On Z- generalized closed sets in topology

A. Zaghdani¹; A. I. EL-Maghrabi ² ⁴; M. Ezzat Mohamed ¹ ³ and A. M. Mubarki ⁴

¹Faculty of Arts and Science, Northern Border University, Rafha, P.O. 840, K. S. A.  
E-mail: hamido20042002@yahoo.fr  
²Department of Mathematics, Faculty of Science, Kafir EL-Sheikh University, Kafir EL-Sheikh, Egypt,  
E-mail: aelmaghribi@yahoo.com  
³Department of Mathematics, Faculty of Science, Fayoum University, Fayoum, Egypt,  
E-mail: mohaezzat@yahoo.com  
⁴Department of Mathematics, Faculty of Science, Taibah University, P.O. Box, 30002, Postal 41477, AL-Madinah AL-Munawarah, K.S.A.  
amaghribi@taibah.edu.sa; alimobarki@hotmail.com.

Abstract. In this paper, we introduce and study the notion of generalized Z-closed sets. Also, the notion of generalized Z-open sets and some of its basic properties are introduced discussed. Further, we introduce the notion of generalized Z-closed functions. Moreover, some characterizations and properties of it are investigated.

Keywords: gZ-closed sets, Z-T₁/₂-spaces, gZ-continuous and ZgZ-continuous functions.

1.Introduction and Preliminary. 
In 2011, EL-Maghrabi and Mubarki [12] introduced and studied the notion of Z-open sets. The class of g-closed sets was investigated by Aull [5]. Maki et al. [14] (resp. Fukutake et al. [17], Dontchev [7]) introduced the concept of gp-closed (resp. gγ-closed, gsp-closed) sets. In this paper, we define and study the notion gZ-closed sets and gZ-open sets which is stronger than the concept of gγ-closed and weaker than the concepts of gp-closed and Z-closed sets. Also, some characterizations of these concepts are discussed. Further, we introduce and study new forms of generalized Z-closed functions. Moreover, some properties of these new forms of generalized Z-closed functions and preservation theorems are discussed.

Throughout this paper (X, τ) and (Y, σ) (Simply, X and Y) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and X \ A denote the closure of A, the interior of A and the complement of A respectively. A point x ∈ X is called a δ-adoherent point of A [18] if 

A ∩ int(cl(V)) ≠ ∅, for every open set V containing x. The set of all δ-adoherent points of A is called the δ-closure of A and is denoted by clδ(A). A subset A of X is called δ-closed if A = clδ(A). The complement of δ-closed set is called δ-open. The δ-interior of set consists of those points x of A such that for some open set U containing x, U ⊆ int(cl(U)) ⊆ A and will be denoted by intδ(A).

Definition 1.1. A subset A of a space (X, τ) is called:
(1) α-open [16] if A ⊆ int(cl(int(A))),
(2) preopen [15] if A ⊆ int(cl(A)),
(3) Z-open [12] if A ⊆ cl(intδ(A)) ∪ int(cl(A)),
(4) b-open [3] or γ-open [10] or sp-open [8] if A ⊆ int(cl(A)) ∪ cl(int(A)),
(5) β-open [1] (= semi-preopen [2]) if A ⊆ cl(int(cl(A))).

The complement of α-open (resp. preopen, Z-open, γ-open, β-open or semi-preopen) sets is called α-closed [16] (resp. pre-closed, Z-closed, γ-closed, β-closed or semi-pre-closed). The intersection of all α-closed (resp. pre-closed, Z-closed, γ-closed, β-closed or semi-pre-closed) sets containing A is called the α-closure (resp. pre-closure, Z-closure, γ-closure, β-closure or semi-pre-closure) of A and denoted by α-cl(A) (resp. pcl(A), Z-cl(A), γ-cl(A), β-cl(A) or sp-cl(A)). The union of all α-open (resp. pre-open, Z-open, γ-open, β-open or semi-pre-open) sets contained in A is called the α-interior (resp. pre-interior, Z-interior, γ-interior, β-interior or semi-pre-interior) of A and denoted by α-int(A) (resp. pint(A), Z-int(A), γ-int(A), β-int(A) or sp-int(A)). The family of all Z-open (resp. Z-closed) sets in a space (X, τ) is denoted by ZO(X, τ) (resp. ZC(X, τ)).

Definition 1.2. A subset A of a space (X, τ) is called:
(1) generalized closed (= g-closed) set [5] if cl(A) ⊆ U whenever A ⊆ U and U is open,
(2) α-g-generalized closed (=αg-closed) set [6] if α-cl(A) ⊆ U whenever A ⊆ U and U is open,
(3) generalized pre-closed (= gp-closed) set [14] if pcl(A) ⊆ U whenever A ⊆ U and U is open,
(4) $\gamma$-generalized closed (=$\gamma g$-closed [17] or $g\gamma$-closed [9]) set if $\gamma \text{-cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open,
(5) generalized semi-pre-closed ($sgsp$-closed) set if $sp\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

The complement of $\gamma$-generalized closed (=$\gamma g$-closed) set is called $\gamma$-generalized open (=$\gamma g$-open).

2. Generalized $Z$-closed sets.
Definition 2.1. A subset $B$ of a topological space $(X, \tau)$ is called a generalized $Z$-closed (=$gZ$-closed) set if $Z\text{-cl}(B) \subseteq U$ whenever $B \subseteq U$ and $U$ is open in $(X, \tau)$.

The family of all generalized $Z$-closed sets of a space $X$ is denoted by $GZC(X)$.

Remark 2.2. The following diagram holds for any subset $A$ of $X$.

$$
\text{closed} \rightarrow \alpha\text{-closed} \rightarrow \text{pre-closed} \rightarrow Z\text{-closed} \rightarrow \gamma\text{-closed} \rightarrow \text{sp-closed} \\
\downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow
\text{g-closed} \rightarrow \alpha g\text{-closed} \rightarrow \text{gp-closed} \rightarrow gZ\text{-closed} \rightarrow \gamma g\text{-closed} \rightarrow gsp\text{-closed}
$$

None of these implications are reversible as is shown by [5, 6, 7, 9, 14, 17] and by the following examples.

Example 2.3. Let $X = \{a, b, c, d, e\}$ with topology $\tau = \{\emptyset, \{a,\}, \{b,\}, \{a, b,\}, \{a, c,\}, \{b, c,\}, \{a, b, c, d,\}, X\}$. Then:
(1) the subset $A=\{c, d\}$ of $X$ is a $gZ$-closed set but not $gp$-closed,
(2) the subset $B=\{a, c, d\}$ of $X$ is a $\gamma g$-closed set but not $gZ$-closed.

Example 2.4. If $X = \{a, b, c, d, e\}$ and $\tau = \{\emptyset, \{a,\}, X\}$, then a subset $A = \{a, c\}$ of $X$ is $gZ$-closed but it is not $Z$-closed.

Theorem 2.5. The arbitrary intersection of any $gZ$-closed subsets of $X$ is $gZ$-closed of $X$.

Proof. Let $\{A_i : i \in I\}$ be any collection of $gZ$-closed subsets of $X$ such that $\bigcap_{i \in I} A_i \subseteq H$ and $H$ be $Z$-open in $X$. Since, $A_i$ is a $gZ$-closed subset of $X$, for each $i \in I$, then $Z\text{-cl}(A_i) \subseteq H$, for each $i \in I$ this implies that $\bigcap_{i \in I} Z\text{-cl}(A_i) \subseteq H$, for each $i \in I$, hence, $Z\text{-cl}(\bigcap_{i \in I} A_i) \subseteq H$. Therefore, $\bigcap_{i \in I} A_i$ is $gZ$-closed of $X$.

Remark 2.6. The union of two $gZ$-closed subsets of $X$ need not be $gZ$-closed of $X$. Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{b, c\}, X\}$. Then two subsets $\{b\}, \{c\}$ of $X$ are $gZ$-closed subsets, but their union $\{b, c\}$ is not $gZ$-closed of $X$.

The following theorem is given the another definition of the concept $gZ$-closed.

Theorem 2.7. A subset $A$ of a space $(X, \tau)$ is $gZ$-closed if and only if, for each $A \subseteq H$ and $H$ is $Z$-open (resp. $\gamma$-open), there exists a $Z$-closed (resp. $\gamma$-closed) set $F$ of $X$ such that $A \subseteq F \subseteq H$.

Proof. We prove that this theorem for the case of $Z$-open. Suppose that $A$ is a $gZ$-closed subset of $X$, $A \subseteq H$ and $H$ is a $Z$-open set. Then $Z\text{-cl}(A) \subseteq H$. If we put $F = Z\text{-cl}(A)$, hence $A \subseteq F \subseteq H$.

Conversely. Assume that $A \subseteq H$ and $H$ is a $Z$-open set. Then by hypothesis, there exists a $Z$-closed set $F$ of $X$ such that $A \subseteq F \subseteq H$. So, $A \subseteq Z\text{-cl}(A) \subseteq F$ and hence $Z\text{-cl}(A) \subseteq H$. Therefore $A$ is $gZ$-closed.

Lemma 2.8. Let $A$ be a $\delta$-closed (resp. closed) and $B$ be a $Z$-closed set of $X$, then $A \cup B$ is $Z$-closed (resp. $\gamma$-closed).

Remark 2.9. The following example is shown that the union of a closed and a $Z$-closed set of $X$ is $\gamma$-closed but it is not $Z$-closed.

Let $X = \{a, b, c, d\}$ with topology $\tau = \{\emptyset, \{a,\}, \{c,\}, \{a, c,\}, \{a, b,\}, \{a, b, c,\}, \{a, c, d,\}, X\}$.

If $A = \{c\}$ is $Z$-closed and $B = \{b\}$ is closed, then $A \cup B = \{b, c\}$ is $\gamma$-closed and it is not $Z$-closed.
Theorem 2.10. If $A$ is a $\delta$-closed (resp. closed) and $B$ is a $gZ$-closed subset of a space $X$, then $A \cup B$ is also $gZ$-closed (resp. $g\gamma$-closed).

Proof. Suppose that $A \cup B \subseteq H$ and $H$ is a $Z$-open set. Then $A \subseteq H$ and $B \subseteq H$. But, $B$ is $gZ$-closed, then $Z-cl(B) \subseteq H$ and hence $A \cup B \subseteq A \cup Z-cl(B) \subseteq H$. But, $A \cup Z-cl(B)$ a $Z$-closed (resp. $\gamma$-closed) set. Hence, there exists a $Z$-closed set $A \cup Z-cl(B)$ of $X$ such that

$$A \cup B \subseteq A \cup Z-cl(B) \subseteq H.$$ Therefore by Theorem 2.7, $A \cup B$ is $gZ$-closed.

Theorem 2.11. For any element $p \in X$ of a space $X$, the set $X \setminus \{p\}$ is $gZ$-closed or $Z$-open.

Proof. Suppose that $X \setminus \{p\}$ is not a $Z$-open set. Then, $X$ is the only $Z$-open set containing $X \setminus \{p\}$. This implies that $Z-cl(X \setminus \{p\}) \subseteq X$. Hence, $X \setminus \{p\}$ is $gZ$-closed in $X$.

Proposition 2.12. If $A$ is a $gZ$-closed set of $X$ such that $A \subseteq B \subseteq Z-cl(A)$, then $B$ is $gZ$-closed in $X$.

Proof. Let $H$ be an open set of $X$ such that $B \subseteq H$. Then $A \subseteq H$. But, $A$ is a $gZ$-closed set of $X$, then $Z-cl(A) \subseteq H$. Now, $Z-cl(B) \subseteq Z-cl(Zcl(A)) = Z-cl(A) \subseteq H$. Therefore $B$ is $gZ$-closed in $X$.

Theorem 2.13. Let $A$ be a $gZ$-closed subset of $(X, \tau)$. Then $Z-cl(A) \setminus A$ does not contain any non-empty closed set of $X$.

Proof. Let $F$ be a closed subset of $Z-cl(A) \setminus A$. Since, $X \setminus F$ is open, $A \subseteq X \setminus F$ and $A$ is $gZ$-closed, it follows that $Z-cl(A) \subseteq X \setminus F$ and thus $F \subseteq X \setminus Z-cl(A)$. This implies that $F \subseteq (X \setminus Z-cl(A)) \cap (Z-cl(A) \setminus A) = \emptyset$ and hence $F = \emptyset$.

Corollary 2.14. A $gZ$-closed subset $A$ of a topological space $X$ is $Z$-closed if and only if $Z-cl(A) \setminus A$ is closed.

Proof. Let $A$ be a $gZ$-closed set of $X$. If $A$ is $Z$-closed, then, by Theorem 2.13, we have $Z-cl(A) \setminus A = \emptyset$ which is closed.

Conversely. Let $Z-cl(A) \setminus A$ be a closed set of $X$. Then, by Theorem 2.13, $Z-cl(A) \setminus A$ does not contain any non-empty closed subset of $X$. Since $Z-cl(A) \setminus A$ is closed, then $Z-cl(A) \setminus A = \emptyset$. This implies that $A = Z-cl(A)$ and so, $A$ is $Z$-closed.

Corollary 2.15. If $A$ is an open and a $gZ$-closed sets of $X$, then $A$ is $gZ$-closed in $X$.

Proof. Let $H$ be any open set of $X$ such that $A \subseteq H$. Since, $A$ is an open and a $gZ$-closed sets of $X$, then $Z-cl(A) \subseteq A$. Then, $Z-cl(A) \subseteq A \subseteq H$. Hence, $A$ is $gZ$-closed in $X$.

Proposition 2.16. If $A$ is both an open and a $gZ$-closed subsets of a topological space $(X, \tau)$, then $A$ is $Z$-closed.

Proof. Assume that $A$ is both an open and a $gZ$-closed subsets of a topological space $(X, \tau)$. Then $Z-cl(A) \subseteq A$. Hence, $A$ is $Z$-closed.

Theorem 2.17. If $A$ is both an open and a $gZ$-closed subsets of $X$ and $F$ is a $\delta$-closed (resp. closed) set of $X$, then $A \cap F$ is $gZ$-closed (resp. $g\gamma$-closed) in $X$.

Proof. Let $A$ be an open and a $gZ$-closed subsets of $X$ and $F$ be a $\delta$-closed (closed) set in $X$. Then by Proposition 2.16, $A$ is $Z$-closed. So, $A \cap F$ is $Z$-closed (resp. $\gamma$-closed). Therefore, $A \cap F$ is a $gZ$-closed (resp. $g\gamma$-closed) set of $X$.

Proposition 2.18. If $A$ is a $\delta$-open (resp. an open) set and $H$ is a $Z$-open set of a topological space $(X, \tau)$, then $A \cap H$ is $Z$-open (resp. $\gamma$-open) in $X$.

Proof. Obvious from Theorem 2.17.

Proposition 2.19. If $A$ is both an open and a $g$-closed subsets of $A$, then $A$ is $gZ$-closed in $X$.

Proof. Let $A$ be an open and a $g$-closed subsets of $X$ and $A \subseteq H$, where $H$ is an open set of $X$.

Then by hypothesis, $Z-cl(A) \subseteq cl(A) \subseteq A$, that is, $Z-cl(A) \subseteq H$. Thus $A$ is $gZ$-closed in $X$. 177
Theorem 2.20. For a topological space \((X, \tau)\), then \(ZO(X, \tau) \subseteq \{F \subseteq X : F \text{ is closed} \}\) if and only if every subset of \(X\) is \(gZ\)-closed of \(X\).

**Proof.** Suppose that \(ZO(X, \tau) \subseteq \{F \subseteq X : F \text{ is closed} \}\). Let \(A\) be any subset of \(X\) such that \(A \subseteq H\), where \(H\) is a \(Z\)-open set of \(X\). Then \(H \in ZO(X, \tau) \subseteq \{F \subseteq X : F \text{ is closed} \}\), that is, \(H \in \{F \subseteq X : F \text{ is closed} \}\). Thus \(H\) is \(Z\)-closed. Then, \(Z-cl(H) = H\). Also, \(Z-cl(A) \subseteq Z-cl(H) \subseteq H\). Hence, \(A\) is a \(gZ\)-closed of \(X\).

Conversely, suppose that every subset of \(X\) is \(gZ\)-closed in \(X\). Let \(H \in ZO(X, \tau)\). Since, \(H \subseteq H\) and \(H\) is \(gZ\)-closed, then \(Z-cl(H) \subseteq H\). Thus, \(Z-cl(H) = H\) and hence, \(H \in \{F \subseteq X : F \text{ is closed} \}\). Therefore, \(ZO(X, \tau) \subseteq \{F \subseteq X : F \text{ is closed} \}\).

Definition 2.21. The intersection of all \(Z\)-open subsets of \((X, \tau)\) containing \(A\) is called the \(Z\)-kernel of \(A\) and is denoted by \(Z-ker(A)\).

Lemma 2.22. For any subset \(A\) of a topological space \((X, \tau)\), then \(A \subseteq Z-ker(A)\).

**Proof.** Follows directly from Definition 2.21.

Lemma 2.23. Let \((X, \tau)\) be a topological space and \(A\) be a subset of \(X\). If \(A\) is a \(Z\)-open set of \(X\), then \(Z-ker(A) = A\).

Theorem 2.24. A subset \(A\) of a topological space \(X\) is \(gZ\)-closed if and only if \(Z-cl(A) \subseteq Z-ker(A)\).

**Proof.** Since, \(A\) is a \(gZ\)-closed set of \(X\), \(Z-cl(A) \subseteq G\), for any open set \(G\) with \(A \subseteq G\). Hence \(Z-cl(A) \subseteq Z-ker(A)\). Conversely, let \(G\) be any open set such that \(A \subseteq G\). Then by hypothesis, \(Z-cl(A) \subseteq Z-ker(A) \subseteq G\). So, \(A\) is \(gZ\)-closed.

3. Some properties of generalized \(Z\)-open sets.

Definition 3.1. A subset \(A\) of a topological space \((X, \tau)\) is called a generalized \(Z\)-open (briefly, \(gZ\)-open) set of \(X\) if \(X \setminus A\) is \(gZ\)-closed in \(X\). We denote the family of all \(gZ\)-open sets of \(X\) by \(GZO(X)\).

Theorem 3.2. Let \((X, \tau)\) be a topological space and \(A \subseteq X\). Then the following statements are equivalent:

1. \(A\) is a \(gZ\)-open set,
2. for each closed set \(F\) contained in \(A\), \(F \subseteq Z-int(A)\),
3. for each closed set \(F\) contained in \(A\), there exists a \(Z\)-open set \(H\) such that \(F \subseteq H \subseteq A\).

**Proof.** (1) \(\rightarrow\) (2). Let \(F \subseteq A\) and \(F\) be a \(Z\)-closed set. Then \(X \setminus A \subseteq X \setminus F\) which is \(Z\)-open of \(X\), hence, \(Z-cl(X \setminus A) \subseteq X \setminus F\). So, \(F \subseteq Z-int(A)\).

(2) \(\rightarrow\) (3). Suppose that \(F \subseteq A\) and \(F\) be a \(Z\)-closed set. Then by hypothesis, \(F \subseteq Z-int(A)\). But, \(H = Z-int(A)\), hence there exists a \(Z\)-open set \(H\) such that \(F \subseteq H \subseteq A\).

(3) \(\rightarrow\) (1). Assume that \(X \setminus A \subseteq V\) and \(V\) is a \(Z\)-open set of \(X\). Then by hypothesis, there exists a \(Z\)-open set \(H\) such that \(X \setminus V \subseteq H \subseteq A\), that is, \(X \setminus A \subseteq X \setminus H \subseteq V\). Therefore, by Theorem 2.7, \(X \setminus A\) is \(gZ\)-closed in \(X\). Then, \(A\) is \(gZ\)-open in \(X\).

Theorem 3.3. If \(A\) is an \(\delta\)-open (resp. \(\gamma\)-open) and \(B\) is a \(gZ\)-open (resp. \(g\gamma\)-open) subset of a space \(X\), then \(A \cap B\) is a \(gZ\)-open (resp. \(g\gamma\)-open).

**Proof.** Follows from Theorem 2.10.

Proposition 3.4. If \(Z-int(A) \subseteq B \subseteq A\) and \(A\) is a \(gZ\)-open set of \(X\), then \(B\) is \(gZ\)-open.
Proposition 3.5. Let \( A \) be a \( Z \)-closed and a \( gZ \)-open sets of \( X \). Then \( A \) is \( Z \)-open.

Proof. Let \( A \) be a \( Z \)-closed and a \( gZ \)-open sets of \( X \). Then \( A \subseteq Z\text{-int}(A) \) and hence \( A \) is \( Z \)-open.

Theorem 3.6. For a space \((X, \tau)\), if \( A \) is a \( gZ \)-closed set of \( X \), then \( Z\text{-cl}(A) \setminus A \) is \( gZ \)-open.

Proof. Suppose that \( A \) is a \( gZ \)-closed set of \( X \) and \( F \) is a \( Z \)-closed set contained in \( Z\text{-cl}(A) \setminus A \). Then by Theorem 2.7, \( F = \emptyset \) and hence \( F \subseteq Z\text{-int}(Z\text{-cl}(A) \setminus A) \). Therefore, \( Z\text{-cl}(A) \setminus A \) is \( gZ \)-open.

Theorem 3.7. If \( A \) is a \( gZ \)-open subset of a space \((X, \tau)\), then \( G = X \), whenever \( G \) is open and \( Z\text{-int}(A) \cup (X \setminus A) \subseteq G \).

Proof. Let \( G \) be an open set of \( X \) and \( Z\text{-int}(A) \cup (X \setminus A) \subseteq G \). Then \( X \setminus G \subseteq (X \setminus Z\text{-int}(A)) \cap A = Z\text{-cl}(X \setminus A) \setminus (X \setminus A) \). Since, \( X \setminus G \) is closed and \( X \setminus A \) is \( gZ \)-closed, by Theorem 2.13, \( X \setminus G = \emptyset \) and hence \( G = X \).

Theorem 3.8. For a topological space \((X, \tau)\), then every singleton of \( X \) is either \( gZ \)-open or \( Z \)-open.

Proof. Let \((X, \tau)\) be a topological space and \( p \in X \). To prove that \( \{ p \} \) is either \( gZ \)-open or \( Z \)-open, that is, to prove \( X \setminus \{ p \} \) is either \( gZ \)-closed or \( Z \)-open which follows directly from Theorem 2.13.

4. **Z-T\(_{1/2} \)** spaces and generalized Z-continuous functions.

Definition 4.1. A space \((X, \tau)\) is called a \( Z\text{-T}_{1/2} \)-space if every \( gZ \)-closed set is \( Z \)-closed.

Theorem 4.2. For a topological space \((X, \tau)\), the following conditions are equivalent:

1. \( X \) is \( Z\text{-T}_{1/2} \).
2. Every singleton of \( X \) is either closed or \( Z \)-open.

Proof. (1) \(\Rightarrow\) (2). Let \( p \in X \) and \( \{ p \} \) be not closed. Then \( X \setminus \{ p \} \) is not open and hence \( X \setminus \{ p \} \) is \( gZ \)-closed. Hence, by hypothesis, \( X \setminus \{ p \} \) is \( Z \)-closed and thus \( \{ p \} \) is \( Z \)-open.

(2) \(\Rightarrow\) (1). Let \( A \subseteq X \) be a \( gZ \)-closed set of \( X \) and \( p \in \text{cl}(A) \). We will show that \( p \in A \). For consider the following two cases:

Case (1). The singleton set \( \{ p \} \) is closed. Then, if \( p \not\in A \), then there exists a closed set of \( \text{cl}(A) \setminus A \). Hence, by Corollary 2.14, \( p \in A \).

Case (2). The singleton set \( \{ p \} \) is \( Z \)-open. Since \( p \in \text{cl}(A) \), then \( \{ p \} \cap \text{cl}(A) \neq \emptyset \). Thus \( p \in A \). So, in both cases, \( p \in A \). This shows that \( \text{cl}(A) \subseteq A \) or equivalently, \( A \) is \( Z \)-closed.

Theorem 4.3. For a topological space \((X, \tau)\), the following statements hold:

1. \( ZO(X, \tau) \subseteq \text{GZO}(X, \tau) \).
2. A space \( X \) is \( Z\text{-T}_{1/2} \) if and only if \( ZO(X, \tau) = \text{GZO}(X, \tau) \).

Proof. (1) Let \( A \) be a \( Z \)-open set. Then \( X \setminus A \) is \( Z \)-closed and so \( gZ \)-closed. This implies that \( A \) is \( gZ \)-open. Hence \( ZO(X, \tau) \subseteq \text{GZO}(X, \tau) \).

(2) The necessity. Let \((X, \tau)\) be a \( Z\text{-T}_{1/2} \) space and let \( A \in \text{GZO}(X, \tau) \). Then \( X \setminus A \) is \( gZ \)-closed. Hence by hypothesis, \( X \setminus A \) is \( Z \)-closed and thus \( A \) is \( Z \)-open this implies that \( A \in \text{ZO}(X, \tau) \). Hence, \( ZO(X, \tau) = \text{GZO}(X, \tau) \).
The sufficiency. Let $ZO(X, \tau) = GZO(X, \tau)$ and let $A$ be a $gZ$-closed set. Then $X \setminus A$ is $gZ$-open. Hence, $X \setminus A \in ZO(X, \tau)$. Thus $A$ is $Z$-closed. Therefore $(X, \tau)$ is $Z_{-T_{1/2}}$.

**Definition 4.4.** A function $f : X \to Y$ is called:

1. $gZ$-continuous if, $f^{-1}(F)$ is $gZ$-closed in $X$, for every closed set $F$ of $Y$,
2. $Z$-$gZ$-continuous if, $f^{-1}(F)$ is $gZ$-closed in $X$, for every $Z$-closed set $F$ of $Y$,
3. $gZ$-irresolute if, $f^{-1}(F)$ is $gZ$-closed in $X$, for every $gZ$-closed set $F$ of $Y$.

**Definition 4.5.** A function $f : X \to Y$ is called:

1. $\gamma g$-continuous [17] if, $f^{-1}(F)$ is $\gamma g$-closed in $X$, for every closed set $F$ of $Y$,
2. $\gamma g$-$g\gamma$-continuous [9] if, $f^{-1}(F)$ is $g\gamma$-closed in $X$, for every $\gamma g$-closed set $F$ of $Y$,
3. $\gamma g$-irresolute [17] if, $f^{-1}(F)$ is $g\gamma$-closed in $X$, for every $g\gamma$-closed set $F$ of $Y$,
4. $Z$-$gZ$-continuous [12] if, $f^{-1}(F)$ is $Z$-closed in $X$, for every closed set $F$ of $Y$,

**Remark 4.6.** The following diagram holds for a function $f : (X, \tau) \to (Y, \sigma)$:

$$
\gamma g \text{-irresoluteness} \rightarrow \gamma g \text{-continuity} \rightarrow \gamma g \text{-continuity}
$$

$$
gZ \text{-irresoluteness} \rightarrow gZ \text{-continuity} \rightarrow gZ \text{-continuity}
$$

$$
Z \text{-irresoluteness} \rightarrow Z \text{-continuity}
$$

The converses of the above implications are not true in general as is shown by [9] and the following example.

**Example 4.7.** In Example 2.3, Let $f : (X, \tau) \to (X, \tau)$ be a function defined by $f(a) = a$, $f(b) = b$, $f(c) = c$, $f(d) = c$ and $f(e) = d$. Then $f$ is $\gamma g$-$g\gamma$-continuous (resp. $\gamma g$-continuous) but it is not $ZgZ$-continuous (resp. $gZ$-continuous).

**Example 4.8.** Let $X = \{a, b, c, d\}$ and $\tau = \sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be a function defined by $f(a) = a$, $f(b) = c$, $f(c) = b$ and $f(d) = d$ Then $f$ is $gZ$-continuous but it is neither $ZgZ$-continuous nor $Z$-continuous.

If we define the function $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$ and $f(d) = b$, then $f$ is $ZgZ$-continuous but it is neither $ZgZ$-irresolute nor $Z$-irresolute.

**Example 4.9.** Let $X = Y = \{a, b, c, d\}$ with $\tau = \sigma = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$. Then $f : (X, \tau) \to (X, \tau)$ be defined by $f(a) = b$, $f(b) = d$, $f(c) = a$ and $f(d) = d$ is $Z$-continuous but it is not $Z$-irresolute.

**Theorem 4.10.** Let $f : X \to Y$ and $h : Y \to Z$ be functions.

1. If, $f$ is $gZ$-irresolute and $h$ is $gZ$-continuous, then the composition $h \circ f : X \to Z$ is $gZ$-continuous.
2. If, $f$ is $gZ$-continuous and $h$ is continuous, then the composition $h \circ f : X \to Z$ is $gZ$-continuous.
3. If, $f$ and $h$ are $gZ$-irresolute, then the composition $h \circ f : X \to Z$ is $gZ$-irresolute.
4. If, $f$ is $gZ$-irresolute and $h$ is $ZgZ$-continuous, then the composition $h \circ f : X \to Z$ is $ZgZ$-continuous.
5. If, $f$ and $h$ are $ZgZ$-continuous and $Y$ is $Z-T_{1/2}$, then the composition $h \circ f : X \to Z$ is $ZgZ$-continuous.

**Theorem 4.11.** If a function $f : X \to Y$ is $ZgZ$-continuous and $Y$ is a $Z-T_{1/2}$ space, then $f$ is $gZ$-irresolute.

**Proof.** Let $F$ be any $gZ$-closed subset of $Y$. Since, $Y$ is a $Z-T_{1/2}$ space, then $F$ is $Z$-closed in $Y$. Hence, $f^{-1}(F)$ is $Z$-closed in $X$. This show that $f$ is $gZ$-irresolute.

**Theorem 4.12.** If a function $f : X \to Y$ is $gZ$-continuous and $X$ is a $Z-T_{1/2}$ space, then $f$ is $Z$-continuous.

**Proof.** Let $F$ be any closed set of $Y$ and $f$ be $gZ$-continuous. Then, $f^{-1}(F)$ is $gZ$-closed in $X$ and hence, $f^{-1}(F)$ is $Z$-closed in $X$. Therefore, $f$ is $Z$-continuous.

**Theorem 4.13.** If a function $f : X \to Y$ is $ZgZ$-continuous and $X$ a $Z-T_{1/2}$ space, then $f$ is $ZgZ$-continuous.

**Proof.** Let $F$ be any $Z$-closed set of $Y$ and $f$ be $ZgZ$-continuous. Then, $f^{-1}(F)$ is $gZ$-closed in $X$ and hence, $f^{-1}(F)$ is $Z$-closed in $X$. Hence, $f$ is $Z$-irresolute.

**Definition 4.14.** A function $f : X \to Y$ is said to be:

1. $gZ$-closed if $f(A)$ is $gZ$-closed in $Y$, for each closed set $A$ of $X$.
2. $ZgZ$-closed if, $f(A)$ is $gZ$-closed in $Y$, for each $Z$-closed set $A$ of $X$.

**Theorem 4.15.** If, $f : X \to Y$ is a closed and a $ZgZ$-continuous functions, then $f^{-1}(K)$ is $gZ$-closed in $X$, for each $gZ$-closed set $K$ of $Y$.

**Proof.** Let $K$ be a $gZ$-closed set of $Y$ and $U$ be an open set of $X$ containing $f^{-1}(K)$. Put,
V = Y \setminus (X \setminus U)$, then $V$ is open in $Y$, $K \subseteq V$ and $f^{-1}(V) \subseteq U$. Therefore, we have $Z\text{-cl}(K) \subseteq V$ and hence $f^{-1}(K) \subseteq f^{-1}(Z\text{-cl}(K)) \subseteq f^{-1}(V) \subseteq U$. Since, $f$ is $Z\text{-gZ}$-continuous, then $f^{-1}(Z\text{-cl}(K))$ is $gZ$-closed in $X$ and hence $Z\text{-cl}(f^{-1}(K)) \subseteq Z\text{-cl}(f^{-1}(Z\text{-cl}(K))) \subseteq U$. This shows that $f^{-1}(K)$ is $gZ$-closed in $X$.

**Corollary 4.16.** If, $f : X \to Y$ is a closed and a $Z$-irresolute functions, then $f^{-1}(K)$ is $gZ$-closed in $X$, for each $gZ$-closed set $K$ of $Y$.

**Theorem 4.17.** If, $f : X \to Y$ is a bijective open and a $Z\text{-gZ}$-continuous functions, then $f^{-1}(K)$ is $gZ$-closed in $X$, for every $gZ$-closed set $K$ of $Y$.

**Proof.** Let $K$ be a $gZ$-closed set of $Y$ and $U$ be an open set of $X$ containing $f^{-1}(K)$. Since, $f$ is a surjective open function, then $K = f(f^{-1}(K)) \subseteq f(U)$ and $f(U)$ is open. Therefore, $Z\text{-cl}(K) \subseteq f(U)$. But, $f$ is an injective, hence $f^{-1}(K) \subseteq f^{-1}(Z\text{-cl}(K)) \subseteq f^{-1}(f(U)) = U$. Since, $f$ is $Z\text{-gZ}$-continuous, then $f^{-1}(Z\text{-cl}(K))$ is $gZ$-closed in $X$ and hence $Z\text{-cl}(f^{-1}(K)) \subseteq Z\text{-cl}(f^{-1}(Z\text{-cl}(K))) \subseteq U$. Therefore $f^{-1}(K)$ is $gZ$-closed in $X$.

**Definition 4.18.** A space $X$ is said to be $Z$-normal if, for any pair of disjoint closed sets $A$, $B$, there exist two disjoint $Z$-open sets $U$, $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Theorem 4.19.** Let $f : X \to Y$ be an injection closed and a $Z\text{-gZ}$-continuous functions. If $Y$ is a $Z$-normal space, then $X$ is $Z$-normal.

**Proof.** Let $N_1$, $N_2$ be disjoint closed sets of $X$ and $f$ be an injection closed function, then $f(N_1)$, $f(N_2)$ are disjoint closed sets of $Y$. Hence by the $Z$-normality of $Y$, there exist disjoint $V_i$, $V_j \in ZO(Y)$ such that $f(N_i) \subseteq V_i$, for $i = 1, 2$. Since, $f$ is $Z\text{-gZ}$-continuous, hence, $f^{-1}(V_i)$, $f^{-1}(V_j)$ are disjoint $Z$-open sets of $X$ and $N_i \subseteq f^{-1}(V_i)$, for $i = 1, 2$. Now, put $U_i = Z\text{-int}(f^{-1}(V_i))$ for $i = 1, 2$. Then, $U_i \in ZO(X)$, $U_i \subseteq U_i$, $U_i \cap U_j = \emptyset$. Therefore, $X$ is $Z$-normal.

**Corollary 4.20.** If, $f : X \to Y$ is an injection closed and a $Z$-irresolute functions and $Y$ is a $Z$-normal space, then $X$ is $Z$-normal.

**Proof.** This is an immediate consequence since every $Z$-irresolute function is $Z\text{-gZ}$-continuous.

**Lemma 4.21.** A surjection function $f : X \to Y$ is $Z\text{-gZ}$-closed if and only if, for each subset $B$ of $Y$ and each $Z$-open set $U$ of $Y$ containing $f^{-1}(B)$, there exists a $Z$-open set of $X$ such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

**Theorem 4.22.** If, $f : X \to Y$ is a surjection continuous and a $Z\text{-gZ}$-closed functions and $X$ is a $Z$-normal space, then $Y$ is $Z$-normal.

**Proof.** Let $A$, $B$ be any disjoint closed sets of $X$. Then $f^{-1}(A)$, $f^{-1}(B)$ are disjoint closed sets of $X$. Since, $X$ is a $Z$-normal space, hence there exist two disjoint $Z$-open sets $U$, $V$ such that $A \subseteq U$ and $f^{-1}(B) \subseteq V$. Hence by Lemma 4.21, there exist two $Z$-open sets $G$, $H$ of $Y$ such that $A \subseteq G$, $B \subseteq H$, $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since, $U$, $V$ are disjoint, then $G$, $H$ are also disjoint. Hence by Theorem 3.2, we have $A \subseteq Z\text{-int}(G)$, $B \subseteq Z\text{-int}(H)$ and $Z\text{-int}(G) \cap Z\text{-int}(H) = \emptyset$. Therefore, $Y$ is $Z$-normal.

**Acknowledgments.**

This paper was supported by Deanship of Scientific Research of Northern Border University, Under grant ( ).

**References.**


The IISTE is a pioneer in the Open-Access hosting service and academic event management. The aim of the firm is Accelerating Global Knowledge Sharing.

More information about the firm can be found on the homepage: http://www.iiste.org

**CALL FOR JOURNAL PAPERS**

There are more than 30 peer-reviewed academic journals hosted under the hosting platform.

**Prospective authors of journals can find the submission instruction on the following page:** [http://www.iiste.org/journals/](http://www.iiste.org/journals/) All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Paper version of the journals is also available upon request of readers and authors.

**MORE RESOURCES**


**IISTE Knowledge Sharing Partners**

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar