Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.2, No.5, 2012

Coincidence Points for Mappings under Generalized Contraction

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Abstract

In this paper we establish some results on the existence of coincidence and fixed points for multi-valued and single valued mappings extending the result of *Feng and Liu [2]* and *Liu et.al [5]*. It is also proved with counter example that our results generalize and extend some well known results.

Key Words: common fixed point, Coincidence point, multi-valued mappings.

1. Introduction and Preliminaries

Generalizing Banach Contraction Principle, *Nadler [3]* introduced the concept of multivalued contraction mapping. Let (X, d) be a metric space. Following *Nadler [3]* and *Liu et.al [5]* we follow following notations throughout this paper.

$CB(X)$ (resp.$CL(X)$) denote the family of all closed and bounded (resp. closed) subsets of X. $C(X)$ represents set of all compact subsets of X. The Hausdorff distance for two subsets A, B of X is defined as:

$H(A, B) = max (\{sup\ d(a, B): a \subset A\}, \{sup\ d(A, b): b \subset B\})$

where $d(a, B) = inf \{d(a, b): b \in B\}$,

It is well known that $CB(X)$ (resp. $CL(X)$) is a metric space with Hausdorff distance function.

Let $T: X \to CL(X)$. Using the concept of Hausdorff distance, *Nadler [3]* defined multivalued contraction as following,

$H(Tx, Ty) \leq \alpha d(x, y)\ \forall x, y \in X$ and $\alpha < 1$.

Nadler proved that for a multivalued contraction in a complete metric space there exists a fixed point. Recently *Feng and Liu [2]* and *Liu et.al [5]* generalized the Nadler's result. *Feng and Liu [2]* gave an example to establish that if the mapping T does not satisfy the above contractive condition even then it has a fixed point. *Feng and Liu [2]* generalized the above contractive condition by considering the point $y \in T(x)$ for any $x \in X$ in place of $x, y \in X$ and proved the following result.

Theorem 1 [4]. Let (X, d) be a complete metric space, and let T be a multivalued mapping from X to $CL(X)$. If there exist constant $b, c \in (0, 1), c < b,$ such that for any $x \in X$ there is $y \in T(x)$ satisfying,

$$bd(x, y) \leq f(x), \quad f(y) \leq cd(x, y) \tag{1.1}$$

then T has a fixed point in X provided the function $f(x) = d(x, T(x)), x \sqsubseteq X$ is lower semi continuous.

Generalizing above result and the result of Ciric [1], *Liu et.al [5]* relax the contractive condition by taking $\alpha(f(x))$ and $\beta(f(x))$ in place of constant b and c, where

$$\alpha: B \to (0,1],\ \beta: B \to [0, 1) \text{ and } B = \begin{cases} [0, sup f(x)], & if\ sup\ f(x) < +\infty \\ [0, +\infty), & if\ sup \emptyset(x) = +\infty \end{cases} \quad \ldots(A)$$

In this paper we extend the result of [2] and [5] for the existence of coincidence points.

2. Main Result

Let (X, d) be a metric space, $T: X \to CL(X)$ and $f: X \to X$. An orbit of the multivalued map T at a point x_0 in X is a sequence $\{x_n : x_n \in Tx_{n-1}, n = 1, 2, 3 \ldots\}$. The space X is T –orbitally complete if every Cauchy sequence of the from $\{x_{n_i} : x_{n_i} \in Tx_{n_i-1}\}$ converges in X. If for a point x_0 in X, there exists a sequence $\{x_n\} \subset X$ such that $fx_{n+1} \in Tx_n$, $n = 0, 1, 2, \ldots$, then $O_f(x_0) = \{fx_n : n = 1, 2 \ldots\}$ is an orbit of (T, f) at x_0. A space X is (T, f) –orbitally complete if every Cauchy sequence of the form $\{fx_{n_i} : fx_{n_i} \in Tx_{n_i-1}\}$ converges in X. A function $\emptyset : X \to R$ such that $\emptyset(x) = d(fx, Tx)$ is called (T, f) -orbitally lower semi continuous if for any point $z \in X$ $\exists$ an orbit $\{f(x_n)\}$ of (T, f) with $\lim fx_n = fz$ implying that $\emptyset(z) \le \lim_{n\to\infty} \emptyset(x_n)$.

Theorem 2.1

Let (X, d) be a metric space. $T: X \to CL(X)$ and $f: X \to X$ such that $T(X) \subseteq f(X)$ and $f(X)$ is (T, f) -orbitally complete. If for any $x \subset X$ there exists $y \subset X$ such that $f(y) \subset T(x)$ and

$\alpha(\emptyset(x))d(fx, fy) \le \emptyset(x)$ and $\emptyset(y) \le \beta((\emptyset(x))d(fx, fy)$

where α and β are defined as (A) satisfying

$$\lim_{r\to 0^+} \alpha(r) > 0,\ \lim_{r\to t^+} \frac{\sup\beta(r)}{\alpha(r)} < 1\ \ \forall t \in [0, \sup\emptyset(x)), \tag{2.1}$$

and the function $\emptyset$ is (T, f) -orbitally lower semi continuous at z. Then there exist a coincidence point z of f and T.

Proof: Let $\gamma(t) = \frac{\beta(t)}{\alpha(t)}$, $\forall\ t \in [0, \sup\emptyset(x))$ (2.2)

Let $x_0 \in X$, since $T(x) \subseteq f(X)$ we choose $x_1 \in X$ so that $fx_1 \in Tx_0$,

and $\alpha(\emptyset(x_0))\, d(fx_0, fx_1) \le \emptyset(x_0) = d(fx_0, Tx_0)$,

$\emptyset(x_1) = d(fx_1, Tx_1) \le \beta(\emptyset(x_0))\, d(fx_0, fx_1)$

implies

$$\emptyset(x_1) \le \beta(\emptyset(x_0)) \frac{\emptyset(x_0)}{\alpha(\emptyset(x_0))}.$$

Using (2.2) we get

$$\emptyset(x_1) \le \beta(\emptyset(x_0)) \frac{\emptyset(x_0)}{\alpha(\emptyset(x_0))} = \gamma(\emptyset(x_0))\,(\emptyset(x_0)$$

continuing the process we get an orbit $\{fx_n\}_{n\ge 0}$ of T satisfying

$\alpha(\emptyset(x_n))\, d(fx_n, fx_{n+1}) \le \emptyset(x_n) = d(fx_n, Tx_n)$ and

$$\emptyset(x_{n+1}) = d(fx_{n+1}, Tx_{n+1}) \le \beta(\emptyset(x_n))\, d(fx_n, fx_{n+1}), \quad \forall n \ge 0. \tag{2.3}$$

Using (2.2) we get

$$\emptyset(x_{n+1}) \le \beta(\emptyset(x_n)) \frac{\emptyset(x_n)}{\alpha(\emptyset(x_n))} = \gamma(\emptyset(x_n))\,(\emptyset(x_n)). \tag{2.4}$$

Since $0 \le \gamma(t) < 1$ and by (2.4) it is clear that $\{\emptyset x_n\}_{n\ge 0}$ is a nonnegative and decreasing sequence. Hence $\emptyset(x_n)$ is convergent.

$$\text{Let } \lim_{n\to\infty} \emptyset(x_n) = a \tag{2.5}$$

where $a \geq 0$, suppose $a > 0$, taking limit $n \to \infty$ in (2.4) and by (2.1), (2.2) and (2.5)

$$a = \lim_{n\to\infty} \sup \emptyset(x_{n+1}) \leq \lim_{n\to\infty}[\gamma(\emptyset(x_n))\emptyset(x_n)]$$

$$\leq \lim_{n\to\infty} \sup \gamma(\emptyset(x_n)) \lim_{n\to\infty} \sup \emptyset(x_n)$$

$$= a \lim_{n\to\infty} \sup \gamma(\emptyset(x_n)) < a$$

Which is a contradiction hence $a = 0$

i.e., $\lim_{n\to\infty} \emptyset(x_n) = 0.$ (2.6)

To prove that $\{fx_n\}$, $n \geq 0$ is a Cauchy sequence.

Let $b = \lim_{n\to\infty} \sup \gamma(\emptyset(x_n))$, $c = \lim_{n\to\infty} \inf \alpha(\emptyset(x_n))$ (2.7)

Then from (2.1), (2.2) and (2.7)
$0 \leq b < 1$, $c > 0$.
Let $p \in (0, c)$, $q \in (b, 1)$ then from (2.7)

$$\gamma(\emptyset(x_n)) < q, \quad \alpha(\emptyset(x_n)) > p, \quad \forall n \geq 0$$

which together with (2.3) and (2.4) gives

$$\emptyset(x_{n+1}) \leq q\emptyset(x_n), \quad d(fx_n, fx_{n+1}) \leq \frac{\emptyset(x_n)}{p}$$

Calculating similar calculation we get

$$\emptyset(x_{n+1}) \leq q^{n+1-n_0} fx_{n_0}, \quad d(fx_n, fx_{n+1}) \leq \frac{\emptyset(x_{n})}{p} q^{n-n_0}.$$

which gives

$$d(fx_n, fx_m) \leq \sum_{k=1}^{n-1} d(fx_k, fx_{k+1}) \leq \frac{\emptyset(x_{n_0})}{p} \sum_{k=n}^{n-1} q^{k-n_0} \leq \frac{\emptyset(x_{n_0})}{p(1-q)} q^{n+1-n_0} \quad (2.8)$$

Since q < 1 therefore (2.8) implies that $\{fx_n\}$ is a Cauchy sequence. And since $f(X)$ is (T, f) -orbitally complete

$\exists\, z \in X$ such that

$\lim_{n\to\infty} f(x_n) = fz$.

Now we will prove that z is coincidence point of f and T.
Since $\emptyset$ is (T, f) -orbitally lower semi continuous therefore
$0 \leq d(fz, Tz) = \emptyset(z) \leq \lim_{n\to\infty} \emptyset(x_n) = 0$ (by 2.6)
$\Longrightarrow \emptyset(z) = 0,$

$\Longrightarrow d(fz, Tz) = 0$

$\Longrightarrow f(z) \in T(z)$ i.e. f and T have a coincidence point .

In Theorem 2.1, taking constants α and β in place of $\alpha(\emptyset(x))$ and $\beta((\emptyset(x))$ respectively we get following result as a corollary. The following corollary is also serves as a generalization of Singh and Kulsrestha[4].

Corollary 2.1.

Let (X, d) be a metric space. $T: X \to CL(X)$ and $f: X \to X$ are mappings such that $T(X) \subseteq f(X)$ and $f(X)$ is (T, f) -orbitally complete. If for any $x \in X$ there is $y \subset X$ such that $f(y) \subset T(x)$ satisfying

$\beta d(fx, fy) \leq \emptyset(x)$ And $\emptyset(y) \leq \alpha d(fx, fy)$

where α, $\beta \in (0, 1)$ and $\alpha < \beta$ and the function $\emptyset$ is is lower semi continuous. Then T and f has a coincidence point in X.

Proof

Let $x_0 \in X$, since $T(x) \subseteq f(X)$ we choose $x_1 \in X$ such that $fx_1 \in Tx_0$.

By the given contractive condition

$\beta\, d(fx_0, fx_1) \leq \emptyset(x_0) = d(fx_0, Tx_0)$ and

$\emptyset(x_1) = d(fx_1, Tx_1) \leq \alpha\, d(fx_0, fx_1)$.

In similar way we choose $x_{n+1} \in X$ such that $f(x_{n+1}) \in T(x_n)$ and

$d(fx_{n+1}, Tx_{n+1}) \leq \alpha\, d(fx_n, fx_{n+1})$, $\beta d(fx_n, fx_{n+1}) \leq \emptyset(x_n) = d(fx_n, Tx_n)$

which implies

$$d(fx_{n+1}, Tx_{n+1}) \leq \frac{\alpha}{\beta} d(fx_n, Tx_n)$$

or $$d(fx_{n+1}, fx_{n+2}) \leq \frac{\alpha}{\beta} d(fx_n, fx_{n+1})$$

$$\Longrightarrow d(fx_n, Tx_n) \leq \left(\frac{\alpha}{\beta}\right)^n d(fx_0, Tx_0)$$

Or $$d(fx_n, fx_{n+1}) \leq \left(\frac{\alpha}{\beta}\right)^n d(fx_0, fx_1). \tag{2.9}$$

Using (2.9) for $m, n \in N$, $m>n$,

$$d(fx_m, fx_n) \leq d(fx_m, fx_{m-1}) + d(fx_{m-1}, fx_{m-2}) + \ldots + d(fx_{n+1}, fx_n)$$

$$\leq \left(\frac{\alpha}{\beta}\right)^{m-1} d(fx_0, fx_1) + \left(\frac{\alpha}{\beta}\right)^{m-2} d(fx_0, fx_1) + \left(\frac{\alpha}{\beta}\right)^{m-3} d(fx_0, fx_1) + \ldots + \left(\frac{\alpha}{\beta}\right)^{n} d(fx_0, fx_1)$$

$\leq \frac{(\frac{\alpha}{\beta})^n}{1-(\frac{\alpha}{\beta})} d(fx_0, fx_1)$ (2.10)

As $n \to \infty$, $(\frac{\alpha}{\beta})^n \to 0$

hence $\{fx_n\}$ is a Cauchy sequence. Since $f(X)$ is (T, f) -orbitally complete there exists $z \in X$ such that

$\{fx_n\}_{n=0}^{\infty}$ converges to fz.

Now using the condition of (T, f) –orbitally lower semi continuity of $\emptyset$ we can be easily prove that z is coincidence point of f and T.

In theorem 2.1 taking C(X) in place of CL(X) we get we get following result as corollary.

Corollary 2.2

Let (X, d) be a metric space. $T: X \to C(X)$ and $f: X \to X$ such that T and f satisfy all conditions as in theorem 2.1 then there exist a coincidence point z of f and T.

Proof.

Proof is same as of theorem 2.1.

In corollary 2.1 taking $C(X)$ in place of $CL(X)$ we get we get following result as corollary.

Corollary 2.3

Let (X, d) be a metric space. $T: X \to C(X)$ and $f: X \to X$ are mappings such that all conditions as in corollary 2.1 are satisfied then T and f has a coincidence point in X.

Proof

Proof is same as of corollary 2.1.

Example

Let $X = \{\frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots\} \cup \{0,1\}$, $d(x, y) = |x - y|$, for $x, y \in X$; then X is complete metric space. Define mapping $T: X \to CL(X)$ as

$$T(x) = \begin{cases} \left\{\frac{1}{2^{2n+2}}, 1\right\}, x = \frac{1}{2^n}, n = 0, 1, 2, \ldots \\ \left\{0, \frac{1}{2}\right\}, x = 0 \end{cases}$$

and $f(x) = x^2$, $x \in X$.

Obviously, T and f does not satisfy hybrid contraction condition [4].

$$H\left(T\left(\frac{1}{2^{2n}}\right), T(0)\right) = \frac{1}{2} \geq \frac{1}{2^{2n}} = \left|\frac{1}{2^{2n}} - 0\right| = d\left(\frac{1}{2^{2n}}, 0\right), n = 1, 2..$$

On the other hand, we have

$$\emptyset(x) = d(fx,\ T(x)) = \begin{cases} \frac{1}{2^{2n+2}}, x = \frac{1}{2^n}, n = 1, 2 \ldots \\ 0, x = 0, 1 \end{cases}$$

It shows that $\emptyset$ is continuous,

Further, there exists $y \in X$ for any $x \in X$ such that

$$\frac{1}{2} d(fx, fy) \leq \emptyset(x)$$

$$d(fy,\ Ty) \leq \frac{1}{2^{\frac{2}{2}}} d(fx, fy).$$

Then from corollary *2*.1 there exist a coincidence point of *f* and *T*.

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