Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4,No.13,2014

A reliable iterative method for Cauchy problems

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Abstract.
In the present paper, the new iterative method proposed by Daftardar-Gejji and Jafari (NIM or DJM) [V. Daftardar-Gejji, H. Jafari, An iterative method for solving non linear functional equations, J. Math. Anal. Appl. 316 (2006) 753-763] is used to solve the Cauchy problems. In this iterative method the solution is obtained in the series form that converge to the exact solution with easily computed components. The results demonstrate that the method has many merits such as being derivative-free, overcome the difficulty arising in calculating calculating Adomian polynomials to handle the nonlinear terms in Adomian Decomposition Method (ADM). It does not require to calculate Lagrange multiplier in Variational Iteration Method (VIM) and no needs to construct a homotopy and solve the corresponding algebraic equations in Homotopy Perturbation Method (HPM) and can be easily comprehended with only a basic knowledge of Calculus. The results show that the present method is very effective, simple and provide the analytic solutions. The software used for the calculations in this study was MATHEMATICA ® 8.0.

Keywords: New iterative method; Cauchy problems; Inviscid Burger's equation; transport equation

1 Introduction

A variety of problems in physics, chemistry and biology have their mathematical setting as linear and nonlinear ordinary or partial differential equations. Many methods have been developed to solve differential equations, especially nonlinear, which are receiving increasing attention.

Many of the phenomena that arise in mathematical physics and engineering fields can be described by partial differential equations (PDEs). Moreover, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, and many other models are formulated by partial differential equations [1].

Due to these huge applications, there is a demand on the development of accurate and efficient analytic or approximate methods able to deal with the PDEs.

Many attempts have been made to develop analytic and approximate methods to solve the Cauchy problems, see [4–6]. Although such methods have been successfully applied but some difficulties have appeared, for examples, in calculating Adomian polynomials to handle the nonlinear terms in Adomian Decomposition Method (ADM) [2] and Modified Adomian Decomposition Method (MADM) [3], evaluating Lagrange multiplier in VIM [4] and in Reconstruction of the variational iteration (RVIM) [5] with some knowledge of the variational theory, construct a homotopy and solve the corresponding algebraic equations in HPM [6] with some knowledge in the deformation from Topology.

Recently, Daftardar-Gejji and Jafari [7] have proposed a new technique for solving linear/nonlinear functional equations namely new iterative method (NIM) or (DJM). The DJM has been extensively used by many researchers for the treatment of linear and nonlinear ordinary and partial differential equations of integer and fractional order, see [8–11]. The method converges to the exact solution if it exists through successive approximations. However, for concrete problems, a few approximations can be used for numerical purposes with high degree of accuracy. The DJM is simple to understand and easy to implement using computer packages and yields better results and does not require any restrictive assumptions for nonlinear terms as required by some existing techniques.

In this paper, the applications of the DJM for Cauchy problems will be presented. Moreover, the results obtained are compared with those obtained by other iterative methods such as VIM [4], RVIM [5] and HPM [6]. Comparisons show that the DJM is effective and convenient to use and overcomes the difficulties arising in others existing techniques.

The present paper has been organized as follows. In section 2 is devoted to the description the basic idea of DJM and its convergence. In section 3 the Cauchy problem is solved by DJM. In section 4 some test examples are solved by DJM to assess the efficiency of the method and finally in section 5 the conclusion is presented.

2 The new iterative method (NIM or DJM)

Consider the following general functional equation:

$$u = N(u) + f, \tag{1}$$

where N is a nonlinear operator from a Banach space $B \to B$ and f is a known function [7–11]. We are looking for a solution u of Eq.(1) having the series form:

$$u=\sum_{i=0}^{\infty}u_i. \tag{2}$$

The nonlinear operator N can be decomposed as

$$N(\sum_{i=0}^{\infty}u_i)=N(u_0)+\sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i}u_j)-N(\sum_{j=0}^{i-1}u_j)\}. \tag{3}$$

From Eqs.(2) and (3), Eq.(1) is equivalent to

$$\sum_{i=0}^{\infty}u_i=f+N(u_0)+\sum_{i=1}^{\infty}\{N(\sum_{j=0}^{i}u_j)-N(\sum_{j=0}^{i-1}u_j)\}. \tag{4}$$

We define the recurrence relation:

$$\begin{aligned}&G_0=u_0=f,\\&G_1=u_1=N(u_0),\\&G_m=u_{m+1}=N(u_0+\cdots+u_m)-N(u_0+\cdots+u_{m-1}),\quad m=1,2,...\end{aligned} \tag{5}$$

Then

$$(u_1+...+u_{m+1})=N(u_1+...+u_m),\quad m=1,2,..., \tag{6}$$

and

$$u(x)=f+\sum_{i=1}^{\infty}u_i. \tag{7}$$

The m-term approximate solution of Eq.(2) is given by $u=\sum\limits_{i=0}^{m-1}u_i$.

2.1 Convergence of the DJM

We present below the condition for convergence of the series $\sum u_i$. For more details we refer the reader to [12].

Theorem 2.1.1 : [12]
If N is $C^{(\infty)}$ in a neighbourhood of u_0 and $\| N^{(n)}(u_0) \|\leq L$, for any n and for some real $L>0$ and $\| u_i \|\leq M<\frac{1}{e}$, $i=1,2,...$, then the series $\sum\limits_{n=0}^{\infty}G_n$ is absolutely convergent and moreover, $\| G_n \|\leq LM^ne^{n-1}(e-1), n=1,2,...$
Theorem 2.1.2 : [12]

If N is $C^{(\infty)}$ and $\|N^{(n)}(u_0)\|\leq M\leq e^{-1}, \forall n$, then the series $\sum\limits_{n=0}^{\infty}G_n$ is absolutely convergent.

3 Solution of Cauchy problem by using DJM

The Cauchy problem of the first-order partial differential equation is given in the form [4–6, 13]

$$u_t(x,t)+a(x,t)u_x(x,t)=\phi(x),\quad x\in R,\quad t>0 \tag{8}$$

with initial condition:

$$u(x,0)=\psi(x),\quad x\in R. \tag{9}$$

When $a(x,t)=a$ is a constant and $\phi(x)=0$, Eq. (8) is a linear equation called the transport equation which can describe many interesting phenomena such as the spread of AIDS, the moving of wind. When $a(x,t)=u(x,t)$, Eq. (8) is a nonlinear equation called the inviscid Burgers' equation arising in one-dimensional stream of particles or fluid having zero viscosity.
Eq. (8) can be written in an operator form as

$$L_tu=\phi(x)-a(x,t)u_x(x,t), \tag{10}$$

where $L_t = \frac{\partial}{\partial t}$. Let us assume the the inverse operator L_t^{-1} exists and it can be take with respect t from 0 to t, i.e.

$$L_t^{-1}(.) = \int_0^t (.)dt \tag{11}$$

Then, by taking the inverse operator L_t^{-1} to both sides of the Eq.(10) and using the initial condition, leads to

$$u(x,t) = \psi(x) + \varphi(x,t) - L_t^{-1}(a(x,t)u_x(x,t)). \tag{12}$$

where the function $\varphi(x,t)$ results from integrating the source term $\phi(x)$ with respect to t from 0 to t. By applying the DJM for Eq.(12) the following recurrence relation for the determination of the components $u_{n+1}(x,t)$ are obtained:

$$u_0(x,,t) = \psi(x) + \varphi(x,t), \tag{13}$$

$$u_1(x,t) = N(u_0) = -L_t^{-1}(a(x,t)u_x(x,t)), \tag{14}$$

$$u_2(x,,t) = N(u_1+u_0) - N(u_0) = -L_t^{-1}(a(x,t)(u_1+u_0)_x) - u_1, \tag{15}$$

$$u_3(x,t) = N(u_2+u_1+u_0) - N(u_1+u_0) = -L_t^{-1}(a(x,t)(u_2+u_1+u_0)_x) + L_t^{-1}(a(x,t)(u_1+u_0)_x), \tag{16}$$

and so on.
Continuing in this manner, the $(n+1)$th approximation of the exact solutions for the unknown functions $u(x,t)$ can be achieved as:

$$\begin{aligned} u_{n+1}(x,t) = N(u_0+\cdots+u_n) - N(u_0+\cdots+u_{n-1}) = -L_t^{-1}(a(x,t)(u_0+\cdots+u_n)_x) - \\ +L_t^{-1}(a(x,t)(u_0+\cdots+u_{n-1})_x), \quad n = 1,2,... \end{aligned} \tag{17}$$

Based on the DJM, we constructed the solution $u(x,t)$ as:

$$u(x,t) = \sum_{k=0}^{n} u_k(x,t) \quad n \geq 0. \tag{18}$$

4 Test examples

In this section, some test examples will be examined to assess the performance of the DJM for Cauchy problem. To verify the convergence of the method, we applied the method to some test problems for which an analytical solution are available.

Example 1: Consider the transport equation [4–6]

$$u_t(x,t) + au_x(x,t) = 0, \quad x \in R, \quad t > 0, \tag{19}$$

with initial condition:

$$u(x,0) = x^2, \quad x \in R.$$

According to the iteration formula in Eq.(17), we have

$$u_0(x,t) = x^2, \tag{20}$$

$$\begin{aligned} u_{n+1}(x,t) = N(u_0+\cdots+u_n) - N(u_0+\cdots+u_{n-1}) = L_t^{-1}(a(u_0+\cdots+u_n)_x) - \\ -L_t^{-1}(a(u_0+\cdots+u_{n-1})_x), \quad n = 1,2,... \end{aligned} \tag{21}$$

According to the DJM, we achieve the following components:

$$u_1(x,t) = N(u_0) = -L_t^{-1}(a(u_0)_x(x,t)) = -2atx, \tag{22}$$

$$u_2(x,t) = N(u_1+u_0) - N(u_0) = -L_t^{-1}(a(u_1+u_0)_x) - u_1 = a^2t^2, \tag{23}$$

$$u_3(x,t) = N(u_2+u_1+u_0) - N(u_1+u_0) = -L_t^{-1}(a(u_2+u_1+u_0)_x) + L_t^{-1}(a(u_1+u_0)_x) = 0, \tag{24}$$

In fact, we have.

$u_j = 0, \;\; \text{for} j \geq 3$

Therefore, according to Eq.(18), we get:

$$u(x,t) = x^2 - 2atx + a^2t^2. \tag{25}$$

which is the exact solution of the problem and it is the same results obtained by VIM [4], RVIM [5] and HPM [6].

Example 2: Consider the Cauchy problem [4–6]

$$u_t(x,t) + xu_x(x,t) = 0, \;\; x \in R, \;\; t > 0, \tag{26}$$

with initial condition:

$$u(x,0) = x^2,$$

Proceeding as before, the recurrence relation

$$u_0(x,t) = x^2, \tag{27}$$

$$u_{n+1}(x,t) = N(u_0+\cdots+u_n) - N(u_0+\cdots+u_{n-1}) = -L_t^{-1}(x(u_0+\cdots+u_n)_x) - +L_t^{-1}(x(u_0+\cdots+u_{n-1})_x), \;\; n = 1,2,... \tag{28}$$

According to the DJM we achieve the following components:

$$u_1(x,t) = N(u_0) = -L_t^{-1}(x(u_0)_x(x,t)) = -2tx^2, \tag{29}$$

$$u_2(x,t) = N(u_1+u_0) - N(u_0) = -L_t^{-1}(x(u_1+u_0)_x) - u_1 = 2t^2x^2, \tag{30}$$

$$u_3(x,t) = N(u_2+u_1+u_0) - N(u_1+u_0) = -L_t^{-1}(x(u_2+u_1+u_0)_x) + L_t^{-1}(x(u_1+u_0)_x) = -\frac{4}{3}t^3x^2, \tag{31}$$

and so on.
Therefore, according to Eq.(18) we have:

$$u(x,t) = x^2\Big(1 - 2t + \frac{(2t^2)^2}{2!} - \frac{(2t^3)^2}{3!} + \frac{(2t^4)^2}{4!} - ...\Big). \tag{32}$$

This has the closed form

$$u(x,t) = x^2e^{-2t}. \tag{33}$$

which is the exact solution of the problem and it is the same results obtained by VIM [4], RVIM [5] and HPM [6].

Example 3: Consider the following non-homogeneous Cauchy problem [4–6]

$$u_t(x,t) + u_x(x,t) = x, \;\; x \in R, \;\; t > 0, \tag{34}$$

with initial condition:

$$u(x,0) = e^x,$$

In this example there is non-homogeneous term x which after integrating it with respect t from 0 to t, leads to xt.

Therefore, proceeding as before, the recurrence relation

$$u_0(x,y,z,t)=e^x+xt, \tag{35}$$

$$u_{n+1}(x,t)=N(u_0+\cdots+u_n)-N(u_0+\cdots+u_{n-1})=-L_t^{-1}((u_0+\cdots+u_n)_x)+ \\ +L_t^{-1}((u_0+\cdots+u_{n-1})_x),\quad n=1,2,... \tag{36}$$

This gives the following components:

$$u_1(x,t)=N(u_0)=-L_t^{-1}((u_0)_x(x,t))=-e^xt-\frac{t^2}{2}, \tag{37}$$

$$u_2(x,t)=N(u_1+u_0)-N(u_0)=-L_t^{-1}((u_1+u_0)_x)-u_1=\frac{e^xt^2}{2}, \tag{38}$$

$$u_3(x,t)=N(u_2+u_1+u_0)-N(u_1+u_0)=-L_t^{-1}((u_2+u_1+u_0)_x)+L_t^{-1}((u_1+u_0)_x)=\frac{-1}{6}e^xt^3, \tag{39}$$

and so on.
Therefore, according to Eq.(18) we have:

$$u(x,t)=e^x+xt-e^xt-\frac{t^2}{2}+\frac{e^xt^2}{2}=t\Big(x-\frac{t}{2}\Big)+e^x\Big(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-...\Big). \tag{40}$$

This has the closed form

$$u(x,t)=t\Big(x-\frac{t}{2}\Big)+e^x\cdot e^{-t}=t\Big(x-\frac{t}{2}\Big)+e^{x-t}. \tag{41}$$

which is the exact solution of the problem and it is the same results obtained by VIM [4], RVIM [5] and HPM [6].

Example 4: Consider the following the non-homogeneous nonlinear Cauchy problem [4–6]

$$u_t(x,t)+u(x,t)u_x(x,t)=x,\quad x\in R,\quad t>0, \tag{42}$$

with initial condition:

$$u(x,0)=x,$$

According to the iteration formula in Eq.(17), we have

$$u_0(x,y,z,t)=x, \tag{43}$$

$$u_{n+1}(x,t)=N(u_0+\cdots+u_n)-N(u_0+\cdots+u_{n-1})=-L_t^{-1}((u_0+\cdots+u_n)(u_0+\cdots+u_n)_x)+ \\ +L_t^{-1}((u_0+\cdots+u_{n-1})(u_0+\cdots+u_{n-1})_x),\quad n=1,2,... \tag{44}$$

This gives the following components:

$$u_1(x,t)=N(u_0)=-L_t^{-1}((u_0)(u_0)_x(x,t))=-tx, \tag{45}$$

$$u_2(x,t)=N(u_1+u_0)-N(u_0)=-L_t^{-1}((u_1+u_0)(u_1+u_0)_x)-u_1=t^2x-\frac{t^3x}{3}, \tag{46}$$

$$u_3(x,t)=N(u_2+u_1+u_0)-N(u_1+u_0)=-L_t^{-1}((u_2+u_1+u_0)(u_2+u_1+u_0)_x)+ \\ +L_t^{-1}((u_1+u_0)(u_1+u_0)_x)=-\frac{2t^3x}{3}+\frac{2t^4x}{3}-\frac{t^5x}{3}+\frac{t^6x}{9}-\frac{t^7x}{63}, \tag{47}$$

Therefore, according to Eq.(18) we have:

$$u(x,t) = \left(x - tx + t^2x - t^3x + t^4x - t^5x + \cdots + (-1)^n t^n x + \cdots\right). \tag{48}$$

This has the closed form

$$u(x,t) = \frac{x}{1-t}. \tag{49}$$

which is the exact solution of the problem and it is the same results obtained by VIM [4], RVIM [5] and HPM [6].

5 Conclusion

In this paper, the reliable iterative method namely (NIM or DJM) is implemented to obtain the exact solution for solving Cauchy problems using the initial condition only. The DJM is simple to understand and easy to implement and does not require any restrictive assumptions as required by some existing techniques. The obtained exact solution for the homogeneous or non-homogeneous linear and nonlinear equations of applying the DJM is in full agreement with the results obtained with those methods available in the literature such as variational iteration method [4], Reconstruction of the variational iteration (RVIM) [5] and homotopy perturbation method [6]. The method gives rapid convergent and can be easily comprehended with only a basic knowledge of Calculus. It is economical in terms of computer power/memory and does not involve tedious calculations. Moreover, by solving some examples, it is seems that the DJM appears to be very accurate to employ with reliable results.

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