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A Complete Market Model for Option Valuation

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Abstract

This paper is an introduction and survey of Black-Scholes Model as a complete model for Option Valuation. It is a Stochastic processes that represent diffusive dynamics, a common and improved modelling assumption for financial systems.

As the markets are frictionless generally, it becomes very necessary for us to use a more convenient and complete method in order to avoid errors for computations. We include a review of Stochastic Differential equations(SDE), the Itô-lemma which gives a clear picture of Log-normal distribution of a Geometrical Brownian Motion path and solution of Black- Scholes Model

Keywords: Stochastic Differential Equations,Itô's lemma, tame and completeness of Black-Scholes Model

Introduction

The standard approach for Option valuation is based on a suitable specification of a stochastic process for the underlying asset. Historically, the protagonist role in describing the evolution of the market prices has been played by the continuous diffusion process. More models for the stochastic dynamics have been proposed market problems, such as Jump-diffusion models and Levy models and so on. We shall focus our attention on the Black-Scholes Model [3].

Financial derivative is core area of financial mathematics. In 1972, Black and Scholes tested the result of their model by using the data of Over-the-Counter Market (OTC), they found that result of their Model give tower values than the actual Market values.

Roll and Shastri (1973) found that these differences created due to the imperfect protection of dividend in the OTC market. Black and Scholes (1973) therefore used Itô's lemma mathematical tools which are used to calculate type of stochastic process, it also provides helps in the derivation of Black-Scholes formula.

In 1973, Black and Scholes then published their analysis of European Call option in a paper tittled "The pricing of Options and Corporate Liabilities. Robert C. M (1975) that

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validity of Black-Scholes Option pricing formula depends on the capability of investors to follow a dynamic portfolio strategy in the stock that replicates the payoff structure to the Option. The critical assumption required for such a strategy to be feasible, is that the underlying stock return dynamics can be described by a stochastic process with a continuous sample path.

Stochastic Differential Equations

The aim of this work is to provide a systematic frame work for an understanding of the basic concepts and mathematical tools needed for the development and implementation of Numerical valuation for financial derivative which has become a standard model for financial quantities such as asset prices, interest rate and their derivatives. Unlike deterministic models such as ODEs, which have a unique solution for each approximate initial condition. SDEs have solutions that are continuous-time stochastic process. It is the equation in which one or more of it terms is a stochastic process, thus, resulting in a solution which itself a stochastic process. It can be use to model the randomness of the underlying asset in financial derivative because they give a formal model of how an underlying asset's price changes overtime [1]

Starting with some fundamental concept from calculus that are needed for this work. We consider a general SDE, of the form

$$dX_t = \mu(t,x)dt + \sigma(t,x)dW_t \quad X(0) = X_0 \; 0 \le t \le T \qquad (1)$$

Where $\mu(t,x)$ is the deterministic or drift coefficient and $\sigma(t,x)$ is the diffusion (Noise) where dW_t is the an innovation term representing unpredictable events that occur during the finitesimal interval dt . while W_t is called a Wiener process [4].

If the diffusion term does not depend on $X_{t,}$ we say that equation (1) is *additive Noise*, otherwise, the equation has *multiplicative Noise.* The Wiener process, named after Norber is an essential instrument for stochastic process by botanist Brown in 1827, commonly called Brownian Motion.

A Wiener process $W = W_t$, $0 \le t \le T$ is a Gaussian process that depends commonly on time such that

1. $W(0) = 0$ (with probability one)
2. For $0 \le t \le T$, $E\big(W(t)\big) = 0$ and for $0 \le t \le T$, $Var\big(W(t) - W(s)\big) = t - s$
3. For $0 \le s < t < u < v \le T$, the increments $W(t) - W(s)$ and $W(v) - W(u)$ are independent. The corresponding stochastic integral to (1) above is

$$X_t = X_0 + \int_{t_\circ}^{t} \mu(s,x)ds \; + \; \int_{t_\circ}^{t} \sigma(s,x)dW_S \qquad (2)$$

Where the last integralS is called Itô integral.

To solve SDEs analytically we introduce the chain's rule for stochastic differential called Itô's lemma [1]

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Itô's lemma

We take into consideration an Itô's lemma by assuming $F(x,t)$ be a twice differentiable function of *t* and of the random process X_t follows the Itô process (1) of the form

$$dX_t = \mu_t dt \; + \; \sigma_t dW_t \qquad t \ge 0 \qquad (3)$$

then

$$dF_t = \frac{\partial F}{\partial X_t} X_t dt + \frac{\partial F}{\partial t} dt \; + \; \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2}\sigma_t^2 dt \qquad (4)$$

Putting (3) into (4) for $X_t dt$ and by using relevant stochastic differential equation, we have

$$dF_t = \left[\frac{\partial F}{\partial S_t}\mu_t \; + \; \frac{\partial F}{\partial t} \; + \; \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2}\sigma_t^2\right] dt \; + \; \frac{\partial F}{\partial X_t}\sigma_t dW_t \qquad (5)$$

Supposing the variance $X(t)$ follows a geometric Brownian motion and obeys a stochastic differential equation (1), then the Itô's lemma for any of the function $F(X,t)$ is given as

$$dF(X,t) = \left[\frac{\partial F}{\partial x}\mu X \; + \; \frac{\partial F}{\partial t} \; + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\sigma^2 X^2\right] dt + \frac{\partial F}{\partial X}\sigma X dW \qquad (6)$$

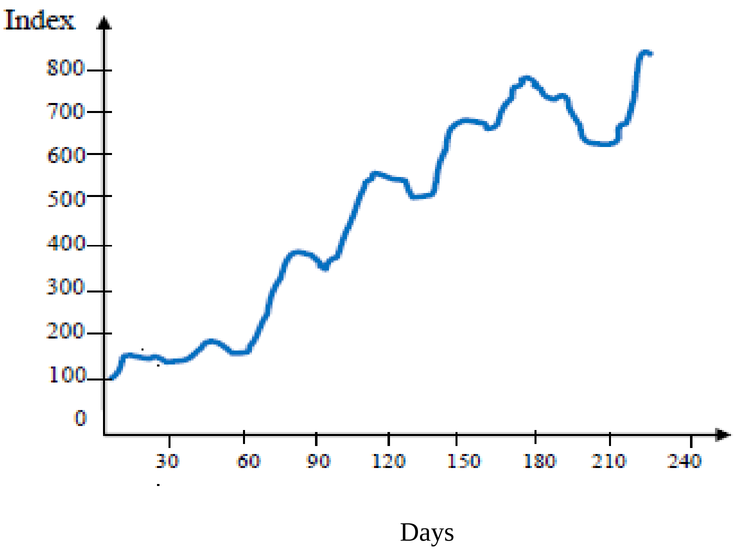
Where μ and σ are constants. We next consider the log-normal distribution, for a stock price process s. Let $F(X,t) = logX$, then

$$\frac{\partial F}{\partial X} = \frac{1}{X}, \quad \frac{\partial F}{\partial t} = 0 \qquad and \quad \frac{\partial^2 F}{\partial x^2} = \frac{-1}{X^2} \tag{7}$$

Putting (7) in (6) and by integration, it is trivial that

$$X_T = X_0 Exp\left[\left(\mu - \frac{\sigma^2}{2}\right)T + \sigma Z\sqrt{T}\right] \tag{8}$$

Where $Z \sim N(0,1)$. Therefore, it is obvious that stock dynamics follows a log-normal distribution [4], which shows the evolution of the stock price in a Geometric Brownian Motion path using (8) and this graph of simulated data below enhances the understanding of the stochastic behaviours of the underlying assets and assumption that stock returns are log-normally distributed.



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Simulation of a geometric Brownian Motion path with $X_o = 100,\ \sigma = 0.20,\ \mu = 0.10,\ and\ N = 300$ as samples drawn from the standard normal distribution

Market Model

As a model of a financial market, we consider a pair of assets: a *nourisky asset* (bank account) B, and a *risk asset* (stock) which may be represented by their prices *B(t)* and *S(t)* ,$t \in \mathbb{R}_+$. In this case, one speaks of a (B,S)-Market with continuous time. Here, the *risky* component of the (B,S)-Market may be Multidimensional [5]

The assets B and S will be called *underlying assets* or *underlying securities*

We describe the dynamic of the processes as follows

$$dB(t) = rB(t)d$$

$$B(0) = 1$$

and

$$dS(t) = S(t)(\mu dt + \sigma dW(t))$$

$$S(0) = S_0 \tag{9}$$

where *r* is the interest rate, μ is the drift parameter and σ is the volatility all assumed to be constant. We get

$$B(t) = e^{rt}$$

and

$$S(t) = S_0 \exp\left[\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W(t)\right] \qquad (10)$$

We consider the risky asset S on a filtered probability space $(\Omega, \beta, \mu, \mathbb{F})$, where the filtration $F = \{\beta_t : t \in \mathbb{R}_+\}$ is given by

$$\beta_t = \sigma(W(s) : 0 \leq s \leq t)$$

slightly enlarged to satisfy the usual conditions. Then, the stochastic process S is *adapted* and strictly positives. We called the market model sketched above the *Black-Scholes Model.* We therefore deduced the Black-Scholes formula as follows.

Black-Scholes Formula

As an illustrative example of the use of SDE for Option pricing, we consider the European Call(Put) whose value at expiration time T, is $Max\{S(T) - K, 0\}$ $(respectively\ Max\{K - S(T), 0\})$

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Where $S(T)$ is the price of the underlying stock, K is the Strike price. The non-arbitrage assumptions of Black-Scholes theory imply that the present values of such an option are

$$C_E(S,T,K) = e^{-rT}\mathbb{E}(Max\{S(T) - K, 0\})$$

and

$$P_E(S,T,K) = e^{-rT}\mathbb{E}(Max\{K - S(T), 0\}) \qquad (11)$$

Where r is the fixed prevailing interest rate during the time interval $[0, T]$, and where the underlying stock price $S(T)$ satisfies the stochastic differential equation (1) of the form

$$dS = rSdt + \sigma SdW_t$$

The value of Call Option can be determined by calculating the expected value (11) explicitly [4], we have

$$C_E(S,T,K) = SN(d_1) - Ke^{-rT}N(d_2)$$

Using Put-Call parity $P_E - C_E = Ke^{-rT} - S$, we have

$$P_E(S,T,K) = Ke^{-rT}N(-d_2) - SN(-d_1)$$

Where

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma\right)T}{\sigma\sqrt{T}} \quad \text{and} \quad d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma\right)T}{\sigma\sqrt{T}}$$

We therefore consider some basic definitions that will aid the completeness of Black-Scholes Model.[4,6]

Definition 1. A trading strategy $\emptyset$ is called *tame* if its associated process is always positive (i.e. $V^{\emptyset}(t) \geq 0 \ \forall \ t \in [0, T]$).

Definition 2. A contingent claim X, for $\mathbb{E}_{\mu^*}(X^2) < \infty$, is *attainable* if there exist at least one tame, self-financing trading strategy $\emptyset$ such that $V^{\emptyset}(T) = T$. If any contingent claim X satisfying $\mathbb{E}_{\mu^*}(X^2) < \infty$ is attainable, then the associated market model is *complete*

Definition 3. Let μ and μ^* be two probability measures on a measurable space (Ω, β) then μ and μ^* are *equivalent* if μ^* is absolutely continuous with respect to μ and μ is absolutely continuous with respect to μ^*.

Definition 4.

(i). The value of a portfolio $\emptyset$, given by $\emptyset(t) = (\beta(t), \xi(t)), t \in [0, T])$ is defined as

$$V^{\emptyset}(t) = \beta(t)B(t) + \xi(t)S(t)$$

The process $V^{\emptyset}$ is called the *value process* or *wealth process* of the trading strategy $\emptyset$.

(ii). The gains process is defined by

$$G^{\emptyset}(t) = \int_0^t \beta(s)dB(s) + \int_0^t \xi(s)dS(s)$$

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(iii). A trading strategy $\emptyset$ is self-financing if the wealth process $V^{\emptyset}$ satisfies

$$V^{\emptyset}(t) = V^{\emptyset}(0) + G^{\emptyset}(t), \qquad t \in [0, T]$$

Complete Market Model

Let X be a contingent claim such that $\mathbb{E}_{\mu^*}(X^2) < \infty$. We consider the martingale

$$M(t) = \mathbb{E}_{\mu^*}(e^{-rT} X \mid B_t), \qquad t \in [0, T]$$

by the martingale representation theorem, we have

$$M(t) = \mathbb{E}_{\mu^*}(M(t) + \int_0^t h(s)d\widetilde{W}(s), \qquad t \in [0, T] \qquad (12)$$

since μ is a μ^*-Martingale, we have $\mathbb{E}_{\mu^*}(M(t)) = M(0)$ and the dynamics of the risk asset S is given by

$$dS(t) = S(t)(rdt + \sigma dW(t)), \qquad t \in [0, T]$$

$$S(0) = S_0$$

$$dB(t) = rB(t)dt$$

$$B(0) = 1$$

then the strategy

$$\emptyset(t) = (\beta(t), \xi(t)) = \left(m(t) - \frac{h(t)}{\sigma}, \frac{h(t)B(t)}{\sigma S(t)}\right), \quad t \in [0, T]$$

is the self-financing strategy that replicate the contingent claim X. To see this, observe that the gains process of the strategy

$$G^{\emptyset}(u) = \int_0^u \beta(t)dB(t) + \int_0^u \xi(t)dS(t)$$

$$= \int_0^u (M(t) - \frac{h(t)}{\sigma})dB(t) + \int_0^u \left(\frac{h(t)}{\sigma}, \frac{h(t)B(t)}{\sigma S(t)}\right) dS(t)$$

$$= \int_0^u M(t)dB(t) - \int_0^u \frac{h(t)}{\sigma} dB(t) + \int_0^u \frac{h(t)B(t)}{\sigma S(t)} S(t)\sigma dt + \int_0^u \frac{h(t)B(t)}{\sigma S(t)} S(t)\sigma d\widetilde{W}(t)$$

$$= \int_0^u M(t)dB(t) + \int_0^u h(t)B(t)\, d\widetilde{W}(t)$$

by equation (12) above, we have

$$G^{\emptyset}(u) = \int_0^u (M(0) + \int_0^t h(s)d\widetilde{W}(s)dB(t) + \int_0^u h(t)B(t)d\,\widetilde{W}(t)$$

$$= M(0)\int_0^u dB(t) + \int_0^u \int_0^t h(s)d\widetilde{W}(s)dB(t) + \int_0^u h(s)B(t)d\widetilde{W}(t)$$

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$$= M(0)(B(u) - B(0)) + \int_0^u \int_s^u h(s)dB(t)d\widetilde{W}(s) + \int_0^u h(s)B(t)d\widetilde{W}(t)$$

$$= M(0)(B(u) - B(0)) + \int_0^u h(s)\int_s^u B(t)d\widetilde{W}(s) + \int_0^u h(s)B(t)d\widetilde{W}(t)$$

$$= M(0)\big(B(u) - B(0)\big) + \int_0^u h(s)\big(B(u) - B(s)\big)d\widetilde{W}(s) + \int_0^u h(s)B(t)d\widetilde{W}(t)$$

$$= M(0)\big(B(u) - B(0)\big) + B(u)\int_0^u h(s)d\widetilde{W}(t)$$

$$= M(0)\big(B(u) - B(0)\big) + B(u)(M(u) - M(0))$$

$$= M(u)B(u) - M(0)B(0)$$

We further consider the wealth process as

$$V^{\emptyset}(u) = \left(M(u) - \frac{h(u)}{\sigma}\right)B(u) + \left(\frac{h(u)B(u)}{\sigma S(u)}\right)S(u)$$

$$= M(u)B(u) \geq 0$$

clearly, $\emptyset$ is a tame strategy and $V^{\emptyset}(0) = M(0)B(0)$. Thus

$$V^{\emptyset}(0) + G^{\emptyset}(u) = M(0)B(0) + M(u)B(u) - M(0)B(0)$$

$$= M(u)B(u)$$

Showing that $\emptyset$ is self financing strategy and replicates X. Then X is attainable and as X was arbitrary, the Black-Scholes Model is complete

Summary and conclusion

In this paper, we investigated the Black-Scholes model for Option valuation and concentrated our investigation on the completeness of the Model

In reality, financial markets are not frictionless generally. This paper has examined the Black and Scholes as one of the numerical method for option valuation assumed to be simplest model for option pricing. This Model therefore attempts to simplify the markets for both financial assets and derivatives into a set of mathematical rules for a trading strategy $V^{\emptyset} \geq 0$ $(Tame)$, that ensure the positive result of any contingent claim

This model also serves as a basis for a wide range of analysis of markets and reduced the risk of the market transaction.

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