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One Condition of *I*-Cofiniteness of Generelazed Local Cohomology Modules

Sara Pooyandeh
Department of Mathematics, Payame Noor University
PO box 19395-3697, Tehran, Iran
E-mail: sareh124@yahoo.com

Abstract

Let I be an ideal of a commutative Noetherian local ring R, M and N two finitely generated modules. Let t be a positive integer. We mainly prove that if $d = pd(M) < \infty$ and $\dim N = n < \infty$, then $H_I^{d+n}(M,N)$ is I-cofinite, which is a generalized of cofinite modules and local cohomology ($H_I^i(M,N) = \lim_{\rightarrow} Ext_R^i$ ($M/I^n M, N$)). In the last part of this note, we also discuss the finiteness of $H_I^i(M,N)$ and prove that if M is a nonzero cyclic R-module, then $H_I^i(N)$ is finitely generated for all $i < t$ if and only if $H_I^i(M,N)$ is finitely generated for all $i < t$.

Keywords: Cofinite, Local cohomology, Noetherian.

1. Introduction

Let R be a commutative Noetherian ring and I a proper ideal of R. The generalized local cohomology module ($H_I^i(M,N) = \lim_{\rightarrow} Ext_R^i(M/I^n M, N)$). for all R-modules M and N was introduced by Herzog in [4]. Clearly, it is a generalization of the usual local cohomology module. The study of generalized local cohomology modules was continued by many authors. For example Asadollahi, Khashyarmanesh and salarian [1] proved that if $H_I^i(M,N)$ is finitely generated for all $i < t$, then $Hom(R/I, H_I^t(M,N))$ is finitely generated. Another, Delfino and Marley [3] proved that if (R,m) be a Noetherian local ring, Ian ideal of R and M finitely generated module ($dimM = n$), then $H_I^n(M)$ is I-cofinite($Ext_R^i(R/I, H_I^n(M))$ is finite for all i).

As an analogue of this result, we show that if $d = pd(M) < \infty$ and $\dim N = n < \infty$, then $H_I^{d+n}(M,N)$ is I-cofinite, which is a generalization of [3, Theorem3]. Throughout this paper (R,m) is a commutative Noetherian local ring (with nozero identity), M and N are finitely generated R-modules and I is proper ideal of R. We refer the reader to [2] for any unexpelained terminology.

2. Results

We begin this section with some lemmas.

2.1 Lemma

Let M be a finitely generated R-module. If L is artinian and I-cofinite, then $Ext_R^i(M,L)$ is I-cofinite for all i.

Proof. Since L is Artinian, $Ext_R^i(M,L)$ is Artinian for all i. By [7, Proposition 4.3], it suffices to prove that $Hom_R(R/I, Ext_R^i(M,L))$ is finitely generated. In the following, we show that

$$Hom_R\left(R/I, Ext_R^i(M,L)\right) \cong Hom_R\left(R/I, Ext_R^i(M,L)\right) \otimes \hat{R} \cong Hom_{\hat{R}}(\hat{R}/I\hat{R}, Ext_{\hat{R}}^i(\hat{M},L))$$

We may assume that R is m-adic complete. Set $E = E(R/m)$, an injective envelope of R/m . By [9, Theorem 11.57],

$$Hom_R\big(Hom_R(R/I, Ext_R^i(M,L)), E\big) \cong R/I \otimes \cong Tor_i^R(M, Hom_R(L,E)).$$

By matlis duality, $R/I \otimes \cong Tor_i^R(M, Hom_R(L,E)$ is finitely generated, so it is enough to show that it is Artinian. Since L is I-cofinite and Artinian, $Hom_R(R/I, L)$ is of finite length, and then $Hom_R(Hom_R(R/I,L), E) \cong R/I \otimes Hom_R(L,E)$ is of finite length. In particular,

$$Supp_R\{R/I \otimes Hom_R(L,E)\} \cong V(I) \cap Supp_R\{Hom_R(L,E)\} = \{m\}.$$

Therefore

$$Supp_R\{R/I \otimes Tor_i^R(M, Hom_R(L,E)\} \cong V(I) \cap Supp_R\{Hom_R(L,E)\} = \{m\}.$$

This complete the proof.

The following lemma is a generalization of [8,Lemma 3.4]

2.2 Lemma

Let M be a finitely generated R-module such that $d = pd(M) < \infty$. Let N be a finitely generated R-module and assume that n is an integer, and $x_1, \dots, x_n$ is an I-filter regular sequence on N. Then $H_I^{i+n}(M,N) \cong H_I^i(M, H^n_{(x_1,\dots,x_n)}(N))$ for all $i \geq d$.

Proof. See [6, Theorem 3.2].

2.3 Proposition

Let I be an ideal of R, and let M, N be two finitely generated R-modules such that $d = pd(M) < \infty$ and $\dim N = n < \infty$. Then $H_I^{d+n}(M,N) \cong Ext_R^d(M, H_I^n(N))$. In particular, $H_I^{d+n}(M,N)$ is Artinian.

Proof. For this integer n, it is well known that there exists a sequence $x_1, \dots, x_n$ in I such that it is an I-filter regular sequence on N. Note that $H^n_{(x_1,\dots,x_n)}(N)$ is Artinian when $n = dimN$. By vitue of [8,Lemma 3.4], $H^n_{(x_1,\dots,x_n)}(N) \cong H_I^0(H^n_{(x_1,\dots,x_n)}(N)) \cong H_I^n(N)$. Therefore, by Lemma 2.6,

$$H_I^{d+n}(M,N) \cong H_I^d(M, H^n_{(x_1,\dots,x_n)}(N)) \cong H_I^d(M, H_I^n(N)) \cong Ext_R^d(M, H_I^n(N)).$$

This completes the proof.

The following theorem is our main result, which generalizes [3, Theorem3].

2.4 Theorem

Let I be an ideal of R, and Let M, N be two finitely generated R-modules such that $d = pd(M) < \infty$ and $\dim N = n < \infty$. Then $H_I^{d+n}(M,N)$ is I-cofinite.

Proof. By [3,Theorem 3], we know that $H_I^n(N)$ is I-cofinite. Then by Lemma 2.1 and proposition 2.3, the result follows.

In the last part of this note, we discuss the finiteness of $H_I^i(M,N)$.

2.5 Lemma

Let N be a finitely generated R-module and M a nonzero cyclic R-module. Let t be a positive integer. If $H_I^i(N)$ is finitely generated for all $i < t$, then $H_I^t(N)$ is finitely generated if and only if $Hom(M, H_I^t(N))$ is finitely generated.

Proof. The 'only if' part is clear. Now suppose that $Hom(M, H_I^t(N))$ is finitely generated. Note that $Hom(M, H_I^t(N))$ is I-tortion, then there exists an integer n such that $I^n Hom(M, H_I^t(N)) = 0$. Assume that M is generated by an element m. For any $x \in H_I^t(N)$, we can find an element $f \in Hom(M, H_I^t(N))$ such that $f(m) = x$. Since $I^n f = 0$, $I^n x = 0$ and so $I^n H_I^t(N) = 0$. Since $H_I^i(N)$ is finitely generated for all $i < t$, by [2, proposition 9.1.2], there exist an integer r, $I^r H_I^i(N) = 0$ for all $i < t$. Thus, $I^r H_I^i(N) = 0$ for all $i < t + 1$. Again by [2, proposition 9.1.2], $H_I^i(N)$ is finitely generated for all $i < t + 1$. In particular, $H_I^t(N)$ is finitely generated.

2.6 Proposition

Let N be a finitely generated R-module and let t be a positive integer. If M is a nonzero cyclic R-module, then $H_I^i(N)$ is finitely generated for all $i < t$ if and only if $H_I^i(M,N)$ is finitely generated for all $i < t$.

Proof. The 'only if' part has been proved in [5, Theorem 1.1($\boldsymbol{iv}$)]. Now we suppose that $H_I^i(M,N)$ is finitely generated fo all $i < t$. By induction on t, we can assume that $H_I^i(N)$ is finitely generated for all $i < t - 1$. Then by [5, Theorem 1.1($\boldsymbol{iii}$)], it followes that $Hom(M, H_I^{t-1}(N))$ is finitely generated from the fact that $H_I^{t-1}(M,N)$ is finitely generated. Then $H_I^{t-1}(N)$ is finitely generated by Lemma 2.5.

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