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On Inhomogeneous Fractional Partial Differential Equations

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Abstract
In this paper, a coupling method of Laplace transform and Homotopy analysis method is applied for solving various inhomogeneous fractional partial differential equations.The proposed algorithm presents a procedure of construct the base function and gives a high order deformation equation in simple form. The purpose of using the Laplace transform is to overcome the deficiency that is mainly caused by unsatisfied conditions in the other analytical techniques. The scheme is tested for some examples to demonstrate the capability of LHAM for fractional partial differential equations.
Keywords: Laplace homotopy analysis method; homotopy analysis method; fractional differential equations; modified Riemann-Liouville derivative; Wave equation; Burger's equation; Klein-Gorden equation:

1. Introduction

In last two decades, fractional differential equations has been given much interest due to exact description of nonlinear phenomena in fluid flow,viscoelasticity, seismology, biology, chemistry,economic, probability and statistics, acoustics, material science, engineering electrical
network, optics and signal processing, electrochemistry, continuum mechanics and so on. However, fractional calculus is three centuries old as the conventional calculus [1].The most recent works on the subject of fractional calculus is the book of Podlubny [2].
The importance of obtaining the exact and approximate solutions offractional linear or nonlinear differential equations is still significant problem that need snew methods to discover the exact and approximate solutions. But these nonlineardifferential equations are difficult to get their exact solutions so numerical methods havebeen used to handle these equations, a wide class of analytical methods have been proposed, suchas Laplace transform method [2,3], differential transform method[4-6], Adomian's decomposition method [7-11], variational iteration method [12-14], homotopy perturbation method [15-16], homotopy perturbation transform method [17]. Another analytical approach that can be applied to solve linear or nonlinear equations ishomotopy analysis method [18-20]. A systematic and clear exposition on HAM is given in [19]. The objective of the paper is to apply the Laplace homotopy analysis method [21], to provide analytic approximate solutions to inhomogeneous fractional partial differential equations.

2. Preliminaries and Notations

We give some basic definitions, notations and properties of the fractional calculus theory which are used further in this paper:

Definition 2.1

Assume $f:R\to R,\ x\to f(x)$ denote a continuous (but not necessarily differentiable) function and let the partition $h>0$ in the interval $[0,1]$. Jumarie's derivative is defined through the fractional difference

$$\Delta^{\alpha}=(FW-1)^{\alpha}\ f(x)=\sum_{k=0}^{\infty}(-1)^{k}\begin{pmatrix}\alpha\\ k\end{pmatrix}f(x+(\alpha-k)) \tag{1}$$

Where $FWf(x)=f(x+h)$. Then the fractional derivative is defined as the following limit.

$$f^{(\alpha)}=\lim_{h\to 0}\frac{\Delta^{\alpha}[f(x)-f(0)]}{h^{\alpha}} \tag{2}$$

This definition is close to the standard definition of derivative, and as a direct result, the αth derivative of a constant $0<\alpha<1$; is zero.

Definition 2.2

The Riemann–Liouville fractional integral operator of order $\alpha\geq 0$ for a function $f\in C_{\mu}$ $\mu\geq -1$is defined as

$$I_{x}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{0}^{x}(x-\xi)^{\alpha-1}f(\xi)d\xi,\qquad \alpha>0,\quad t>0\,(3) \tag{3}$$

Definition 2.3

The Jumarie's modified Riemann–Liouville derivative is defined as

$$I_x^{\alpha} f(x)=\frac{1}{\Gamma(m-\alpha)}\frac{d^m}{d\,x^m}\int_0^x (x-\xi)^{m-\alpha}\,(f(\xi)-f(0))\,d\xi, \tag{4}$$

Where, $x\in[0,1]$, $m-1<\alpha\le m,\quad m\ge 1$.

The proposed modified Riemann–Liouville derivative as shown in Eq. (4) is strictly equivalent to Equation. (2).

Definition 2.4

Fractional derivative of compounded functions is defined as

$$d^{\alpha} f(x)\cong\Gamma(1+\alpha)\,df,\qquad 0<\alpha<1 \tag{5}$$

Definition 2.5

The integral with respect to $(d\,\xi)^{\alpha}$ is defined as the solution of fractional differential equation given by eq.

$$d\,y\cong f(x)(d\,x)^{\alpha},\quad x\ge 0,\quad y(0)=0,\quad 0<\alpha<1 \tag{6}$$

$$y\cong\int_0^x f(\xi)(d\,\xi)^{\alpha}=\alpha\int_0^x (x-\xi)^{\alpha-1} f(\xi)\,d\xi,\qquad 0<\alpha\le 1 \tag{7}$$

For example $f(x)=x^{\beta}$ in eq (2), we have

$$\int_0^x \xi^{\beta}(d\,\xi)^{\alpha}=\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}x^{\alpha+\beta},\qquad 0<\alpha\le 1 \tag{8}$$

Definition 2.6

If $m-1<\alpha\le m$, $m\in N$, then the Laplace transform of the fractional derivative $D_*^{\alpha} f(t)$ is

$$L\left(D_*^{\alpha} f(t)\right)=s^{\alpha}\,\bar{f}(s)-\sum_{k=0}^{m-1} f^{(k)}\left(0^{+}\right)s^{\alpha-k-1},\qquad t>0 \tag{9}$$

3. Analysis of Laplace Homotopy Analysis Method

In order to elucidate the solution procedure of the LHAM, we consider the following fractional differential equation:

$$D_t^{\alpha} u(x,t)=f\left(u,u_x,u_{xx}\right),\quad 1<\alpha\le 2,\quad t>0,\quad x\in R, \tag{10}$$

Subject to the initial conditions

$$u(x,0)=f_1(x),\quad u_T(x,0)=f_2(x) \tag{11}$$

Where f a linear or nonlinear is function, and D_t^{α} is a fractional differential operator. The operator form of equation (10) is

$$D_t^{\alpha} u(x,t)=A\left(u,u_x,u_{xx}\right)+B\left(u,u_x,u_{xx}\right)+C(x,t) \tag{12}$$

Where A and B are linear and nonlinear operators respectively, which might include other fractional derivatives of order less than α and C is the known analytic function.

Now Taking the Laplace transform of both sides of eq. (10) and using (11), we have

$$L\left(D_t^{\alpha} u(x,t)\right)=L\left(A\left(u,u_x,u_{xx}\right)+B\left(u,u_x,u_{xx}\right)+C(x,t)\right) \tag{13}$$

$$\bar{u}(x,s)=\frac{f_1(x)}{s}+\frac{f_2(x)}{s^2}+\frac{1}{s^{\alpha}}L\left(A\left(u,u_x,u_{xx}\right)+B\left(u,u_x,u_{xx}\right)+C(x,t)\right) \tag{14}$$

The so-called zero-order deformation equation of equation (14) is given by

$$(1-p)\left[\bar{\phi}(x,s;p)-\bar{u}_0(x,s)\right]=ph\,[\bar{\phi}(x,s;p)-\frac{f_1(x)}{s}-\frac{f_2(x)}{s^2}$$
$$-\frac{1}{s^\alpha}L\big(A(\phi(x,t),\ \phi_x(x,t),\phi_{xx}(x,t))+B(\phi(x,t),\ \phi_x(x,t),\phi_{xx}(x,t))+C(x,t)\big)] \tag{15}$$

Where $p\in[0,1]$ is an embedding parameter, when *p =0* and *p=1*, we have $\bar{\phi}(x,s;0)=\bar{u}_0(x,s)$ and $\bar{\phi}(x,s;1)=\bar{u}(x,s)$ respectively. Thus, as *p* increasing from 0 to 1 $\bar{\phi}(x,s;p)$ varies from $\bar{u}_0(x,s)$ to $\bar{u}(x,s)$.

Expanding $\bar{\phi}(x,s;p)$ in Taylor series with respect to *p*, we have

$$\bar{\phi}(x,s;p)=\bar{u}_0(x,s)+\sum_{m=1}^{\infty}\bar{u}_m(x,s)\,p^m \tag{16}$$

Where

$$\bar{u}_m(x,s)=\left[\frac{1}{m!}\frac{\partial^m}{\partial x^m}\bar{\phi}(x,s;p)\right]_{p=0} \tag{17}$$

We define the vector

$$\vec{\bar{u}}_0(x,s)=\{\bar{u}_0(x,s),\bar{u}_1(x,s),\bar{u}_2(x,s),\ldots\bar{u}_m(x,s)\}$$
(18)

Differentiating Equation (15) m times with respect to the embedding parameter *p*, and then Setting *p* =0, h= -1 and finally dividing them by $m!$, we have the so-called mth-order deformation equation

$$\bar{u}_m(x,s)=\chi_m\,\bar{u}_{m-1}(x,s)-R_m\left(\vec{\bar{u}}_{m-1}(x,s)\right) \tag{19}$$

where

$$R_m\left(\vec{\bar{u}}_{m-1}(x,s)\right)=\vec{\bar{u}}_{m-1}(x,s)-\left[\frac{1}{m!}\frac{\partial^{m-1}}{\partial x^{m-1}}F(x,t;p)\right]_{p=0}-\left[\frac{f_1(x)}{s}+\frac{f_2(x)}{s^2}+\frac{1}{s^\alpha}L\big(C(x,t)\big)\right](1-\chi_m)$$

(20)

Where

$$F(x,t;p)=L\big(A(\phi(x,t),\ \phi_x(x,t;p),\phi_{xx}(x,t;p))+B(\phi(x,t),\ \phi_x(x,t;p),\phi_{xx}(x,t;p))\big)$$

and $\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$ (21)

Applying the inverse Laplace transform of both sides of (19), then we have a power series $u(x,t)=\sum_{n=0}^{\infty}u_n(x,t)$ of equation (10).

4. Numerical Examples

Example 1. We consider the following one-dimensional linear inhomogeneous fractional wave equation

$$D_t^\alpha u+u_x=\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\sin x+t\cos,\qquad t>0,\qquad x\in R,\qquad 0<\alpha\le 1, \tag{22}$$

with the initial condition

$$u(x,0)=0. \tag{23}$$

Taking the Laplace transform of both sides of equation (22) and using (23), we have

$$\bar{u}(x,s)=\frac{1}{s^{\alpha}}L\left(-\frac{\partial u}{\partial x}\right)+\frac{\sin x}{s^{2}}+\frac{\cos x}{s^{\alpha+1}} \tag{24}$$

In view of equations (19) and (20), we have

$$\bar{u}_m(x,s)=\chi_m\,\bar{u}_{m-1}(x,s)-\left[\bar{u}_{m-1}(x,s)+\frac{1}{s^{\alpha}}L\left(-\frac{\partial u_{m-1}(x,t)}{\partial x}\right)-\left(\frac{\sin x}{s^{2}}+\frac{\cos x}{s^{\alpha+1}}\right)(1-\chi_m)\right]$$

(25)

$$\bar{u}_0(x,s)=\frac{\sin x}{s^{2}}+\frac{\cos x}{s^{\alpha+1}}$$

$$\bar{u}_1(x,s)=-\frac{\cos x}{s^{\alpha+2}}+\frac{\sin x}{s^{2\alpha+1}}+\frac{\sin x}{s^{2}}+\frac{\cos x}{s^{\alpha+1}}$$

It is easily observed that the self canceling "noise" terms appear in the components $\bar{u}_0(x,s)$ and $\bar{u}_1(x,s)$. Canceling the noise terms, we have

$$\bar{u}(x,s)=\frac{\sin x}{s^{2}}$$

(26)

Taking the inverse Laplace transform of equation (26), we have

$u(x,t)=t\sin x$.

Which is the exact solution of equation (22) for $\alpha=1$.

Example 2. We consider the one-dimensional linear inhomogeneous fractional Burger's equation given by

$$D_t^{\alpha}u+u_x-u_{xx}=\frac{2t^{2-\alpha}}{\Gamma(3-\alpha)}+2x-2\,,\qquad t>0,\quad x\in R,\qquad 0<\alpha\le 1, \tag{27}$$

with the initial condition

$$u(x,0)=x^{2}. \tag{28}$$

Taking the Laplace transform of both sides of equation (27) and using (28), we have

$$\bar{u}(x,s)=\frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u}{\partial x^2}-\frac{\partial u}{\partial x}\right)+\frac{x^2}{s}+\frac{2}{s^3}+\frac{2x-2}{s^{\alpha+1}}$$

(29)

In view of equations (19) and (20), we have

$$\bar{u}_m(x,s)=\chi_m\,\bar{u}_{m-1}(x,s)-\left[\bar{u}_{m-1}(x,s)-\frac{1}{s^{\alpha}}L\left(\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2}-\frac{\partial u_{m-1}(x,t)}{\partial x}\right)-\left(\frac{x^2}{s}+\frac{2}{s^3}+\frac{2x-2}{s^{\alpha+1}}\right)(1-\chi_m)\right]$$

(30)

$$\bar{u}_0(x,s)=\frac{x^2}{s}+\frac{2}{s^3}+\frac{2x-2}{s^{\alpha+1}}$$

$$\bar{u}_1(x,s)=-\frac{2x-2}{s^{\alpha+1}}-\frac{2}{s^{2\alpha+1}}$$

It is easily observed that the self canceling "noise" terms appear in the components $\bar{u}_0(x,s)$ and $\bar{u}_1(x,s)$. Canceling the noise terms, we have

$$\bar{u}(x,s)=\frac{x^2}{s}+\frac{2}{s^3}$$

(31)

Taking the inverse Laplace transform of equation (31), we have

$u(x,t)=x^{2}+t^{2}$.

Which is the exact solution of equation (27) for $\alpha = 1$.

Example 3. Finally, we consider the one-dimensional linear inhomogeneous fractional Klein-Gorden Equation

$$D_t^\alpha u - u_{xx} + u = 6x^3 t + \left(x^3 - 6x\right)t^3, \qquad t > 0, \qquad x \in R, \qquad 1 < \alpha \le 2, \tag{32}$$

with the initial conditions

$$u(x,0) = 0, \qquad u_t(x,0) = 0. \tag{33}$$

Taking the Laplace transform of both sides of equation (27) and using (28), we have

$$\bar{u}(x,s) = \frac{1}{s^\alpha} L\left(\frac{\partial^2 u}{\partial x^2} - u\right) + \frac{6x^3}{s^{\alpha+2}} + \frac{6\left(x^3 - 6x\right)}{s^{\alpha+4}}$$

(34)

In view of equations (19) and (20), we have

$$\bar{u}_m(x,s) = \chi_m \bar{u}_{m-1}(x,s) - \left[\bar{u}_{m-1}(x,s) - \frac{1}{s^\alpha} L\left(\frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} - u_{m-1}(x,t)\right) - \left(\frac{6x^3}{s^{\alpha+2}} + \frac{6\left(x^3 - 6x\right)}{s^{\alpha+4}}\right)\left(1 - \chi_m\right)\right]$$

(35)

$$\bar{u}_0(x,s) = \frac{6x^3}{s^{\alpha+2}} + \frac{6\left(x^3 - 6x\right)}{s^{\alpha+4}}$$

$$\bar{u}_1(x,s) = \frac{12}{s^{2\alpha+2}} + \frac{36x}{s^{2\alpha+4}} - \frac{6x^2}{s^{2\alpha+2}} - \frac{6\left(x^3 - 6x\right)}{s^{\alpha+4}}$$

It is easily observed that the self canceling "noise" terms appear in the components $\bar{u}_0(x,s)$ and $\bar{u}_1(x,s)$. Canceling the noise terms, we have

$$\bar{u}(x,s) = \frac{6x^3}{s^{\alpha+2}}$$

(36)

Taking the inverse Laplace transform of equation (36), and setting $\alpha = 2$, we have

$$u(x,t) = x^3 t^3.$$

This is the exact solution of equation (32).

5. Conclusion

In this paper, the main objective is to apply the Laplace homotopy analysis method (LHAM) to construct solutions for inhomogeneous partial differential equations of fractional order with constant coefficients. A general analysis of the LHAM for the analytical treatment of fractional partial differential equations is presented. Also its small size of computation in comparison with the computational size required in other numerical methods and its rapid convergence shows that the LHAM is reliable and introduces a significant improvement in solving partial differential equations over existing methods.

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