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Solution of Telegraph Equation by Modified of Double Sumudu Transform "Elzaki Transform"

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Abstract

In this paper, we apply modified version of double Sumudu transform which is called double Elzaki transform to solve the general linear telegraph equation. The applicability of this new transform is demonstrated using some functions, which arise in the solution of general linear telegraph equation.

Keywords: Double Elzaki Transform, modified of double Sumudu transforms, Double Laplace transform, Telegraph Equation.

1. Introduction:

Partial differential equations are very important in mathematical physic [7], the wave equation is known as one of the fundamental equations in mathematical physics is occur in many branches of physics, for example, in applied mathematics and engineering.

A lot of problems have been solved by integral transforms such as Laplace [7], Fourier, Mellin, and Sumudu [9, 10]. Also these problems have been solved by differential transform method [13-20] and homotopy perturbation [22-25] an ingenious solution to visualizing the Elzaki transform was proposed originally by Tarig M. Elzaki [1-4], this new transform rivals Sumudu transform in problem solving.

In this paper we derive, we believe for the first time and solve telegraph and wave equations by using modified of double Sumudu transform [8] "double Elzaki transform".

We write that Laplace transform is defined by:

$$L\left[f(t)\right]=\int_0^\infty e^{-st}f(t)dt \qquad , \quad s>0 \qquad (1\text{-}1)$$

Where that Elzaki transform is defined over the set of functions:

$$A=\left\{f(t):\exists M,k_1,k_2>0,\left|f(t)\right|>Me^{\frac{t}{k_j}},t\in(-1)^j\mathrm{X}[0,\infty)\right\}$$

By

$$E\left[f(t)\right]=u\int_0^\infty f(t)e^{-\frac{t}{u}}dt, \qquad u\in(k_1,k_2). \tag{1-2}$$

By analogy with the double Laplace transform, we shall denote the double Elzaki transform.

1.1 ***Double Elzaki Transform:***

The double Laplace transform of a function of two variables is given by:

$$L_2\left[f(x,t)\right]=F(p,s)=\int_0^\infty\int_0^\infty f(x,t)e^{-(px+st)}dxdt \tag{1-3}$$

Where p,s are the transform variables for x,t respectively.

Definition:

Let $f(x,t)$, $t,x\in R^+$, be a function which can be expressed as a convergent infinite series, then, its double Elzaki transform, given by

$$E_2\left[f(x,t),u,v\right]=T(u,v)=uv\int_0^\infty\int_0^\infty f(x,t)e^{-(\frac{x}{u}+\frac{t}{v})}dxdt, \quad x,t>0 \tag{1-4}$$

Where u,v are complex values. To find the solution of telegraph and wave equations by double Elzaki transform, first we must find double Elzaki transform of partial derivatives as follows:

Double Laplace transform of the first and second order partial derivatives are given by:

$$L_2\left[\frac{\partial f}{\partial x}\right]=pF(p,s)-F(0,s) \qquad L_2\left[\frac{\partial^2 f}{\partial x^2}\right]=p^2F(p,s)-pF(0,s)-\frac{\partial F(0,s)}{\partial x}$$

$$L_2\left[\frac{\partial f}{\partial t}\right]=sF(p,s)-F(p,0) \qquad L_2\left[\frac{\partial^2 f}{\partial t^2}\right]=s^2F(p,s)-sF(p,0)-\frac{\partial F(p,0)}{\partial t}$$

$$L_2\left[\frac{\partial^2 f}{\partial x\,\partial t}\right]=psF(p,s)-pF(p,0)-sF(0,s)-F(0,0)$$

Similarly double Elzaki transform for first and second partial derivatives are given by:

$$E_2\left[\frac{\partial f}{\partial x}\right]=\frac{1}{u}T(u,v)-uT(0,v) \qquad E_2\left[\frac{\partial^2 f}{\partial x^2}\right]=\frac{1}{u^2}T(u,v)-T(0,v)-u\frac{\partial T(0,v)}{\partial x}$$

$$E_2\left[\frac{\partial f}{\partial t}\right]=\frac{1}{v}T(u,v)-vT(u,0) \qquad E_2\left[\frac{\partial^2 f}{\partial t^2}\right]=\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial T(u,0)}{\partial t}$$

$$E_2\left[\frac{\partial^2 f}{\partial x\,\partial t}\right]=\frac{1}{v}T(u,v)-vT(u,0)-\frac{u}{v}T(0,v)+uvT(0,0)$$

Proof:

$$E_2\left[\frac{\partial f}{\partial x}\right]=uv\int_0^\infty\int_0^\infty e^{-(\frac{x}{u}+\frac{t}{v})}\frac{\partial}{\partial x}f(x,t)dxdt=v\int_0^\infty e^{-\frac{t}{v}}\left\{u\int_0^\infty e^{-\frac{x}{u}}\frac{\partial}{\partial x}f(x,t)dx\right\}dt$$

The inner integral gives: $\frac{1}{u}T(u,t)-uf(0,t)$, and then:

$$E_2\left[\frac{\partial f}{\partial x}\right]=\frac{u}{v}\int_0^\infty e^{-\frac{t}{v}}T(u,t)dt-uv\int_0^\infty e^{-\frac{t}{v}}f(0,t)dt=\frac{1}{u}T(u,v)-uT(0,v)$$

Also $E_2\left[\frac{\partial f}{\partial t}\right]=\frac{1}{v}T(u,v)-vT(u,0)$

We can prove another derivative easily by using the same method.

2. Applications:

In this section we establish the validity of the double Elzaki transform by applying it to solve the general linear telegraph equations.

To solve partial differential equations by double Elzaki transform, we need the following steps.

(i) Take the double Elzaki transform of partial differential equations.

(ii) Take the single Elzaki transform of the conditions.

(iii) Substitute (ii) in (i) and solve the algebraic equation.

(iv) Take the double inverse of Elzaki transform to get the solution

Here we need the main equation:

$$E_2\left[e^{ax+bt}\right]=\frac{u^2v^2}{(1-au)(1-bv)}$$

Consider the general linear telegraph equation in the form:

$$U_{tt}+aU_t+bU=c^2U_{xx} \tag{2-1}$$

With the boundary conditions:

$$U(0,t)=f_1(t) \quad , \quad U_x(0,t)=g_1(t)$$

And the initial conditions:

$$U(x,0)=f_2(x) \quad , \quad U_t(x,0)=g_2(x)$$

Solution:

Take the double Elzaki transform of equation (2-1) and single Elzaki transform of conditions, and then we have:

$$\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial T(u,0)}{\partial t}+\frac{a}{v}T(u,v)-avT(u,0)+bT(u,v)$$
$$+\frac{c^2}{u^2}T(u,v)-c^2T(0,v)-c^2u\frac{\partial T(0,v)}{\partial x}=0$$

(2-2)

And:

$$T(0,v)=F_1(v) \quad , \quad \frac{\partial T(0,v)}{\partial x}=G_1(v)$$
$$T(u,0)=F_2(u) \quad , \quad \frac{\partial T(u,0)}{\partial t}=G_2(u) \tag{2-3}$$

Substituting (2-3) in (2-2), we obtain:

$$T(u,v)=\frac{u^2v^2F_2(u)+u^2v^3G_2(u)+av^3u^2F_2(u)+c^2v^2u^2F_1(v)+c^2v^2u^3G_1(v)}{1+avu^2+bv^2u^2+c^2v^2}=H(u,v)$$

Take double inverse Elzaki transform to obtain the solution of general linear telegraph equation (2-1) in the form:

$$U(x,t)=E_2^{-1}\left[H(u,v)\right]=K(x,t)$$

Assumed that the double inverse Elzaki transform is exists.

Example 2.1:

Consider the telegraph equation

$$U_{xx}=U_{tt}+U_t+U \tag{2-4}$$

With the boundary conditions:

$$U(0,t)=e^{-t}, \qquad U_x(0,t)=e^{-t} \tag{2-5}$$

And the initial conditions:

$$U(x,0)=e^{x}, \qquad U_t(x,0)=-e^{x} \tag{2-6}$$

The exact solution is $U(x,t)=e^{x-t}$

Solution

Take the double Elzaki transform of equation (2-4), and single Elzaki transform of conditions (2-5), (2-6), and then we have:

$$\frac{1}{u^2}T(u,v)-T(0,v)-u\frac{\partial T(0,v)}{\partial x}=\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial T(u,0)}{\partial t}$$
$$+\frac{1}{v}T(u,v)-vT(u,0)+T(u,v)$$

(2-7)

And,

$$T(0,v)=\frac{v^2}{1+v} \quad , \quad \frac{\partial T(0,v)}{\partial x}=\frac{v^2}{1+v} \tag{2-8}$$

$$T(u,0)=\frac{u^2}{1-u} \quad , \quad \frac{\partial T(u,0)}{\partial t}=\frac{-u^2}{1-u} \tag{2-9}$$

Substituting (2-8) and (2-9) in (2-7), we obtain:

$$\frac{1}{u^2}T-\left[\frac{v^2}{1+v}\right]-u\left[\frac{v^2}{1+v}\right]=\frac{1}{v^2}T-\left[\frac{u^2}{1-u}\right]+\frac{u^2v}{1-u}+\frac{1}{v}T-v\left[\frac{u^2}{1-u}\right]=T$$

Or

$$(v^2-u^2-u^2v-u^2v^2)T=\frac{v^3u^4}{1-u}-\frac{v^3u^4}{1-u}-\frac{v^2u^4}{1-u}+\frac{u^2v^4}{1+v}+\frac{u^3v^4}{1+v}$$

$$And \qquad T(u,v)=\frac{u^2v^2(-u^2-vu^2+v^2-u^2v^2)}{(1+v)(1-u)(-u^2-vu^2+v^2-u^2v^2)}=\frac{u^2v^2}{(1+v)(1-u)}$$

Inversion to find the solution of equation (2-4) in the form:

$$U(x,t)=e^x.e^{-t}=e^{x-t}$$

Example 2.2:

Consider the telegraph equation

$$U_{xx}=U_{tt}+U_t-U \tag{2-10}$$

With the boundary conditions:

$$U(0,t)=e^{-2t}, \qquad U_x(0,t)=e^{-2t} \tag{2-11}$$

And the initial conditions:

$$U(x,0)=e^x, \qquad U_t(x,0)=-2e^x \tag{2-12}$$

The exact solution is $U(x,t)=e^{x-2t}$

Solution

Take the double Elzaki transform of eq (2-10), and single Elzaki transform of conditions (2-11), (2-12), and then we have:

$$\frac{1}{u^2}T(u,v)-T(0,v)-u\frac{\partial T(0,v)}{\partial x}=\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial T(u,0)}{\partial t}$$
$$+\frac{1}{v}T(u,v)-vT(u,0)-T(u,v) \tag{2-13}$$

And

$$T(0,v)=\frac{v^2}{1+2v} \quad , \quad \frac{\partial T(0,v)}{\partial x}=\frac{v^2}{1+2v} \tag{2-14}$$

$$T(u,0)=\frac{u^2}{1-u} \quad , \quad \frac{\partial T(u,0)}{\partial t}=\frac{-2u^2}{1-u} \tag{2-15}$$

Substituting (2-14) and (2-15) in (2-13), to find:

$$(v^2-u^2-u^2v+u^2v^2)T=\frac{2v^3u^4}{1-u}-\frac{v^3u^4}{1-u}-\frac{v^2u^4}{1-u}+\frac{u^2v^4}{1+2v}+\frac{u^3v^4}{1+2v}$$

$$And \qquad T(u,v)=\frac{u^2v^2(-u^2-vu^2+v^2+u^2v^2)}{(1+2v)(1-u)(-u^2-vu^2+v^2+u^2v^2)}=\frac{u^2v^2}{(1+2v)(1-u)}$$

The inverse of the last equation gives the solution of equation (2-10) in the form: $U(x,t)=e^{x-2t}$

Example 2.3:

Let us the telegraph equation

$$U_{xx}=U_{tt}+4U_t+4U \tag{2-16}$$

With the boundary conditions:

$$U(0,t)=1+e^{-2t}, \qquad U_x(0,t)=2 \tag{2-17}$$

And the initial conditions:

$$U(x,0)=1+e^{2x}, \qquad U_t(x,0)=-2 \tag{2-18}$$

The exact solution is $U(x,t)=e^{2x}+e^{-2t}$

Solution

Applying double Elzaki transform to eq (2-16), and single Elzaki transform to conditions (2-17), (2-18), we get:

$$\frac{1}{u^2}T(u,v)-T(0,v)-u\frac{\partial T(0,v)}{\partial x}=\frac{1}{v^2}T(u,v)-T(u,0)-v\frac{\partial T(u,0)}{\partial t}$$
$$+\frac{4}{v}T(u,v)-4vT(u,0)+4T(u,v)$$

(2-19)

And the transform of conditions are,

$$T(0,v)=\frac{2v^2+2v^3}{1+2v} \quad , \quad \frac{\partial T(0,v)}{\partial x}=2v^2 \tag{2-20}$$

$$T(u,0)=\frac{2u^2-2u^3}{1-2u} \quad , \quad \frac{\partial T(u,0)}{\partial t}=-2u^2 \tag{2-21}$$

By the same method in examples (2-4) and (2-5), substituting (2-20) and (2-21) in (2-19) to find:

$$T(u,v)=\frac{u^2+2vu^2+v^2-2uv^2)}{(1+2v)(1-2u)}=\frac{u^2}{(1-2u)}+\frac{v^2}{1+2v}$$

Take the double inverse of Elzaki transform to get the solution of equation (2-16) in the form: $U(x,t)=e^{2x}+e^{-2t}$

3. Conclusion:

In this work, double Elzaki transform is applied to obtain the solution of general linear telegraph. It may be concluded that double Elzaki transform is very powerful and efficient in finding the analytical solution for a wide class of partial differential equations.

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References

[1] Tarig M. Elzaki, The New Integral Transform “Elzaki Transform” Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1(2011), pp. 57-64.

[2] Tarig M. Elzaki & Salih M. Elzaki, Application of New Transform “Elzaki Transform” to Partial Differential Equations, Global Journal of Pure and Applied Mathematics, ISSN 0973-1768,Number 1(2011), pp. 65-70.

[3] Tarig M. Elzaki & Salih M. Elzaki, On the Connections Between Laplace and Elzaki transforms, Advances in Theoretical and Applied Mathematics, ISSN 0973-4554 Volume 6, Number 1(2011),pp. 1-11.

[4] Tarig M. Elzaki & Salih M. Elzaki, On the Elzaki Transform and Ordinary Differential Equation With Variable Coefficients, Advances in Theoretical and Applied Mathematics. ISSN 0973-4554 Volume 6,

Number 1(2011),pp. 13-18.

[5] Tarig M. Elzaki, Adem Kilicman, Hassan Eltayeb. On Existence and Uniqueness of Generalized Solutions for a Mixed-Type Differential Equation, Journal of Mathematics Research, Vol. 2, No. 4 (2010) pp. 88-92.

[6] Tarig M. Elzaki, Existence and Uniqueness of Solutions for Composite Type Equation, Journal of Science and Technology, (2009). pp. 214-219.

[7] Lokenath Debnath and D. Bhatta. Integral transform and their Application second Edition, Chapman & Hall /CRC (2006).

[8] A.Kilicman and H.E.Gadain. An application of double Laplace transform and Sumudu transform, Lobachevskii J. Math.30 (3) (2009), pp.214-223.

[9] J. Zhang, A Sumudu based algorithm m for solving differential equations, Comp. Sci. J. Moldova 15(3) (2007), pp – 303-313.

[10] Hassan Eltayeb and Adem kilicman, A Note on the Sumudu Transforms and differential Equations, Applied Mathematical Sciences, VOL, 4,2010, no.22,1089-1098

[11] Kilicman A. & H. ELtayeb. A note on Integral transform and Partial Differential Equation, Applied Mathematical Sciences, 4(3) (2010), PP.109-118.

[12] Hassan ELtayeh and Adem kilicman, on Some Applications of a new Integral Transform, Int. Journal of Math. Analysis, Vol, 4, 2010, no.3, 123-132.

[13] Abdel – Hassan, I.H, 2004 Differential transformation technique for solving higher-order initial value problem. Applied math .Comput, 154-299-311.

[14] Ayaz.F.2004 solution of the system of differential equations by differential transforms method .Applied math.Comput, 147: 54767.

[15] Bongsoo Jang: Solving linear and nonlinear initial value proplems by the projected differential transform method. Ulsan National Institute of Science and Technology (UNIST), Banyeon-ri-100, Ulsan 689-798 korea. Compu. Phys. Communication(2009)

[16] C. Hchen, S. H. Ho. Solving Partial differential by two dimensional differential transform method, APPL. Math .Comput.106 (1999)171-179.

[17] Fatma Ayaz-Solution of the system of differential equations by differential transform method .Applied .math. Comput. 147(2004)547-567.

[18] F. Kanglgil. F. Ayaz. Solitary wave Solution for kdv and M kdv equations by differential transform method, chaos solutions and fractions do1:10.1016/j. Chaos 2008.02.009.

[19]Hashim,IM.SM.Noorani,R.Ahmed.S.A.Bakar.E.S.I.Ismailand A.M.Zakaria,2006.Accuracy of the Adomian decomposition method applied to the Lorenz system chaos 2005.08.135.

[20] J. K. Zhou, Differential Transformation and its Application for Electrical eructs .Hunzhong university press, wuhan, china, 1986.

[21] Montri Thong moon. Sasitornpusjuso.The numerical Solutions of differential transform method and the Laplace transform method for a system of differential equation. Nonlinear Analysis. Hybrid systems (2009) d0I:10.1016/J.nahs 2009.10.006.

[22] N.H. Sweilam, M.M. Khader. Exact Solutions of some capled nonlinear partial differential equations using the homotopy perturbation method. Computers and Mathematics with Applications 58 (2009) 2134-

2141.
[23] P.R. Sharma and Giriraj Methi. Applications of Homotopy Perturbation method to Partial differential equations. Asian Journal of Mathematics and Statistics 4 (3): 140-150, 2011.
[24] M.A. Jafari, A. Aminataei. Improved Homotopy Perturbation Method. International Mathematical Forum, 5, 2010, no, 32, 1567-1579.
[25] Jagdev Singh, Devendra, Sushila. Homotopy Perturbation Sumudu Transform Method for Nonlinear Equations. Adv. Theor. Appl. Mech., Vol. 4, 2011, no. 4, 165-175.

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