Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.2, No.5, 2012

Characterization of Trees with Equal Total Edge Domination and Double Edge Domination Numbers

M.H.Muddebihal A.R.Sedamkar*
Department of Mathematics, Gulbarga University, Gulbarga-585106, Karnataka, INDIA
E-mail of the corresponding author: mhmuddebihal@yahoo.co.in

Abstract

A total edge dominating set of a graph G is a set D of edges of G such that the sub graph $\langle D \rangle$ has no isolated edges. The total edge domination number of G denoted by $\gamma_t^{'}(G)$, is the minimum cardinality of a total edge dominating set of G. Further, the set D is said to be double edge dominating set of graph G. If every edge of G is dominated by at least two edges of D. The double edge domination number of G, denoted by, $\gamma_d^{'}(G)$, is the minimum cardinality of a double edge dominating set of G. In this paper, we provide a constructive characterization of trees with equal total edge domination and double edge domination numbers.

Key words: Trees, Total edge domination number, Double edge domination number.

1. Introduction:

In this paper, we follow the notations of [2]. All the graphs considered here are simple, finite, non-trivial, undirected and connected graph. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively.

In general, we use $\langle X \rangle$ to denote the sub graph induced by the set of vertices X and $N(v)$ and $N[v]$ denote the open and closed neighborhoods of a vertex v, respectively.

The degree of an edge $e = uv$ of G is defined by $\deg e = \deg u + \deg v - 2$, is the number of edges adjacent to it. An edge e of degree one is called end edge and neighbor is called support edge of G.

A strong support edge is adjacent to at least two end edges. A star is a tree with exactly one vertex of degree greater than one. A double star is a tree with exactly one support edge.

For an edge e is a rooted tree T, let $C(e)$ and $S(e)$ denote the set of childrens and descendants of e respectively. Further we define $S[e]=S(e)\cup\{e\}$. The maximal sub tree at e is the sub tree of T induced by $S[e]$, and is denoted by T_e.

A set $D \subseteq E$ is said to be total edge dominating set of G, if the sub graph $\langle D \rangle$ has no isolated edges. The total edge domination number of G, denoted by $\gamma_t^{'}(G)$, is the minimum cardinality of a total edge dominating set of G. Total edge domination in graphs was introduced by S.Arumugam and S.Velammal [1].

A set $S \subseteq V$ is said to be double dominating set of G, if every vertex of G is dominated by at least two vertices of S. The double domination number of G, denoted by $\gamma_d(G)$, is the minimum cardinality of a double dominating set of G. Double domination is a graph was introduced by F. Harary and T. W. Haynes [3]. The concept of domination parameters is now well studied in graph theory (see [4] and [5]).

Analogously, a set $D \subseteq E$ is said to be double edge dominating set of G, if every edge of G is dominated by at least two edges of D. The double edge domination number of G, denoted by $\gamma_d^{'}(G)$, is the minimum cardinality of a double edge dominating set of G.

In this paper, we provide a constructive characterization of trees with equal total edge domination and double edge domination numbers.

2. Results:

Initially we obtain the following Observations which are straight forward.

Observation 2.1: Every support edge of a graph G is in every $\gamma_t^{'}(G)$ set.

Observation 2.2: Suppose every non end edge is adjacent to exactly two end edge, then every end edge of a graph G is in every $\gamma_d^{'}(G)$ set.

Observation 2.3: Suppose the support edges of a graph G are at distance at least three in G, then every support edge of a graph G is in every $\gamma_d^{'}(G)$.

3. Main Results:

Theorem 3.1: For any tree T, $\gamma_d^{'}(T) \geq \gamma_t^{'}(T)$.

Proof: Let q be the number of edges in tree T. We proceed by induction on q. If $diam(T) \leq 3$. Then T is either a star or a double star and $\gamma_d^{'}(T) = 2 = \gamma_t^{'}(T)$. Now assume that $diam(T) \geq 4$ and the Theorem is true for every tree $T^{'}$ with $q^{'} < q$. First assume that some support edge of T, say e_x is strong. Let e_y and e_z be the end edges adjacent to e_x and $T^{'} = T - e$. Let $D^{'}$ be any $\gamma_d^{'}(T^{'})$- set. Clearly $e_x \in D^{'}$, where $D^{'}$ is a total edge dominating set of tree T. Therefore $\gamma_t^{'}(T^{'}) \leq \gamma_t^{'}(T)$. Now let S be any $\gamma_d^{'}(T)$ - set. By observations 2 and 3, we have $e_y, e_x, e_z \in S$. Clearly, $S - \{e_y\}$ is a double edge dominating set of tree $T^{'}$. Therefore $\gamma_d^{'}(T^{'}) \leq \gamma_d^{'}(T) - 1$. Clearly, $\gamma_d^{'}(T) \geq \gamma_d^{'}(T^{'}) + 1 \geq \gamma_t^{'}(T^{'}) + 1 \geq \gamma_t^{'}(T) + 1 \geq \gamma_t^{'}(T)$, a contradiction. Therefore every support edge of T is weak.

Let T be a rooted tree at vertex r which is incident with edge e_r of the $diam(T)$. Let e_t be the end edge at maximum distance from e_r, e be parent of e_t, e_u be the parent of e and e_w be the parent of e_u in the rooted tree. Let T_{e_x} denotes the sub tree induced by an edge e_x and its descendents in the rooted tree.

Assume that $\deg_T(e_u) \geq 3$ and e_u is adjacent to an end edge e_x. Let $T^{'} = T - e$ and $D^{'}$ be the $\gamma_t^{'}(T^{'})$- set. By Observation 1, we have $e_u \in D^{'}$. Clearly, $D^{'} \cup \{e\}$ is a total edge dominating set of tree T. Thus $\gamma_t^{'}(T) \leq \gamma_t^{'}(T^{'}) + 1$. Now let S be any $\gamma_d^{'}(T)$- set. By Observations 2 and 3, $e_t, e_x, e, e_u \in S$. Clearly, $S - \{e, e_t\}$ is a double edge dominating set of tree $T^{'}$. Therefore $\gamma_d^{'}(T^{'}) \leq \gamma_d^{'}(T) - 2$. It follows that $\gamma_d^{'}(T) \geq \gamma_d^{'}(T^{'}) + 2 \geq \gamma_t^{'}(T^{'}) + 2 \geq \gamma_t^{'}(T) + 1 \geq \gamma_t^{'}(T)$.

Now assume that among the decedents of e_u there is a support edge say e_x, which is different from e. Let $T' = T - e$ and D' be the $\gamma_t'(T')$ - set containing no end edges. Since e_x must have a neighbor in D', thus $e_u \in D'$. Clearly $D' \cup \{e\}$ is a total edge dominating set of tree T and hence $\gamma_t'(T) \le \gamma_t'(T') + 1$. Now let S be any $\gamma_d'(T)$ - set. By Observations 2 and 3, we have $e_t, e, e_x \in S$. If $e_u \in S$, then $S - \{e, e_t\}$ is the double edge dominating set of tree T'. Further assume that $e_u \notin S$. Then $S \cup \{e_u\} - \{e, e_t\}$ is a double edge dominating set of tree T. Therefore $\gamma_d'(T') \le \gamma_d'(T) - 1$. Clearly, it follows that, $\gamma_d'(T) \ge \gamma_d'(T') + 1 \ge \gamma_t'(T') + 1 \ge \gamma_t'(T)$. Therefore, we obtain $\gamma_d'(T) \ge \gamma_t'(T)$.

To obtain the characterization, we introduce six types of operations that we use to construct trees with equal total edge domination and double edge domination numbers.

Type 1: Attach a path P_1 to two vertices u and w which are incident with e_u and e_w respectively of T where e_u, e_w located at the component of $T - e_x e_y$ such that either e_x is in γ_d' set of T or e_y is in γ_d' - set of T.

Type 2: Attach a path P_2 to a vertex v incident with e of tree T, where e is an edge such that $T - e$ has a component P_3.

Type 3: Attach $k \ge 1$ number of paths P_3 to the vertex v which is incident with an edge e of T where e is an edge such that either $T - e$ has a component P_2 or $T - e$ has two components P_2 and P_4, and one end of P_4 is adjacent to e is T.

Type 4: Attach a path P_3 to a vertex v which is incident with e of tree T by joining its support vertex to v, such that e is not contained is any γ_t' - set of T.

Type 5: Attach a path $P_4(n)$, $n \ge 1$ to a vertex v which is incident with an edge e, where e is in a γ_d' - set of T if $n = 1$.

Type 6: Attach a path P_5 to a vertex v incident with e of tree T by joining one of its support to v such that $T-e$ has a component $H \in \{P_3, P_4, P_6\}$.

Now we define the following families of trees

Let $\Im$ be the family of trees with equal total edge domination number and double edge domination number. That is $\Im = \{T / T$ is a tree satisfying $\gamma_t^{'}(T) = \gamma_d^{'}(T)\}$.

We also define one more family as

$\Re = \{T / T$ is obtained from P_3 by a finite sequence of type - i operations where

$1 \le i \le 5\}$.

We prove the following Lemmas to provide our main characterization.

Lemma 3.2: If $T^{'} \in \Im$ and T is obtained from $T^{'}$ by Type-1 operation, then $T \in \Im$.

Proof: Since $T^{'} \in \Im$, we have $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$. By Theorem 1, $\gamma_t^{'}(T) \le \gamma_d^{'}(T)$, we only need to prove that $\gamma_t^{'}(T) \ge \gamma_d^{'}(T)$. Assume that T is obtained from $T^{'}$ by attaching the path P_1 to two vertices u and w which are incident with e_u and e_w as $e_u^{'}$ and $e_w^{'}$ respectively. Then there is a path $e_x e_y$ in $T^{'}$ such that either e_x is in $\gamma_d^{'}$ - set of T and $T^{'} - e_x e_y$ has a component $P_5 = e_u e e_w e_x$ or e_y is in $\gamma_d^{'}$ - set of $T^{'}$ and $T^{'} - e_x e_y$ has a component $P_6 = e_u e e_w e_x e_x^{'}$. Clearly, $\gamma_t^{'}(T^{'}) = \gamma_t^{'}(T) - 1$.

Suppose $T^{'} - e_x e_y$ contains a path $P_5 = e_u e e_w e_x$ then $S^{'}$ be the $\gamma_d^{'}$ - set of $T^{'}$ containing e_x. From Observation 2 and by the definition of $\gamma_d^{'}$ - set, we have $S^{'} \cap \{e_u, e, e_w, e_x\} = \{e_u, e_w\}$ or $\{e_u, e\}$. Therefore

$S=(S^{'}-\{e_u,e,e_w\})\bigcup\{e_u^{'},e,e_w^{'}\}$ is a double edge dominating set of T with $|S|=|S^{'}|+1=\gamma_d^{'}(T^{'})+1$. Clearly, $\gamma_t^{'}(T)=\gamma_t^{'}(T^{'})+1=\gamma_d^{'}(T^{'})+1=|S|>\gamma_d^{'}(T)$.

Now, if $T^{'}-e_xe_y$ contains a path $P_6=e_ue\,e_we_xe_x^{'}$. Then $S^{'}$ be the $\gamma_d^{'}$ - set of $T^{'}$ containing e_y. By Observation 2 and by definition of $\gamma_d^{'}$ - set, we have $S^{'}\bigcap\{e_u,e,e_w,e_x,e_x^{'}\}=\{e_u,e_w,e_x^{'}\}$. Therefore $S=(S^{'}-\{e_u,e_w\})\bigcup\{e_u^{'},e,e_w^{'}\}$ is a double edge dominating set of T with $|S|=|S^{'}|+1=\gamma_d^{'}(T^{'})+1$. Clearly, $\gamma_t^{'}(T)=\gamma_t^{'}(T^{'})+1=\gamma_d^{'}(T^{'})+1=|S|\geq\gamma_d^{'}(T)$.

Lemma 3.3: If $T^{'}\in\Im$ and T is obtained from $T^{'}$ by Type-2 operation, then $T\in\Im$.

Proof: Since $T^{'}\in\Im$, we have $\gamma_t^{'}(T^{'})=\gamma_d^{'}(T^{'})$. By Theorem 1, $\gamma_t^{'}(T)\leq\gamma_d^{'}(T)$, we only need to prove that $\gamma_t^{'}(T)\geq\gamma_d^{'}(T)$. Assume that T is obtained from $T^{'}$ by attaching a path P_2 to a vertex v which is incident with e of $T^{'}$ where $T^{'}-e$ has a component $P_3=e_we_x$. We can easily show that $\gamma_t^{'}(T)=\gamma_t^{'}(T^{'})+1$. Now by definition of $\gamma_d^{'}$ - set, there exists a $\gamma_d^{'}$ - set, $D^{'}$ of $T^{'}$ containing the edge e. Then $D^{'}\bigcup\{e_u^{'}\}$ forms a double edge dominating set of T. Therefore $\gamma_t^{'}(T)=\gamma_t^{'}(T^{'})+1=\gamma_d^{'}(T^{'})+1=|D^{'}\bigcup\{e_u^{'}\}|\geq\gamma_d^{'}(T)$.

Lemma 3.4: If $T^{'}\in\Im$ and T is obtained from $T^{'}$ by Type - 3 operation, then $T\in\Im$.

Proof: Since $T^{'}\in\Im$, we have $\gamma_t^{'}(T^{'})=\gamma_d^{'}(T^{'})$. By Theorem 1, $\gamma_t^{'}(T)\leq\gamma_d^{'}(T)$, hence we only need to prove that $\gamma_t^{'}(T)\geq\gamma_d^{'}(T)$. Assume that T is obtained from $T^{'}$ by attaching $m\geq 1$ number of paths P_3 to a vertex v which is incident with an edge e of $T^{'}$ such that $T^{'}-e$ has a component P_3 or two components P_2 and P_4. By definition of $\gamma_t^{'}$- set and $\gamma_d^{'}$- set, we can easily show that $\gamma_t^{'}(T)\geq\gamma_t^{'}(T^{'})+2m$ and $\gamma_d^{'}(T^{'})+2m\geq\gamma_d^{'}(T)$. Since $\gamma_t^{'}(T^{'})=\gamma_d^{'}(T^{'})$, it follows that $\gamma_t^{'}(T)\geq\gamma_t^{'}(T^{'})+2m=\gamma_d^{'}(T^{'})+2m\geq\gamma_d^{'}(T)$.

Lemma 3.5: If $T^{'} \in \Im$ and T is obtained from $T^{'}$ by Type - 4 operation, then $T \in \Im$.

Proof: Since $T^{'} \in \Im$, we have $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$. By Theorem 1, $\gamma_t^{'}(T) \le \gamma_d^{'}(T)$, hence we only need to prove that $\gamma_t^{'}(T) \ge \gamma_d^{'}(T)$. Assume that T is obtained from $T^{'}$ by attaching path P_3 to a vertex v incident with e in $T^{'}$ such that e is not contained in any $\gamma_t^{'}$-set of $T^{'}$ and $T^{'} - e$ has a component P_4. For any $\gamma_d^{'}$ - set, $S^{'}$ of $T^{'}$, $S^{'} \cup \{e_x, e_z\}$ is a double edge dominating set of T. Hence $\gamma_d^{'}(T^{'}) + 2 \ge \gamma_d^{'}(T)$. Let D be any $\gamma_t^{'}$ - set of T containing the edge e_u, which implies $e_y \in D$ and $|D \cap \{e, e_x, e_z\}| = 1$.

If $e \notin D$, then $|D \cap E(T^{'})| = |D| - 2 = \gamma_t^{'}(T) - 2 \ge \gamma_t^{'}(T^{'})$, since $D \cap E(T^{'})$ is a total edge dominating set of $T^{'}$. Further since $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$, it follows that $\gamma_t^{'}(T) \ge \gamma_t^{'}(T^{'}) + 2 = \gamma_d^{'}(T^{'}) + 2 \ge \gamma_d^{'}(T)$.

If $e \in D$, then $D \cap \{e, e_x, e_z\} = \{e\}$ and $|D \cap E(T^{'})| = |D| - 1 = \gamma_t^{'}(T) - 1 \ge \gamma_t^{'}(T^{'})$, since $D \cap E(T^{'})$ is a total edge dominating set of $T^{'}$. Suppose $\gamma_t^{'}(T) \le \gamma_d^{'}(T) - 1$, then by $\gamma_d^{'}(T^{'}) = \gamma_t^{'}(T^{'})$, it follows that $\gamma_d^{'}(T) \ge \gamma_t^{'}(T) + 1 \ge \gamma_t^{'}(T^{'}) + 2 = \gamma_d^{'}(T^{'}) + 2 \ge \gamma_d^{'}(T^{'})$. Clearly, $|D \cap E(T^{'})| = \gamma_t^{'}(T) - 1 = \gamma_t^{'}(T^{'})$ and $D \cap E(T^{'})$ is a total edge dominating set of $T^{'}$ containing e. Therefore, it gives a contradiction to the fact that e is not in any total edge dominating set of $T^{'}$ and hence $\gamma_t^{'}(T) \ge \gamma_d^{'}(T)$.

Lemma 3.6: If $T^{'} \in \Im$ and T is obtained from $T^{'}$ by Type - 5 operation, then $T \in \Im$.

Proof: Since $T^{'} \in \Im$, we have $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$. By Theorem 1, $\gamma_t^{'}(T) \le \gamma_d^{'}(T)$, hence we only need to prove that $\gamma_t^{'}(T) \ge \gamma_d^{'}(T)$. Assume that T is obtained from $T^{'}$ by attaching path $P_4(n)$, $n \ge 1$ to a vertex v incident with e in $T^{'}$ such that e is in $\gamma_d^{'}$ - set for $n=1$. Clearly, $\gamma_t^{'}(T) \geq \gamma_t^{'}(T^{'}) + 2n$. If $n \geq 2$, then by $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$, it is obvious that $\gamma_t^{'}(T) \geq \gamma_t^{'}(T^{'}) + 2n = \gamma_d^{'}(T^{'}) + 2n \geq \gamma_d^{'}(T)$. If $n=1$, then $D^{'}$ be a $\gamma_d^{'}$ - set of $T^{'}$ containing e. Thus $D^{'} \cup \{e_z, e_x\}$ is a double edge dominating set of T. Hence $\gamma_t^{'}(T) \geq \gamma_t^{'}(T^{'}) + 2 = \gamma_d^{'}(T^{'}) + 2 = \left| S^{'} \cup \{e_z, e_x\} \right| \geq \gamma_d^{'}(T)$.

Lemma 3.7: If $T^{'} \in \Im$ and T is obtained from $T^{'}$ by Type - 6 operation, then $T \in \Im$.

Proof: Since $T^{'} \in \Im$, we have $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$. By Theorem 1, $\gamma_t^{'}(T) \leq \gamma_d^{'}(T)$, hence we only need to prove that $\gamma_t^{'}(T) \geq \gamma_d^{'}(T)$. Assume that T is obtained from $T^{'}$ by attaching path a path P_5 to a vertex v which is incident with e. Then there exists a subset D of $E(T)$ as $\gamma_t^{'}$- set of T such that $D \cap N_{T^{'}}(e) \neq \phi$ for $n=1$. Therefore $D \cap E(T^{'})$ is a total edge dominating set of $T^{'}$ and $\left| D \cap E(T^{'}) \right| \geq \gamma_t^{'}(T^{'})$. By the definition of double edge dominating set, we have $\gamma_d^{'}(T^{'}) + 3 \geq \gamma_d^{'}(T)$. Clearly, it follows that

$$\gamma_t^{'}(T) = |D| = |D \cap E(P_6)| + |D \cap E(T^{'})| > 3 + \gamma_t^{'}(T^{'}) = 3 + \gamma_d^{'}(T^{'}) \geq \gamma_d^{'}(T).$$

Now we define one more family as

Let T be the rooted tree. For any edge $e \in E(T)$, let $C(e)$ and $F(e)$ denote the set of children edges and descendent edges of e respectively. Now we define

$$C^{'}(e) = \{e_u \in C(e) \mid \text{every edge of } F[e_u] \text{ has a distance at most two from } e \text{ in } T\}.$$

Further partition $C^{'}(e)$ into $C_1^{'}(e)$, $C_2^{'}(e)$ and $C_3^{'}(e)$ such that every edge of $C_i^{'}(e)$ has edge degree i in T, $i = 1, 2$ and 3.

Lemma 3.8: Let T be a rooted tree satisfying $\gamma_t^{'}(T) = \gamma_d^{'}(T)$ and $e_w \in E(T)$. We have the following conditions:

1. If $C^{'}(e_w) \neq \phi$, then $C_1^{'}(e_w) = C_3^{'}(e_w) = \phi$.
2. If $C_3^{'}(e_w) \neq \phi$, then $C_1^{'}(e_w) = C_2^{'}(e_w) = \phi$ and $|C_3^{'}(e_w)| = 1$.
3. If $C(e_w) = C^{'}(e_w) \neq C_1^{'}(e_w)$, then $C_1^{'}(e_w) = C_3^{'}(e_w) = \phi$.

Proof: Let $C_1^{'}(e_w) = \{e_{x_1}, e_{x_2}, ...e_{x_l}\}$, $C_2^{'}(e_w) = \{e_{y_1}, e_{y_2}, ...e_{y_m}\}$ and $C_3^{'}(e_w) = \{e_{z_1}, e_{z_2}, ...e_{z_n}\}$ such that $|C_1^{'}(e_w)| = l$, $|C_2^{'}(e_w)| = m$ and $|C_3^{'}(e_w)| = n$. For every $i = 1, 2, ..., n$, let e_{u_i} be the end edge adjacent to e_{z_i} in T and $T^{'} = T - \{e_{x_1}, e_{x_2}, ..., e_{x_l}, e_{u_1}, e_{u_2}, ..., e_{u_n}\}$.

For (1): We prove that if $m \geq 1$, then $l + n = 0$. Assume $l + n \geq 1$. Since $m \geq 1$, we can have a $\gamma_d^{'}$ - set S of T such that $e_w \in S$ and a $\gamma_t^{'}$ - set $D^{'}$ of $T^{'}$ such that $e_w \in D^{'}$. Clearly $S - \{e_{x_1}, e_{x_2}, ..., e_{x_l}, e_{u_1}, e_{u_2}, ..., e_{u_n}\}$ is a double edge dominating set of $T^{'}$ and $D^{'}$ is a total edge dominating set of T. Hence $\gamma_t^{'}(T^{'}) = |D^{'}| \geq \gamma_t^{'}(T) = \gamma_d^{'}(T) = |S| > |S - \{e_{x_1}, e_{x_2}, ..., e_{x_l}, e_{u_1}, e_{u_2}, ..., e_{u_n}\}| \geq \gamma_d^{'}(T^{'})$, it gives a contradiction with Theorem 1.

For (2) and (3): Either if $C_3^{'}(e_w) \neq \phi$ or if $C(e_w) = C^{'}(e_w) \neq C_1^{'}(e_w)$. Then for both cases, $m + n \geq 1$. Now select a $\gamma_t^{'}$ - set $D^{'}$ of $T^{'}$ such that $e_w \in D^{'}$. Then $D^{'}$ is also a total edge dominating set of T. Hence $\gamma_t^{'}(T^{'}) = |D^{'}| \geq \gamma_t^{'}(T)$. Further by definition of $\gamma_d^{'}$ - set and by Observation 2, there exists a $\gamma_d^{'}$ -set S of T which satisfies $S \cap \{e_{y_1}, e_{y_2}, ..., e_{y_m}, e_{z_1}, e_{z_2}, ..., e_{z_n}\} = \phi$. Then $(S \cap E(T^{'})) \cup \{e_w\}$ is a $\gamma_d^{'}$ -set of $T^{'}$. Hence $\gamma_d^{'}(T^{'}) \leq |(S \cap E(T^{'})) \cup \{e_w\}| \leq |S| - (l + n) + 1 = \gamma_d^{'}(T) - (l + n) + 1 = \gamma_t^{'}(T) - (l + n) + 1$.

If $n \geq 1$, then $\gamma_d^{'}(T^{'}) \leq \gamma_t^{'}(T) \leq \gamma_t^{'}(T^{'}) \leq \gamma_d^{'}(T^{'})$, the last inequality is by Theorem 1. It follows that $l + n = 1$ and $e_w \notin S$. Therefore $l = 0$ and $n = 1$. From Condition 1, we have $m = 0$. Hence 2 follows.

If $C(e_w)=C^{'}(e_w)\neq C_1^{'}(e_w)$, then $m+n\geq 1$. By conditions 1 and 2, $l=0$. Now we show that $n=0$. Otherwise, similar to the proof of 2, we have $e_w\notin S$, $n=1$ and $m=0$. Since $C(e_w)=C^{'}(e_w)$ and $\deg(e_w)=2$, for double edge domination, $e_w, e_z\in S$, a contradiction to the selection of S.

Lemma 3.9: If $T\in\Im$ with at least three edges, then $T\in\Re$.

Proof: Let $q=|E(T)|$. Since $T\in\Im$, we have $\gamma_t^{'}(T)=\gamma_d^{'}(T)$. If $diam(T)\leq 3$, then T is either a star or a double star and $\gamma_t^{'}(T)=2=\gamma_d^{'}(T)$. Therefore $T\in\Re$. If $diam(T)\geq 4$, assume that the result is true for all trees $T^{'}$ with $|E(T^{'})|=q^{'}<q$.

We prove the following Claim to prove above Lemma.

Claim 3.9.1: If there is an edge $e_a\in E(T)$ such that $T-e_a$ contains at least two components P_3, then $T\in\Re$.

Proof: Assume that $P_3=e_b\,e_b^{'}$ and $e_c\,e_c^{'}$ are two components of $T-e_a$. If $T^{'}=T-\{e_b,e_b^{'}\}$, then use $D^{'}$ and S to denote the $\gamma_t^{'}$-set of $T^{'}$ containing e_a and $\gamma_d^{'}$ - set of T, respectively. Since $e_a\in D^{'}$, $D^{'}\cup\{e_b\}$ is a total edge dominating set of T and hence $\gamma_t^{'}(T^{'})\geq\gamma_t^{'}(T)-1$. Further since S is a $\gamma_d^{'}$ - set of T, $S\cap\{e_a,e_b,e_b^{'}\}=\{e_a,e_b^{'}\}$ by the definition of $\gamma_d^{'}$ - set. Clearly, $S\cap E(T^{'})$ is a double edge dominating set of $T^{'}$ and hence $\gamma_t^{'}(T^{'})\geq$ $\gamma_t^{'}(T)-1=\gamma_d^{'}(T)-1=|S\cap E(T^{'})|\geq\gamma_d^{'}(T^{'})$. By using Theorem 1, we get $\gamma_t^{'}(T^{'})=\gamma_d^{'}(T^{'})$ and so $T^{'}\in\Im$. By induction on $T^{'}$, $T^{'}\in\Re$. Now, since T is obtained from $T^{'}$ by type - 2 operation, $T\in\Re$.

By above claim, we only need to consider the case that, for the edge e_a, $T-e_a$ has exactly one component P_3. Let $P=e_u e e_w e_x e_y e_z\ldots e_r$ be a longest path in T having root at vertex r which is incident with e_r.

Clearly, $C(e_w)=C^{'}(e_w)\neq C_1^{'}(e_w)$ .By 3 of Lemma 6, $C_1^{'}(e_w)=C_3^{'}(e_w)=\phi$. Hence $P_4=e_u e e_w$ is a component of $T-e_x$. Let n be the number of components of P_4 of $\langle S(e_x)\rangle$ in T such that an end edge of every

P_4 is adjacent to e_x. Suppose $\langle S(e_x)\rangle$ in T has a component P_4 with its support edge is adjacent to e_x. Then it consists of j number of P_3 and k number of P_2 components. By Lemma 6, $m, j \in \{0,1\}$ and $k \in \{0,1,2\}$. Denoting the n components P_4 of the sub graph $\langle S(e_x)\rangle$ in T with one of its end edges is adjacent to an edge e_x in T by $P_4 = e_{u_i} e_i e_{w_i}$, $1 \le i \le n$. We prove that result according to the values of $\{m, j, k\}$.

Case 1: Suppose $m = j = k = 0$. Then $\langle S(e_x)\rangle = P_4(n), n \ge 1$ in T. Further assume that $T^{'} = T - S[e_x]$, then $2 \le |E(T^{'})| < q$. Clearly, $\gamma_t^{'}(T^{'}) \ge \gamma_t^{'}(T) - 2n$. Let S be a $\gamma_d^{'}$ - set of T such that S contains as minimum number of edges of the sub graph $\langle S(e_x)\rangle$ as possible. Then $e_x \notin S$ and $|S \cap S[e_x]| = 2n$ by the definition of $\gamma_d^{'}$ - set. Clearly $S \cap E(T^{'})$ is a double edge dominating set of $T^{'}$ and hence $\gamma_t^{'}(T^{'}) \ge \gamma_t^{'}(T) - 2n = \gamma_d^{'}(T) - 2n = |S \cap E(T^{'})| \ge \gamma_d^{'}(T)$. By Theorem 1, $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$ and $S \cap E(T^{'})$ is a double edge dominating set of $T^{'}$. Hence $T^{'} \in \Im$. By applying the inductive hypothesis to $T^{'}$, $T^{'} \in \Re$.

If $n \ge 2$, then it is obvious that T is obtained from $T^{'}$ by type - 5 operation and hence $T \in \Re$.

If $n = 1$. Then $\langle S(e_x)\rangle = P_4 = e_u e e_w$ in T which is incident with x of an edge e_x and $S \cap \{e_u, e, e_w, e_x\} = \{e_u, e_w\}$. To double edge dominate, $e_x, e_y \in S$ and so $e_y \in S \cap E(T^{'})$, which implies that e_y is in some $\gamma_d^{'}$ - set of $T^{'}$. Hence T is obtained from $T^{'}$ by type-5 operation and $T \in \Re$.

Case 2: Suppose $m \ne 0$ and by the proof of Lemma 6, $m = 1$ and $j = k = 0$. Denote the component P_4 of $\langle S(e_x)\rangle$ in T whose support edge is adjacent to e_x in T by $P_4 = e_a e_b e_c$ and if $T^{'} = T - \{e_a, e_b, e_c\}$. Then, clearly $3 \le |E(T^{'})| \le q$. Let S be a $\gamma_d^{'}$ - set of T which does not contain e_b.

Now we claim that e_x is not in any $\gamma_t^{'}$- set of $T^{'}$. Suppose that $T^{'}$ has a $\gamma_t^{'}$- set containing e_x which is denoted by $D^{'}$, then $D^{'}\bigcup\{e_b\}$ is a total edge dominating set of T. Clearly, $\gamma_t^{'}(T^{'})\geq\gamma_t^{'}(T)-1$. Since $e_b\notin S$ then $S\bigcap E(T^{'})$ is a double edge dominating set of $T^{'}$. Hence $\gamma_t^{'}(T^{'})\geq\gamma_t^{'}(T)-1=\gamma_d^{'}(T)-1=\left|S\bigcap E(T^{'})\right|+1\geq\gamma_d^{'}(T^{'})+1$, which gives a contradiction to the fact that $\gamma_t^{'}(T^{'})\leq\gamma_d^{'}(T^{'})$. This holds the claim and therefore T can be obtained from $T^{'}$ by type-4 operation.

Now we prove that $T^{'}\in\Re$. Let $D^{'}$ be any $\gamma_t^{'}$- set of $T^{'}$. By above claim, $e_x\notin D^{'}$. Since $D^{'}\bigcup\{e_x,e_b\}$ is a total edge dominating set of T, $\gamma_t^{'}\left(T^{'}\right)\geq\gamma_t^{'}\left(T\right)-2$. Further since $e_b\notin S$, $S\bigcap E(T^{'})$ is a double edge dominating set of $T^{'}$, $\gamma_t^{'}(T^{'})\geq\gamma_t^{'}(T)-2 =\gamma_d^{'}(T)-2 =\left|S\bigcap E(T^{'})\right|\geq\gamma_d^{'}(T^{'})$. Therefore by Theorem 1, we get $\gamma_t^{'}(T^{'})=\gamma_d^{'}(T^{'})$, which implies that $T^{'}\in\Im$. Applying the inductive hypothesis on $T^{'}$, $T^{'}\in\Re$ and hence $T\in\Re$.

Case 3: Suppose $j\neq0$ and by the proof of Lemma 6, $m=k=0$. Let $T^{'}=T-\bigcup_{i=1}^{n}\left\{e_{u_i},e_i,e_{w_i}\right\}$. Clearly, $3\leq\left|E(T^{'})\right|<q$ and T is obtained from $T^{'}$ by type - 3 operation.

We only need to prove that $T^{'}\in\Re$. Suppose $D^{'}\subset E\left(T^{'}\right)$ be a $\gamma_t^{'}$ - set of $T^{'}$. Then $D^{'}\bigcup\left(\bigcup_{i=1}^{n}\left\{e_i,e_{w_i}\right\}\right)$ is a total edge dominating set of T and hence. $\gamma_t^{'}(T^{'})\geq\gamma_t^{'}(T)-2n$. Since $T-e_x$ has a component $P_3=e_a e_b$, we can choose $S\subseteq E(T)$ as a $\gamma_d^{'}$ - set of T containing e_x. Then $S\bigcap E(T^{'})$ is a $\gamma_d^{'}$ - set of $T^{'}$ and hence $\gamma_d^{'}(T)=\left|S\right| =2n+ \left|S\bigcap E(T^{'})\right|\geq2n+\gamma_d^{'}(T^{'})$. Clearly, it follows that, $\gamma_t^{'}(T^{'})\geq\gamma_d^{'}(T^{'})$. Therefore, by Theorem 1, we get, $\gamma_t^{'}(T^{'})=\gamma_d^{'}(T^{'})$ and hence $T^{'}\in\Im$. Applying the inductive hypothesis on $T^{'}$, we get $T^{'}\in\Re$.

Case 4: Suppose $k \neq 0$. Then by Lemma 6, $k \in \{1,2\}$ and so $m = j = 0$. We claim $k = 1$. If not, then $k = 2$. We denote the two components P_2 of $\langle S(e_x) \rangle$ by $e_x^{'}$ and $e_x^{''}$ in T. Let $T^{'} = T - e_x^{''}$. Clearly, $\gamma_t^{'}(T^{'}) = \gamma_t^{'}(T)$. Let S be a $\gamma_d^{'}$ - set of T containing $\{e_{w_1}, e_{w_2}, ..., e_{w_n}\}$. By Observation 2, $\{e_x^{'}, e_x^{''}\} \subseteq S$. Since $S \cap E(T^{'})$ is a double edge dominating set of $T^{'}$ with $|S \cap E(T^{'})| = \gamma_d^{'}(T) - 1$, we have $\gamma_t^{'}(T^{'}) = \gamma_t^{'}(T) = \gamma_d^{'}(T) > \gamma_d^{'}(T) - 1 \geq \gamma_d^{'}(T^{'})$, which is a contradiction to the fact that $\gamma_t^{'}(T^{'}) \leq \gamma_d^{'}(T^{'})$.

Sub case 4.1: For $n \geq 2$. Suppose $T^{'} = T - \bigcup_{i=1}^{n} \{e_{u_i}, e_i, e_{w_i}\}$. Then T is obtained from $T^{'}$ by type - 3 operation. Now by definition of $\gamma_t^{'}$ - set and $\gamma_d^{'}$ - set, it is easy to observe that $\gamma_t^{'}(T^{'}) + 2(n-1) = \gamma_t^{'}(T)$ and $\gamma_d^{'}(T^{'}) + 2(n-1) = \gamma_d^{'}(T)$. Hence $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$ and $T^{'} \in \Im$. Applying the inductive hypothesis on $T^{'}$, $T^{'} \in \Re$ and hence $T \in \Re$.

Sub case 4.2: For $n = 1$. Denote the component P_2 of $\langle S(e_x) \rangle$ by $e_x^{'}$ in T. Suppose $\langle S(e_y) - S[e_x] \rangle$ has a component $H \in \{P_3, P_4, P_6\}$ in T, then $T^{'} = T - S[e_x]$. Therefore we can easily check that T is obtained from $T^{'}$ by type-6 operation. Now by definition of $\gamma_d^{'}$ - set, $\gamma_d^{'}(T^{'}) + 3 = \gamma_d^{'}(T)$. For any $\gamma_t^{'}$ - set $D^{'}$ of $T^{'}$, $D^{'} \cup \{e, e_w, e_x\}$ is a total edge dominating set of T. Clearly, $\gamma_t^{'}(T^{'}) \geq \gamma_t^{'}(T) - 3 = \gamma_d^{'}(T) - 3 = \gamma_d^{'}(T^{'})$. By Theorem 1, we get $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$ and $T^{'} \in \Im$. Applying inductive hypothesis on $T^{'}$, $T^{'} \in \Re$ and hence $T \in \Re$.

Now if the sub graph $\langle S(e_y) - S[e_x] \rangle$ has no components P_3, P_4 or P_6. Then we consider the structure of $\langle S(e_y) \rangle$ in T. By above discussion, $\langle S(e_y) \rangle$ consists of a component $P_6 = e_u\, e\, e_w\, e_x\, e_x^{'}$ and g number of components of P_2, denoted by $\{e_1, e_2, ..., e_g\}$. Assume $l = 2$. Then, let $T^{'} = T - S[e_y]$. It can be easily checked that $\gamma_t^{'}(T^{'}) + 4 \geq \gamma_d^{'}(T) = \gamma_d^{'}(T^{'}) + 5$, which is a contradiction to the fact that $\gamma_t^{'}(T^{'}) \leq \gamma_d^{'}(T^{'})$. Hence $g \leq 1$.

Suppose $T^{'} = T - \{e_u, e_x^{'}\}$. Here we can easily check that $\gamma_t^{'}(T^{'}) + 1 = \gamma_t^{'}(T)$. Let S be a $\gamma_d^{'}$- set of T such that S contains as minimum edges of $S[e_y]$ as possible and $S \cap S[e_x] = \{e_u, e_w e_x^{'}\}$. Then $S^{'} = (S - \{e_u, e_w e_x^{'}\}) \cup \{e, e_x\}$ is a double edge dominating set of $T^{'}$. Therefore $\gamma_t^{'}(T^{'}) = \gamma_t^{'}(T) - 1 = \gamma_d^{'}(T) - 1 = \left| S^{'} \right| \geq \gamma_d^{'}(T^{'})$, which implies that $\gamma_t^{'}(T^{'}) = \gamma_d^{'}(T^{'})$ where $S^{'}$ is a double edge dominating set of $T^{'}$. Hence $T^{'} \in \Im$. Applying inductive hypothesis to $T^{'}$, $T^{'} \in \Re$.

If $g = 0$, then $\deg_T(e_y) = 2$. Since $e_x \notin S$, to double edge dominate e_y, $e_y \in S$. Therefore e_y is in the double edge dominating set $D^{'}$ of $T^{'}$. Hence T is obtained from $T^{'}$ by type-1 operation. Thus $T \in \Re$.

If $g = 1$, then $\deg_T(e_y) = 3$. Since $e_x \notin S$ to double edge dominate e_y, we have $e_y \notin S$ and $e_z \in S$, by the selection of S. Therefore e_z is in the double edge dominating set $S^{'}$ of $T^{'}$. Hence T is obtained from $T^{'}$ by type-1operation. Thus $T \in \Re$.

By above all the Lemmas, finally we are now in a position to give the following main characterization.

Theorem 3.10: $\Im = \Re \cup \{P_3\}$

References

S. Arumugan and S. Velammal, (1997), "Total edge domination in graphs", *Ph.D Thesis, Manonmaniam Sundarnar University, Tirunelveli, India.*

F. Harary, (1969), "Graph theory", *Adison-wesley, Reading mass.*

F. Harary and T. W. Haynes, (2000), "Double domination in graph", *Ars combinatorics*, 55, pp. 201-213.

T.W. Haynes, S. T. Hedetniemi and P. J. Slater, (1998), "Fundamentals of domination in graph", *Marcel Dekker, New York.*

T. W. Haynes, S. T. Hedetniemi and P. J. Slater, (1998), "Domination in graph, Advanced topics", *Marcel Dekker, New York.*

Biography

A. **Dr.M.H.Muddebihal** born at: Babaleshwar taluk of Bijapur District in 19-02-1959.

Completed Ph.D. education in the field of Graph Theory from Karnataka. Currently working as a Professor in Gulbarga University, Gulbarga-585 106 bearing research experience of 25 years. I had published over 50 research papers in national and international Journals / Conferences and I am authoring two text books of GRAPH THEORY.

B. **A.R.Sedamkar** born at: Gulbarga district of Karnataka state in 16-09-1984.
Completed PG education in Karnataka. Currently working as a Lecturer in Government Polytechnic Lingasugur, Raichur Dist. from 04 years. I had published 10 papers in International Journals in the field of Graph Theory.