

On Z- generalized closed sets in topology

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Abstract. In this paper, we introduce and study the notion of generalized Z-closed sets. Also, the notion of generalized Z-open sets and some of its basic properties are introduced discussed. Further, we introduce the notion of generalized Z-closed functions. Moreover, some characterizations and properties of it are investigated.
Keywords: gZ-closed sets, Z-$T_{1/2}$-spaces, gZ-continuous and ZgZ-continuous functions.

1.Introduction and Preliminaers.

In 2011, EL-Magharabi and Mubarki [12] introduced and studied the notion of Z-open sets. The class of g-closed sets was investigated by Aull [5]. Maki et.al [14] (resp. Fukutake et.al [17], Dontchev [7]) introduced the concept of gp-closed (resp. γg-closed, gsp-closed) sets. In this paper, we define and study the notion gZ-closed sets and gZ-open sets which is stronger than the concept of γg-closed and weaker than the concepts of gp-closed and Z-closed sets. Also, some characterizations of these concepts are discussed. Further, we introduce and study new forms of generalized Z-closed functions. Moreover, some properties of these new forms of generalized Z-closed functions and preservation theorems are discussed.

Throughout this paper (X, τ) and (Y, σ) (Simply, X and Y) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space (X, τ), cl(A), int(A) and $X \setminus A$ denote the closure of A, the interior of A and the complement of A respectively. A point $x \in X$ is called a δ-adherent point of A [18] if

$A \cap \text{int}(\text{cl}(V)) \neq \emptyset$, for every open set V containing x. The set of all δ-adherent points of A is called the δ-closure of A and is denoted by $\text{cl}_\delta(A)$. A subset A of X is called δ-closed if $A= \text{cl}_\delta(A)$). The complement of δ-closed set is called δ-open. The δ-interior of set consists of those points x of A such that for some open set U containing x, $U\subseteq \text{int}(\text{cl}(U)) \subseteq A$ and will be denoted by $\text{int}_\delta(A)$.

Definition 1.1. A subset A of a space (X, τ) is called:

(1) α-open [16] if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$,

(2) preopen [15] if $A \subseteq \text{int}(\text{cl}(A))$,

(3) Z-open [12] if $A \subseteq \text{cl}(\text{int}_\delta(A)) \cup \text{int}(\text{cl}(A)))$,

(4) b-open [3] or γ-open [10] or sp-open [8] if $A \subseteq \text{int}(\text{cl}(A)) \cup \text{cl}(\text{int}(A))$,

(5) β-open [1] (= semi-preopen [2]), if $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$.

The complement of α-open (resp. preopen, Z-open, γ-open, β-open or semi-preopen) sets is called α-closed [16] (resp. pre-closed, Z-closed, γ-closed, β-closed). The intersection of all α-closed (resp. pre-closed, Z-closed, γ-closed, β-closed or semi-preclosed) sets containing A is called the α-closure (resp. pre-closure, Z-closure, γ-closure, β-closure or semi-preclosure) of A and denoted by α-cl(A) (resp. pcl(A), Z-cl(A), γ-cl(A), β-cl(A) or sp-cl(A)). The union of all α-open (resp. preopen,Z-open, γ-open, β-open or semi-preopen) sets contained in A is called the α-interior (resp. pre-interior, Z-interior, γ-interior, β-interior or semi-pre-interior) of A and denoted by α-int(A) (resp. pint(A),Z-int(A),γ-int(A), β-int(A) or sp-int(A)). The family of all Z-open (resp. Z-closed) sets in a space (X, τ) is denoted by $ZO(X, \tau)$ (resp. $ZC(X, \tau)$).

Definition 1.2. A subset A of a space(X, τ) is called:
(1) generalized closed (= g-closed)set [5] if $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
(2) α-generalized closed (=αg-closed) set [6] if $\alpha\text{-cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open,
(3) generalized pre-closed (= gp-closed) set [14] if $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open,

(4) γ-generalized closed (= γg-closed [17] or gγ-closed [9]) set if γ-cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open,
(5) generalized semi-pre-closed (=gsp-closed) set [7] if sp-cl(A)$\subseteq$U whenever A $\subseteq$U and U is open.

The complement of γ- generalized closed (=γg-closed) set is called γ-generalized open (= γg-open).

2. Generalized Z-closed sets.

Definition 2.1. A subset B of a topological space (X,τ) is called a generalized Z-closed (= gZ-closed) set if Z-cl(B) $\subseteq$ U whenever B $\subseteq$ U and U is open in (X, τ).

The family of all generalized Z-closed sets of a space X is denoted by GZC(X).

Remark 2.2. The following diagram holds for any a subset A of X.

$$\begin{array}{ccccccccccc}
\text{closed} & \to & \alpha\text{-closed} & \to & \text{pre-closed} & \to & \text{Z-closed} & \to & \gamma\text{-closed} & \to & \text{sp-closed} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{g-closed} & \to & \alpha\text{g-closed} & \to & \text{gp-closed} & \to & \text{gZ-closed} & \to & \gamma\text{g-closed} & \to & \text{gsp-closed}
\end{array}$$

None of these implications are reversible as is shown by[5, 6, 7, 9, 14, 17] and by the following examples.

Example 2.3. Let X = {a, b, c, d, e} with topology τ = {∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}, {a, b, c, d}, X}. Then:

(1) the subset A={c, d}of X is a gZ-closed set but not gp-closed,

(2) the subset B ={a, c, d} of X is a γg-closed set but not gZ-closed.

Example 2.4. If X = {a, b, c, d, e} and τ = {∅, {a}, X}, then a subset A = {a, c} of X is gZ-closed but it is not Z-closed.

Theorem 2.5. The arbitrary intersection of any gZ-closed subsets of X is gZ-closed of X.

Proof. Let $\{A_i : i \in I\}$ be any collection of gZ-closed subsets of X such that $\bigcap_{i=1} A_i \subseteq H$ and H be Z-open in X Since, A_i is a gZ-closed subset of X, for each $i \in I$, then Z-cl(A_i) $\subseteq$ H, for each $i \in I$ this implies that $\bigcap_{i=1}$ Z-cl(A_i) $\subseteq$ H, for each $i \in I$, hence, Z-cl($\bigcap_{i=1} A_i$) $\subseteq$ H. Therefore, $\bigcap_{i=1} A_i$ is gZ-closed of X.

Remark 2.6. The union of two gZ-closed subsets of X need not be gZ-closed of X. Let

X = {a, b, c} with topology τ = {∅, {b, c}, X}. Then two subsets {b} ,{c} of X are gZ-closed subsets , but their union {b, c} is not gZ-closed of X.

The following theorem is given the another definition of the concept gZ-closed.

Theorem 2.7. A subset A of a space (X, τ) is gZ-closed if and only if, for each A $\subseteq$ H and H is

Z-open (resp.γ-open), there exists a Z-closed (resp. γ-closed) set F of X such that A $\subseteq$ F $\subseteq$ H.

Proof. We prove that this theorem for the case of Z-open. Suppose that A is a gZ-closed subset of X, A $\subseteq$ H and H is a Z-open set. Then Z-cl(A) $\subseteq$ H. If we put F = Z-cl(A) , hence A $\subseteq$ F $\subseteq$ H.

Conversely. Assume that A $\subseteq$ H and H is a Z-open set. Then by hypothesis, there exists a Z-closed set F of X such that A $\subseteq$ F $\subseteq$ H. So, A $\subseteq$ Z-cl(A) $\subseteq$ F and hence Z-cl(A) $\subseteq$ H. Therefore A is gZ-closed.

Lemma 2.8. Let A be a δ-closed (resp. closed) and B be a Z-closed set of X, then A $\cup$ B is

Z-closed (resp. γ-closed).

Remark. 2.9. The following example is shown that the union of a closed and a Z-closed set of X is γ-closed but it is not Z-closed.

Let X = {a, b, c, d} with topology τ = {∅, {a}, {c}, {a, c}, {a, b}, {a, b, c}, {a, c, d}, X}.

If A = {c} is Z-closed and B = {b} is closed, then A $\cup$ B = {b, c} is γ-closed and it is not Z-closed.

Theorem 2.10. If A is a δ-closed (resp. closed) and B is a gZ-closed subset of a space X, then $A \cup B$ is also gZ-closed (resp. γg-closed).

Proof. Suppose that $A \cup B \subseteq H$ and H is a Z-open set. Then $A \subseteq H$ and $B \subseteq H$. But, B is gZ-closed, then Z-cl(B) $\subseteq$ H and hence $A \cup B \subseteq A \cup$ Z-cl(B) $\subseteq$ H. But, $A \cup$ Z-cl(B) a Z-closed (resp. γ-closed) set. Hence, there exists a Z-closed set $A \cup$ Z-cl(B) of X such that

$A \cup B \subseteq A \cup$ Z-cl(B) $\subseteq$ H. Therefore by Theorem 2.7, $A \cup B$ is gZ-closed.

Theorem 2.11. For any element $p \in X$ of a space X , the set $X \setminus \{p\}$ is gZ-closed or Z-open.

Proof. Suppose that $X \setminus \{p\}$ is not a Z-open set. Then, X is the only Z-open set containing $X / \{p\}$. This implies that Z-cl($X \setminus \{p\}$) $\subseteq$ X . Hence, $X \setminus \{p\}$ is gZ-closed in X.

Proposition 2.12. If A is a gZ-closed set of X such that $A \subseteq B \subseteq$ Z-cl(A), then B is gZ-closed in X.

Proof. Let H be an open set of X such that $B \subseteq H$. Then $A \subseteq H$. But, A is a gZ-closed set of X, then ,Z-cl(A) $\subseteq$ H. Now, Z-cl(B) $\subseteq$ Z-cl(Zcl(A))= Z-cl(A) $\subseteq$ H. Therefore B is gZ-closed in X.

Theorem 2.13. Let A be a gZ-closed subset of (X, τ). Then Z-cl(A) \ A does not contain no non-empty closed set of X.

Proof. Let F be a closed subset of Z-cl(A) \ A. Since, X \ F is open, $A \subseteq X \setminus F$ and A is gZ-closed, it follows that Z-cl(A) $\subseteq$ X \ F and thus $F \subseteq$ X \ Z-cl(A). This implies that $F \subseteq$ (X \ Z-cl(A)) $\cap$ (Z-cl(A) \ A) = $\emptyset$ and hence F = $\emptyset$.

Corollary 2.14. A gZ-closed subset A of a topological space X is Z-closed if and only if Z-cl(A) \ A is closed.

Proof. Let A be a gZ-closed set of X. If A is Z-closed, then, by Theorem 2.13, we have Z-cl(A) \ A = $\emptyset$ which is closed.

Conversely. Let Z-cl(A) \ A be a closed set of X . Then, by Theorem 2.13, Z-cl(A) \ A does not contain any non-empty closed subset set of X. Since Z-cl(A) \ A is closed, then Z-cl(A) \ A = $\emptyset$. This implies that A = Z-cl(A) and so, A is Z-closed.

Corollary 2.15. If A is an open and a gZ-closed sets of X, then A is gZ-closed in X.

Proof . Let H be any open set of X such that $A \subseteq H$. Since, A is an open and a gZ- closed sets of

X, then Z-cl(A) $\subseteq$ A. Then, Z-cl(A) $\subseteq A \subseteq H$. Hence, A is gZ-closed in X.

Proposition 2.16. If A is both an open and a gZ-closed subsets of a topological space (X, τ), then A is Z-closed.

Proof . Assume that A is both an open and a gZ-closed subsets of a topological space (X, τ). Then Z-cl(A) $\subseteq$ A. Hence, A is Z-closed.

Theorem 2.17. If A is both an open and a gZ-closed subsets of X and F is a δ-closed (resp. closed) set of X, then $A \cap F$ is gZ-closed (resp. γg-closed) in X.

Proof . Let A be an open and a gZ-closed subsets of X and F be a δ-closed (closed) set in X. Then by Proposition 2.16, A is Z-closed. So, $A \cap F$ is Z-closed (resp. γ-closed). Therefore, $A \cap F$ is a gZ-closed (resp. γg-closed) set of X.

Proposition 2.18. If A is a δ-open (resp. an open) set and H is a Z-open set of a topological space (X, τ), then $A \cap H$ is Z-open (resp. γ-open) in X.

Proof. Obvious from Theorem 2.17.

Proposition 2.19. If A is both an open and a g-closed subsets of A, then A is gZ-closed in X.

Proof. Let A be an open and a g-closed subsets of X and $A \subseteq H$, where H is an open set of X.

Then by hypothesis, Z-cl(A) $\subseteq$ cl(A) $\subseteq$ A , that is, Z-cl(A) $\subseteq$ H . Thus A is gZ-closed in X.

Theorem 2.20. For a topological space (X, τ), then ZO(X, τ) $\subseteq$ {F $\subseteq$ X: F is closed} if and only if

every subset of X is gZ-closed of X.

Proof. Suppose that ZO(X, τ) $\subseteq$ {F $\subseteq$ X: F is closed}. Let A be any subet of X such that A $\subseteq$ H, where H is a Z-open set of X. Then H $\in$ ZO(X, τ) $\subseteq$ {F $\subseteq$ X: F is closed}, that is, H $\in$ {F $\subseteq$ X: F is closed}.Thus H is Z-closed . Then, Z-cl(H) = H . Also, Z-cl(A) $\subseteq$ Z-cl(H) $\subseteq$ H . Hence, A is a gZ-closed of X.

Conversely. Suppose that every subset of X is gZ-closed in X . Let H $\in$ ZO(X, τ). Since, H $\subseteq$ H and H is gZ-closed, then Z-cl(H) $\subseteq$ H. Thus, Z-cl(H) = H and hence, H $\in$ {F $\subseteq$ X: F is closed}. Therefore, ZO(X, τ) $\subseteq$ {F $\subseteq$ X: F is closed}.

Definition 2.21. The intersection of all Z-open subsets of (X, τ) containing A is called the Z-kernel of A and is denoted by Z-ker(A).

Lemma 2.22. For any subset A of a toplogical space (X, τ), then A $\subseteq$ Z-ker(A).

Proof. Follows directly from Definition 2.21.

Lemma 2.23. Let (X, τ) be a topological space and A be a subset of X. If A is a Z-open set of X X, then, Z-ker(A) = A.

Theorem 2.24. A subset A of a topological space X is gZ-closed if and only if Z-cl(A) $\subseteq$ Z-ker(A).
Proof. Since, A is a gZ-closed set of X, Z-cl(A) $\subseteq$ G, for any open set G with A $\subseteq$ G . Hence
Z-cl(A) $\subseteq$ Z-ker(A).
Conversely. Let G be any open set such that A $\subseteq$ G. Then by hypothesis, Z-cl(A) $\subseteq$ Z-ker(A) $\subseteq$ G . So, A is gZclosed.

3. Some properties of generalized Z-open sets.

Definiton 3.1. A subset A of a topological space (X, τ) is called a generalized Z-open (breifly, gZ-

open) set of X if X \ A is gZ-closed in X.We denote the family of all gZ-open sets of X by GZO(X).

Theorem 3.2. Let (X, τ) be a topological space and A $\subseteq$ X. Then the following statements are equivalent:

(1) A is a gZ-open set,

(2) for each closed set F contained in A, F $\subseteq$ Z-int(A),

(3) for each closed set F contained in A, there exists a Z-open set H such that F $\subseteq$ H $\subseteq$ A.

Proof. (1) $\rightarrow$(2). Let F $\subseteq$ A and F be a Z-closed set. Then X \ A $\subseteq$ X \ F which is Z-open of X , hence, Z-cl(X \ A) $\subseteq$ X \ F. So, F $\subseteq$ Z-int(A).

(2) $\rightarrow$(3). Suppose that F $\subseteq$ A and F be a Z-closed set. Then by hypothesis, F $\subseteq$ Z-int(A). But,

H = Z-int(A), hence there exists a Z-open set H such that F $\subseteq$ H $\subseteq$ A.

(3) $\rightarrow$ (1). Assume that X \ A $\subseteq$ V and V is a Z-open set of X. Then by hypothesis, there exists a

Z-open set H such that X \ V $\subseteq$ H $\subseteq$ A, that is, X \ A $\subseteq$ X \ H $\subseteq$ V. Therefore, by Theorem 2.7,

X \ A is gZ-closed in X . Then, A is gZ-open in X.

Theorem 3.3. If A is an δ-open (resp. open) and B is a gZ-open (resp. γg-open) subset of a space X, then $A \cap B$ is gZ-open (resp.γg-open).

Proof. Follows from Theorem 2.10.

Proposition 3.4. If Z-int(A) $\subseteq$ B $\subseteq$ A and A is a gZ-open set of X, then B is gZ-open.

Proposition 3.5. Let A be a Z-closed and a gZ-open sets of X. Then A is Z-open.

Proof. Let A be a Z-closed and a gZ-open sets of X. Then A $\subseteq$ Z-int(A) and hence A is Z-open.

Theorem 3.6. For a space (X, τ), if A is a gZ-closed set of X, then Z-cl(A) \ A is gZ-open.

Proof . Suppose that A is a gZ-closed set of X and F is a Z-closed set contained in Z-cl(A) \ A.

Then by Theorem 2.7 , F = $\emptyset$ and hence F $\subseteq$ Z-int(Z-cl(A) \ A). Therefore, Z-cl(A) \ A is gZ-open.

Theorem 3.7. If A is a gZ-open subset of a space (X, τ), then G = X, whenever G is open and

Z-int(A) $\cup$ (X \ A) $\subseteq$ G.

Proof. Let G be an open set of X and Z-int(A) $\cup$ (X \ A) $\subseteq$ G. Then X \ G $\subseteq$ (X \ Z-int(A)) $\cap$ A = Z-cl(X \ A) \ (X \ A). Since, X \ G is closed and X \ A is gZ-closed, by Theorem 2. 13, X \ G = $\emptyset$ and hence G = X.
Theorem 3. 8. For a topological space (X, τ), then every singleton of X is eiher gZ-open or Z-open.

Proof.L et (X,τ) be a topological space and p$\in$X. To prove that $\{p\}$ is eiher gZ-open or Z-open,

that is, to prove $X \setminus\{p\}$ is eiher gZ-closed or Z-open which follows directly from Theorem 2.13.

4. Z-$T_{1/2}$ spaces and generalized Z-continuous functions.
Definition 4.1. A space (X, τ) is called a Z-$T_{1/2}$ -space if every gZ-closed set is Z-closed.

Theorem 4.2. For a topological space (X, τ), the following conditions are equivalent:

(1) X is Z-$T_{1/2}$.

(2) Every singleton of X is either closed or Z-open.

Proof. (1)$\rightarrow$ (2). Let $p \in X$ and $\{p\}$ be not closed. Then $X \setminus\{p\}$ is not open and hence $X \setminus\{p\}$ is gZ-closed. Hence, by hypothesis, $X \setminus\{p\}$ is Z-closed and thus $\{p\}$ is Z-open.

(2) $\rightarrow$ (1). Let A $\subseteq$ X be a gZ-closed set of X and p $\in$Z-cl(A). We will show that $p \in A$. For consider the following two cases:

Case (1). The singleton set $\{p\}$ is closed. Then, if $p \notin A$, then there exists a closed set of

Z-cl(A) \ A. Hence, by Corollary 2.14, $p \in A$.

Case (2). The singleton set $\{p\}$ is Z-open. Since p $\in$Z-cl(A), then $\{p\}$ $\cap$ Z-cl(A) $\neq$ $\emptyset$ .Thus $p \in A$. So, in both cases, $p \in A$. This shows that Z-cl(A) $\subseteq$ A or equivalently, A is Z-closed.

Theorem 4.3. For a topological space (X, τ), the following statements are hold:

(1) ZO(X, τ) $\subseteq$ GZO(X,τ),

(2) a space X is Z-$T_{1/2}$ if and only if ZO(X, τ) = GZO(X,τ).

Proof. (1) Let A be a Z-open set. Then X \ A is Z-closed and so gZ-closed . This implies that A is

gZ-open. Hence ZO(X, τ) $\subseteq$ GZO(X, τ) .

(2) The necessity. Let (X, τ) be a Z-$T_{1/2}$ space and let A$\in$GZO(X, τ). Then X \ A is gZ-closed. Hence by hypothesis, X \ A is Z-closed and thus A is Z-open this implies that A $\in$ ZO(X, τ). Hence, ZO(X, τ) = GZO(X, τ).

The sufficiency. Let $ZO(X, \tau) = GZO(X, \tau)$ and let A be a gZ-closed set. Then $X \setminus A$ is gZ-open. Hence, $X \setminus A \in ZO(X, \tau)$. Thus A is Z-closed. Therefore (X, τ) is Z-$T_{1/2}$.

Definition 4.4. A function $f: X \to Y$ is called:
(1) gZ-continuous if, $f^{-1}(F)$ is gZ-closed in X, for every closed set F of Y,
(2) Z-gZ-continuous if, $f^{-1}(F)$ is gZ-closed in X, for every Z-closed set F of Y,
(3) gZ-irresolute if, $f^{-1}(F)$ is gZ-closed in X, for every gZ-closed set F of Y.
Definition 4.5. A function $f: X \to Y$ is called:
(1) γg-continuous [17] if, $f^{-1}(F)$ is γg-closed in X, for every closed set F of Y,
(2) γ-gγ-continuous [9] if, $f^{-1}(F)$ is gγ-closed in X, for every γ-closed set F of Y,
(3) γg-irresolute [17] if, $f^{-1}(F)$ is gγ-closed in X, for every gγ-closed set F of Y,
(4) Z-continuous [12] if, $f^{-1}(F)$ is Z-closed in X, for every closed set F of Y,
(5) Z-irresolute [13] if, $f^{-1}(F)$ is Z-closed in X, for every Z-closed set F of Y.
Remark 4.6. The following diagram holds for a function $f: (X, \tau) \to (Y, \sigma)$:

$$\begin{array}{ccccc} \gamma g\text{-irresoluteness} & \rightarrow & \gamma\text{-}g\gamma\text{-continuity} & \rightarrow & \gamma g\text{-continuity} \\ & & \uparrow & & \uparrow \\ gZ\text{-irresoluteness} & \rightarrow & Z\text{-}gZ\text{-continuity} & \rightarrow & gZ\text{-continuity} \\ & & \uparrow & & \uparrow \\ & & Z\text{-irresoluteness} & \rightarrow & Z\text{-continuity} \end{array}$$

The converses of the above implications are not true in general as is shown by [9] and the following example.
Example 4.7. In Example 2.3, Let $f: (X, \tau) \to (X, \tau)$ be a function defined by f(a) = a, f(b) = b, f(c)= c, f(d) = e and f(e) = d. Then f is γgγ-continuous (resp. γg-continuous) but it is not ZgZ-continuous (resp. gZ-continuous).
Example 4.8. Let X = {a, b, c, d} and $\tau = \sigma = \{\emptyset, \{b\}, \{d\}, \{b, d\}, \{a, b\}\}$. Let $f: (X, \tau) \to (Y, \sigma)$ be a function defined by f(a) = a, f(b) = c, f(c)= b and f(d) = d Then f is gZ-continuous but it is neither Z-gZ-continuous nor Z-continuous.
If we define the function $f: (X, \tau) \to (Y, \sigma)$ as follows: f(a) = a, f(b) = b, f(c) = d and f(d) = b, then f is Z-gZ-continuous but it is neither gZ-irresolute nor Z-irresolute.
Example 4.9. Let X = Y = {a, b, c, d} with $\tau = \sigma = \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b ,c\}, \{a, c, d\}, X\}$ and $f :(X, \tau) \to (X, \tau)$ be defined by f(a) = b, f(b) = d, f(c) = a and f(d) = d is Z-continuous but it is not Z-irresolute.

Theorem 4.10. Let $f : X \to Y$ and $h : Y \to Z$ be functions.
(1) If, f is gZ-irresolute and h is gZ-continuous, then the composition $h \circ f: X \to Z$ is gZ-continuous.
(2) If, f is gZ-continuous and h is continuous, then the composition $h \circ f: X \to Z$ is gZ-continuous.
(3) If, f and h are gZ-irresolute, then the composition $h \circ f : X \to Z$ is gZ-irresolute.
(4) If, f is gZ-irresolute and h is Z-gZ-continuous, then the composition $h \circ f: X \to Z$ is Z-gZ-continuous.
(5) If, f and h are Z-gZ-continuous and Y is Z-$T_{1/2}$, then the composition $h \circ f : X \to Z$ is Z-gZ-continuous.
Theorem 4.11. If a function $f : X \to Y$ is Z-gZ-continuous and Y is a Z-$T_{1/2}$ space, then f is gZ-irresolute.
Proof. Let F be any gZ-closed subset of Y. Since, Y is a Z-$T_{1/2}$ space , then F is Z-closed in Y. Hence, $f^{-1}(F)$ is Z-closed in X. This show that f is gZ-irresolute.
Theorem 4.12. If a function $f: X \to Y$ is gZ-continuous and X is a Z-$T_{1/2}$ space , then, f is Z-continuous.
Proof. Let F be any closed set of Y and f be gZ-continuous. Then, $f^{-1}(F)$ is gZ-closed in X and hence , $f^{-1}(F)$ is Z-closed in X. Therefore, f is Z-continuous.
Theorem 4.13. If a function $f: X \to Y$ is Z-gZ-continuous and X a Z-$T_{1/2}$ space , then, f is Z-irresolute.
Proof. Let F be any Z-closed set of Y and f be Z-gZ-continuous. Then, $f^{-1}(F)$ is gZ-closed in X and hence, $f^{-1}(F)$ is Z-closed in X. Hence, f is Z-irresolute.
Definition 4.14. A function $f : X \to Y$ is said to be:
(1) gZ-closed if ,f(A) is gZ-closed in Y, for each closed set A of X.
(2) Z-gZ-closed if, f(A) is gZ-closed in Y, for each Z-closed set A of X.
Theorem 4.15. If, $f : X \to Y$ is a closed and a Z-gZ-continuous functions, then $f^{-1}(K)$ is gZ-closed in X, for each gZ-closed set K of Y.
Proof. Let K be a gZ-closed set of Y and U be an open set of X containing $f^{-1}(K)$. Put,

$V = Y \setminus f(X \setminus U)$, then V is open in Y, $K \subseteq V$ and $f^{-1}(V) \subseteq U$. Therefore, we have Z-cl(K) $\subseteq$ V and hence $f^{-1}(K) \subseteq f^{-1}$ (Z-cl(K)) $\subseteq f^{-1}(V) \subseteq U$. Since, f is Z-gZ-continuous, then f^{-1} (Z-cl(K)) is gZ-closed in X and hence Z-cl(f^{-1}(K)) $\subseteq$ Z-cl(f^{-1} (Z-cl(K))) $\subseteq$ U. This shows that f^{-1} (K) is gZ-closed in X.

Corollary 4.16. If, $f : X \to Y$ is a closed and a Z-irresolute functions, then f^{-1}(K) is gZ-closed in X, for each gZ-closed set K of Y.

Theorem 4.17. If, $f : X \to Y$ is a bijective open and a Z-gZ-continuous functions, then f^{-1}(K) is gZ-closed in X, for every gZ-closed set K of Y.

Proof. Let K be a gZ-closed set of Y and U be an open set of X containing f^{-1}(K). Since, f is a surjective open function, then K = f(f^{-1}(K)) $\subseteq$ f(U) and f(U) is open. Therefore, Z-cl(K) $\subseteq$ f(U). But, f is an injective, hence $f^{-1}(K) \subseteq f^{-1}$(Z-cl(K)) $\subseteq f^{-1}$(f(U)) = U. Since, f is Z-gZ-continuous, then f^{-1}(Z-cl(K)) is gZ-closed in X and hence Z-cl(f^{-1}(K)) $\subseteq$ Z-cl(f^{-1}(Z-cl(K))) $\subseteq$ U. Therefore f^{-1}(K) is gZ-closed in X.

Definition 4.18. A space X is said to be Z-normal if, for any pair of disjoint closed sets A , B, there exist two disjoint Z-open sets U , V such that $A \subseteq U$ and $B \subseteq V$.

Theorem 4.19. Let $f : X \to Y$ be an injection closed and a Z-gZ-continuous functions. If Y is a Z-normal space, then X is Z-normal.

Proof. Let N_1 , N_2 be disjoint closed sets of X and f be an injection closed function , then f(N_1) , f(N_2) are disjoint closed sets of Y. Hence by the Z-normality of Y, there exist disjoint $V_1, V_2 \in$ ZO(Y) such that $f(N_i) \subseteq V_i$, for i = 1, 2. Since, f is Z-gZ-continuous, hence, $f^{-1}(V_1)$, $f^{-1}(V_2)$ are disjoint gZ-open sets of X and $N_i \subseteq f^{-1}(V_i)$, for i = 1, 2. Now, put U_i = Z-int($f^{-1}(V_i)$) ,for i = 1, 2. Then, $U_i \in ZO(X, \tau)$, $N_i \subseteq U_i$ and $U_1 \cap U_2 = \emptyset$. Therefore, X is Z-normal.

Corollary 4.20. If, $f: X \to Y$ is an injection closed and a Z-irresolute functions and Y is a Z-normal space, then X is Z-normal.

Proof. This is an immediate consequence since every Z-irresolute function is Z-gZ-continuous.

Lemma 4.21. A surjection function $f : X \to Y$ is Z-gZ-closed if and only if, for each subset B of Y and each Z-open set U of X containing f^{-1}(B), there exists a gZ-open set of V of Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Theorem 4.22. If, $f : X \to Y$ is a surjection continuous and a Z-gZ-closed functions and X is a Z-normal space, then Y is Z-normal.

Proof. Let A , B be any disjoint closed sets of Y. Then f^{-1}(A) , f^{-1}(B) are disjoint closed sets of X. Since, X is a Z-normal space, hence there exist two disjoint Z-open sets U , V such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. Hence by Lemma 4.21, there exist two gZ-open sets G , H of Y such that $A \subseteq G$, $B \subseteq H$, $f^{-1}(G) \subseteq U$ and $f^{-1}(H) \subseteq V$. Since, U ,V are disjoint, then G , H are also disjoint. Hence by Theorem 3.2, we have $A \subseteq$ Z-int(G), $B \subseteq$ Z-int(H) and Z-int(G) $\cap$ Z-int(H) = $\emptyset$. Therefore, Y is Z-normal.

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