i\, \delta_i^2\, nqp \sum_{i=1}^{nqp} \frac{nqp\sigma_\delta^2}{1+nqp\sigma_\delta^2} = O(nqp\, S_{nqp,1})$, then $w_{nqp} = O(nqp^2)$.

Now we can consider $\delta_m^2 > 0$, where $m \geq 1$ is a fixed positive integer and a sufficiently large nqp with $\frac{nqp\sigma_\delta^2}{1+nqp\sigma_\delta^2} \geq \frac{1}{2}$. Then for such nqp and m.

$w_{nqp} = \sum_{i=1}^{nqp-(1+q+p+qp)} \frac{nqp^2\sigma_\delta^2\delta_i^2}{1+nqp\sigma_\delta^2} \geq \frac{nqp^2\sigma_\delta^2}{1+nqp\sigma_\delta^2}\delta_m^2$

$= nqp \frac{nqp\sigma_\delta^2}{1+nqp\sigma_\delta^2}\delta_m^2 \geq nqp\frac{\delta_m^2}{2}$.

Similarly to the previous calculation, we have

$$E(Y_{nqp})^T Q_{nqp}^2 E(Y_{nqp}) = (\mu, \tau^T, \gamma^T, (\tau\gamma)^T, \delta_{nqp-(1+q+p+qp)}^T) M_{nqp}^T \frac{1}{(nqp)^3} M_{nqp}$$

$$\begin{bmatrix} 0 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} & 0_{1\times nq} \\ 0_{q\times 1} & 0_{q\times q} & 0_{q\times p} & 0_{q\times qp} & 0_{q\times nq} \\ 0_{p\times 1} & 0_{p\times q} & 0_{p\times p} & 0_{p\times qp} & 0_{q\times nq} \\ & & 0_{qp\times p} & 0_{qp\times qp} & 0_{qp\times nq} \\ 0_{qp\times 1} & 0_{qp\times q} & & & \\ 0_{nq\times 1} & 0_{nq\times q} & 0_{nq\times p} & 0_{nq\times qp} & \left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta)\sigma_\epsilon^2}\right)^2 I_{nqp-(1+q+p+qp)} \end{bmatrix} M_{nqp}^T M_{nqp}\left(\mu,\tau^T,\gamma^T,(\tau\gamma)^T,\delta^T_{nqp-(1+q+p+qp)}\right)^T$$

$$= \frac{1}{nqp}\delta^T_{nqp-((1+q+p+qp)}\left(\frac{(nqp)^2\sigma_\delta^2}{(\sigma_\epsilon^2+nqp\sigma_\delta^2)\sigma_\epsilon^2}\right)^2 I_{nqp-(1+q+p+qp)}\delta_{nqp-(1+q+p+qp)}$$

$$= \frac{1}{nqp}\sum_{i=1}^{nqp-(1+q+p+qp)}\frac{(nqp)^4\sigma_\delta^4\delta_i^2}{(1+nqp\sigma_\delta^2)^2}.$$

Since $E(Y_{nqp})^T Q_{nqp}^2\, E(Y_{nqp}) \le \sum_{i=1}^{nqp}\frac{(nqp)^3\sigma_\delta^4\delta_i^2}{(1+nqp\sigma_\delta^2)^2} \le nqp\ sup_i\ \delta_i^2 \sum_{i=1}^{nqp}\frac{(nqp)^2\sigma_\delta^4}{(1+nqp\sigma_\delta^2)^2} = O(\,nqpS_{nqp,2})$ (as in case of $E(Y_{nqp})^T Q_{nqp}\, E(Y_{nqp})$).

Now for previous nqp and m we can find

$$E(Y_{nqp})^T Q_{nqp}^2\, E(Y_{nqp}) = \sum_{i=1}^{nqp-(1+q+p+qp)}\frac{(nqp)^3\sigma_\delta^4\delta_i^2}{(1+nqp\sigma_\delta^2)^2} \ge nqp\ \delta_m^2\left(\frac{nqp\sigma_\delta^2}{1+nqp\sigma_\delta^2}\right)^2 \ge \frac{\delta_m^2}{4}\ nqp.$$

By combining value of $S_{nqp,1}$ and the previous result, get

$var(Y_{nqp}^T Q_{nqp} Y_{nqp}\,) \le\ O(nqp\ S_{nqp,1})$.

Furthermore, note that $E(Y_{nqp})^T Q_{nqp}\, E(Y_{nqp}) \ \ge\ E(Y_{nqp})^T {Q_{nqp}}^2\, E(Y_{nqp})$.

By the Chebshev inequality[13] get

$$pr\left\{\left|\frac{1}{w_{nqp}}Y_{nqp}^T Q_{nqp} Y_{nqp} - \frac{E(Y_{nqp}^T Q_{nqp} Y_{nqp})}{w_{nqp}}\right| > \varepsilon\right\} \le \frac{var(Y_{nqp}^T Q_{nqp} Y_{nqp})}{w_{nqp}^2\varepsilon^2} = O(nqp^{-2}) \xrightarrow{nqp\to\infty} 0, \text{ where } \varepsilon > 0\,.$$

Since $\lim_{nqp\to\infty}\frac{E(Y_{nqp}^T Q_{nqp} Y_{nqp})}{w_{nqp}} = \lim_{nqp\to\infty}\left(\frac{S_{nqp,1}}{w_{nqp}} + \frac{w_{nqp}}{w_{nqp}}\right) = 1$

$\therefore \lim_{nqp\to\infty}\frac{1}{w_{nqp}}Y_{nqp}^T Q_{nqp} Y_{nqp} = 1$, in probability ■

Lemma(5):

Suppose we have the mixed one- way repeated measurements model, then

$$\lim_{nqp\to\infty}\frac{1}{w_{nqp}}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z\Sigma_1Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)} = 0$$

Proof:

Since $\det(\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z\Sigma_1Z^T) \le \prod_{i=1}^{nqp}(\sigma_\epsilon^2 + \sigma_\mu^2x_i^2 + \sum_{k=1}^{q}\sigma_\tau^2x_{ik}^2 + \sum_{k=1}^{p}\sigma_\gamma^2x_{ik}^2 + \sum_{k=1}^{qp}\sigma_{(\tau\gamma)}^2x_{ik}^2 + \sum_{k=1}^{nq}\sigma_\delta^2\,x_{ik}^2)$ (from the property of the determinant of the positive definite matrix)

$$= (nqp)^{nqp}\prod_{i=1}^{nqp}\left(\frac{\sigma_\epsilon^2}{nqp} + \frac{(\sigma_\mu^2x_i^2+\sum_{k=1}^{q}\sigma_\tau^2x_{ik}^2+\sum_{k=1}^{p}\sigma_\gamma^2x_{ik}^2+\sum_{k=1}^{qp}\sigma_{(\tau\gamma)}^2x_{ik}^2+\sum_{k=1}^{nq}\sigma_\delta^2x_{ik}^2}{nqp}\right).$$

And we have

$\det(\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T) = (nqp)^{nqp}(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp} + \sigma_{(\tau\gamma)}^2)^{qp}(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\delta^2)^{nqp-(1+q+p+qp)}$ (from lemma 1).

$\to \log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z\Sigma_1Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)} = \log\det(\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z\Sigma_1Z^T) - \log\det(\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)$

$\le \left\{(nqp)\log(nqp) + \sum_{i=1}^{nqp}\log\left(\frac{\sigma_\epsilon^2}{nqp} + \frac{(\sigma_\mu^2x_i^2+\sum_{k=1}^{q}\sigma_\tau^2x_{ik}^2+\sum_{k=1}^{p}\sigma_\gamma^2x_{ik}^2+\sum_{k=1}^{qp}\sigma_{(\tau\gamma)}^2x_{ik}^2+\sum_{k=1}^{nq}\sigma_\delta^2x_{ik}^2}{nqp}\right)\right\} - \left\{nqp\log(nqp) + \log\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\mu^2\right) + q\log\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\tau^2\right) + p\log\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\gamma^2\right) + qp\log\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_{(\tau\gamma)}^2\right) + nqp - (1 + q + p + qp)\log\left(\frac{\sigma_\epsilon^2}{nqp} + \sigma_\delta^2\right)\right\} = O(1) - O(1) = O(1).$

Then

$$\frac{1}{w_{nqp}}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z\Sigma_1Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)} \xrightarrow{nqp\to\infty} 0$$

Theorem(2):

Suppose that the true model is the mixed one- way repeated measurements model (the model under H_1 in 5), and let P_1^{nqp} denote the true distribution of the data, with probability density function $p_1(y_{ijk}|\mu\tau_j, \delta_{i(j)}, \gamma_k, (\tau\gamma)_{jk}) = \Phi\{(y_{ijk} - (\mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk}))/\sigma_\epsilon\}$, where $\Phi(\cdot)$ is the standard normal density. Then the Bayes factor is consistent under the alternative hypothesis H_1 i.e
$\lim_{nqp\to\infty} B_{01} = 0$ in probability P_1^{nqp}.

Proof:

$$\log B_{01} = \frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z\Sigma_1Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T)} - \frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z\Sigma_1Z^T)^{-1})Y_{nqp}$$

$$= \frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z\Sigma_1Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)} + \frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T)} - \frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1})Y_{nqp} - \frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z\Sigma_1Z^T)^{-1})Y_{nqp}$$

$$= \log\tilde{B}_{01} - \frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1})Y_{nqp} - \frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z\Sigma_1Z^T)^{-1})Y_{nqp}$$, where

$$\log\tilde{B}_{01} = \frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T)} - \frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1})Y_{nqp}.$$

From lemma (1), we have

$$\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T)} = \frac{(nqp)^{nqp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2)^{nqp-(1+q+p+qp)}}{nqp^{1+q+p+qp}(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2)(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2)^q(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2)^p(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2)^{qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}}$$

$$= \frac{nqp^{nqp-(1+q+p+qp)}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2\right)^{nqp-(1+q+p+qp)}}{\sigma_\epsilon^{2\,nqp-(1+q+p+qp)}} = \left(\frac{\sigma_\epsilon^2+nqp\sigma_\delta^2}{\sigma_\epsilon^2}\right)^{nqp-(1+q+p+qp)}$$

$$= \left(1 + \frac{nqp\sigma_\delta^2}{\sigma_\epsilon^2}\right)^{nqp-(1+q+p+qp)} = O(nqp)$$

$$\Longrightarrow \log\left(1 + \frac{nqp\sigma_\delta^2}{\sigma_\epsilon^2}\right)^{nqp-(1+q+p+qp)} = O(nqp).$$

And from lemma (4) $w_{nqp} = O(nqp^2)$. Then

$$\frac{1}{w_{nqp}}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T)} = O(nqp^{-1}) \xrightarrow{nqp\to\infty} 0.$$

Then,

$$\frac{1}{w_{nqp}}\log\tilde{B}_{01} = \frac{1}{2}\left(\frac{1}{w_{nqp}}\log\frac{\det(V_{1,nqp})}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T)} - \frac{1}{w_{nqp}}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1})Y_{nqp}\right).$$

Since $(\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1} = Q_{nqp}$, then $\frac{1}{w_{nqp}}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1})Y_{nqp} = \frac{1}{w_{nqp}}Y_{nqp}^TQ_{nqp}Y_{nqp}$. from lemma (4), $\lim_{nqp\to\infty}\frac{1}{w_{nqp}}Y_{nqp}^TQ_{nqp}Y_{nqp} = 1$, then

$$\frac{1}{w_{nqp}}\log\tilde{B}_{01} \xrightarrow{nqp\to\infty} = \frac{1}{2}\left(\frac{1}{w_{nqp}}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T)} - \frac{1}{w_{nqp}}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1})Y_{nqp}\right) \xrightarrow{nqp\to\infty} = \frac{1}{2}(0-1) = \frac{-1}{2}$$

$\therefore \frac{1}{w_{nqp}}\log\tilde{B}_{01} \xrightarrow{nqp\to\infty} \frac{-1}{2}$, in probability P_1^{nqp}.

Now from lemma (5),

$\frac{1}{w_{nqp}}\frac{1}{2}\log\frac{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z\Sigma_1Z^T)}{\det(\sigma_\epsilon^2 I_{nqp}+X\Sigma_0X^T+Z_r\Sigma_{1,r}Z_r^T)} \xrightarrow{nqp\to\infty} 0$, and $-\frac{1}{2}Y_{nqp}^T((\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z\Sigma_1Z^T)^{-1})Y_{nqp}$ is nonpositive random variable since $(\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z_r\Sigma_{1,r}Z_r^T)^{-1} - (\sigma_\epsilon^2 I_{nqp} + X\Sigma_0X^T + Z\Sigma_1Z^T)^{-1}$ is a nonnegative definite matrix.

Therefore, by combining the above results, there exist $C_2 > 0$ such that
$\lim_{nqp\to\infty} \sup \frac{1}{w_{nqp}} \log B_{01} \le \frac{-C_2}{2}$, with probability tending to 1.
Then , $B_{01} \xrightarrow{nqp\to\infty} 0$, in probability P_1^{nqp}. ■

4.Conclusions

1- The posterior distribution $\pi_1(\xi_{nqp}|Y_{nqp})$ is the multivariate normal with
$E(\xi_{nqp}|Y_{nqp},\sigma_\epsilon^2) = \sum_2(\sum_2 + \sigma_\epsilon^2 I_N)^{-1} Y_{nqp}$, where N=1+q+p+qp+nq and
$Var(\xi_{nqp}|Y_{nqp},\sigma_\epsilon^2) = (\sum_2^{-1} + (\sigma_\epsilon^2 I_N)^{-1})^{-1} = \sum_2(\sum_2 + \sigma_\epsilon^2 I_N)^{-1}\sigma_\epsilon^2 I_N$, where
$\xi_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk}$,for, $i = 1,2,\dots,n, j = 1,\dots,q$ and $k = 1,\dots,p$, and

$$\sum_2 = X\sum_0 X^T + Z\sum_1 Z^T = X\begin{bmatrix} \sigma_\mu^2 & 0_{1\times q} & 0_{1\times p} & 0_{1\times qp} \\ 0_{q\times 1} & \sigma_\tau^2 I_{q\times q} & 0_{q\times p} & 0_{q\times qp} \\ 0_{p\times 1} & 0_{p\times q} & \sigma_\gamma^2 I_{p\times p} & 0_{p\times qp} \\ 0_{qp\times 1} & 0_{qp\times q} & 0_{qp\times p} & \sigma_{(\tau\gamma)}^2 I_{qp\times qp} \end{bmatrix} X^T + Z$$

$$\begin{bmatrix} \sigma_{\delta_{11}}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{\delta_{12}}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{\delta_{nq}}^2 \end{bmatrix} Z^T$$

2-The Bayes factor for testing problem H_0: $y_{ijk} = \mu + \tau_j + \gamma_k + (\tau\gamma)_{jk} + e_{ijk}$ versus
H_1: $y_{ijk} = \mu + \tau_j + \delta_{i(j)} + \gamma_k + (\tau\gamma)_{jk} + e_{ijk}$ is given by:

$$B_{01} = \frac{\sqrt{(nqp)^{nqp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2\right)\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2\right)^q\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2\right)^p\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2\right)^{qp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2\right)^{nq}}}{\sqrt{nqp^{1+q+p+qp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2\right)\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2\right)^q\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2\right)^p\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2\right)^{qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}}} \exp\{-\frac{1}{2nqp^2} Y_{nqp}^T$$

$$(M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + C_{nqp}\right]^{-1} M_{nqp}^T - M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1} M_{nqp}^T) Y_{nqp}\}.$$

3-The approximation of B_{01} with design matrix of the random effects (Z) to the first nqp terms as following

$$\tilde{B}_{01} = \frac{\sqrt{(nqp)^{nqp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2\right)\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2\right)^q\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2\right)^p\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2\right)^{qp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\delta^2\right)^{nqp-(1+q+p+qp)}}}{\sqrt{nqp^{1+q+p+qp}\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\mu^2\right)\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\tau^2\right)^q\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_\gamma^2\right)^p\left(\frac{\sigma_\epsilon^2}{nqp}+\sigma_{(\tau\gamma)}^2\right)^{qp}(\sigma_\epsilon^2)^{nqp-(1+q+p+qp)}}} \exp\{-\frac{1}{2nqp^2} Y_{nqp}^T$$

$$(M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + C_{nqp}\right]^{-1} M_{nqp}^T - M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T) Y_{nqp}\}.$$

4- If we have the one- way repeated measurements model, then

$$\frac{1}{s_{nqp,1}} tr\left(\frac{1}{nqp^2} M_{nqp}\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + D_{nqp}\right]^{-1} M_{nqp}^T - [X\ Z]\left[\frac{\sigma_\epsilon^2}{nqp} I_{nqp} + G_{nqp}\right]^{-1} [X\ Z]^T\right) \xrightarrow{nqp\to\infty} 0$$

5-If the true model is the one-way repeated measurement model with fixed effects only then the Bayes factor is consistent under the null hypothesis H_0 . This implies that, $\lim_{nqp\to\infty} B_{01} = \infty$ in probability P_0^{nqp}.

6- If we have the mixed one- way repeated measurements model, then

$$\lim_{nqp\to\infty} \frac{1}{w_{nqp}} \log \frac{\det(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z\sum_1 Z^T)}{\det(\sigma_\epsilon^2 I_{nqp} + X\sum_0 X^T + Z_r\sum_{1,r} Z_r^T)} = 0.$$

7- If the true model is the mixed one- way repeated measurements model, then the Bayes factor is consistent under the alternative hypothesis H_1.This implies that, $\lim_{nqp\to\infty} B_{01} = 0$ in probability P_1^{nqp}.

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