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APPLICATION OF MAX-FLOW MIN-CUT THEOREM IN BIPARTITE GRAPHS TO OBTAIN MAXIMUM FLOW

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Abstract−The Max-Flow Min-Cut Theorem is the most efficient result which can be used to determine the maximum value of flow by minimum value of capacities of all the cut sets in the network flows. In this paper we show that this theorem implies the some important results for bipartite graphs to obtain maximum flow in graph theory.

Keywords−Network flow, Augmenting path, Maximum flow, Minimum cut, Maximum matching, Minimum covering, Bipartite graph.

1. Introduction to Network Flows

The maximum flow problem in network flows implies that there is wide range of applications such as power transmissions, telecommunications, road networking, circuits, transportations, pipelines, traffics etc Graph theory [**1**] provides a framework for discussing systems in which it is possible to travel between discrete vertices. [**2**]If we extend a directed graph to a network flow by assigning a capacity and a flow value to every edge, then this flow can be used to model any number of systems in which a resource travels from one point to another. Our goal is to push as much flow as possible from S to T in the graph.The rules are that no edge can have flow exceeding its capacity, and for any vertex except for S and T, the flow in to the vertex must equal the flow out from the vertex. That is,

Capacity constraint: On any edge e we have

$$f\ (e)\ \leq\ c(e)$$

Flow conservation: For any vertex $v\ \notin\ \{S,T\}$, flow in equals flow out:

$$\sum_u f\ (u,v)\ =\sum_u f\ (v,u)$$

Subject to these constraints, we want to maximize the total flow into T. For instance, imagine we want to route message traffic from the source to the sink, and the capacities tell us how much bandwidth we are allowed on each edge, this disconnects the source from the sink. Thepoint is that any unit of flow going from s to t must take up at least 1 unit of capacity in these pipes. So, in general, an important property of flow is that the maximum $S-T$ flow ≤the capacity of the minimum $S-T$ cut.This is called the Max-flow Min-Cut Theorem.

2. Preliminaries

In this section, we will introduce some basic concepts and results on maximum flow minimum cut problems which we need in this paper.

Definition 2.1: The capacity of a cut (S,T) is the sum of capacities of edges in the cutor in the formal viewpoint, it is the sum of capacities of all edges going from S to T. (Don't include the edges from T to S).

Definition 2.2: A cut of a network with vertex set V is a partition of V into two disjoint sets $\{S,T\}$such that the source is in one and the sink is in the other. The capacity $c(S,T)$ of the cut $\{S,T\}$is the sum of the capacities of all edges directed from a vertex in S to a vertex in T

Definition (Flow Augmenting Paths) 2.3: Consider a path connecting s and t in a network. (Don't worry about the directions of the edges.) In traversing the path from s to t we will sometimes be going with the direction of edges, these are forward edges, and will sometimes be going against the direction of edges, these are reverse edges. A path from s to t is a flow augmenting path if

- $f(e) < c(e)$ if e is a forward edge, and
- $f(e) > 0$ if e is a reverse edge.

The worth w of a flow augmenting path is given by

$$w = min\{c(e_i) - f(e_i) : e_i\ is\ forward;\ f(e_i) : e_i\ is\ reverse\}$$

In other words, the worth of a flow augmenting path is found by looking at the difference between flow and capacity in each forward edge, looking at the flow in each reverse edge, and taking the minimum of these numbers.

Theorem 2.4: Let $\{S,T\}$be a cut and f be a flow in a network with value F. Then,

$$F = \sum_{(x,y)\in E: x\in S, y\in T} f(x,y) - \sum_{(y,x)\in E: x\in S, y\in T} f(y,x)$$

This just says. Take any cut $\{S,T\}$. Then the value of the flow is the amount flowing from S to T minus the amount flowing from T to S. **[3]**

Theorem 2.5: For every flow with value F, and every cut $\{S,T\}$ of the network, $F \leq c(S,T)$.

Theorem tells us that in a network, values of flows are always less than or equal to capacities of cuts. This is a mini-max situation. If we can find a flow and a cut such that the value of the flow is equal to the capacity of the cut then we know that the flow has maximum value. **[4]**

Theorem 2.6: Consider a network with a flow of value F, and a flow augmenting path P of worth w. Add w to the flow in each forward edge of P, and subtract w from the flow in each reverse edge of P. The result is a new flow whose worth is $F + w$.This means that whenever we can find a flow augmenting path we can increase the flow. **[5]**

Theorem 2.7 (Max-flow Min-cut): In a network, the maxi-mum value of a flow from s to t is equal to the minimum value of a cut. **[6]**

3. MAIN RESULTS:

The Max-Flow Min-Cut Theorem is a fundamental result within the field of network flows, but it can also be used to show some profound results in graph theory. First given any graph with at least two vertices, designate some vertex x the source and vertex y the sink and let all arcs have unit capacity then a flow on this network counts (via its value) a number of arc disjoint directed x, y- paths and a cut counts (via its capacity) a number of arcs whose deletion destroys all x, y- paths.

Theorem 3.1: For any finite undirected graph $G = (V, E)$ with vertices x and y, the minimum vertex cut of x and y is equal to the number of pair-wise internally-disjoint paths (i.e. the number of paths that pair-wise share no edges) from x to y.

Proof: The maximal set of pair-wise internally-disjoint paths from x to y as P, with $|P|$ = n. The flow f in the network N is defined as:

(a) Let x be the source and y be the sink.

(b) Extend G to a network N by the capacity function c (u, v) = 1 for each edge $\overline{uv}$∈ E.

(c) The flow along an edge e in E is 1 if $e \in p$ for some path p ∈ P and e is forward-oriented from x to y and 0 otherwise.

(d) If a vertex v is not x or y and it is part of a path p in P, erase all edges entering and leaving v that are not in p.

First, we must show that this flow must satisfy the capacity restraint and the Conversion of flow restraint. The capacity restraint is trivially satisfied, as every edge has capacity 1 and flow 0 or 1. The conversion of flow restraint is also satisfied: pick any v in V such that v is not x or y. If v is not a member of any path in P, then no flow passes through it. If v is in at least one member of P, then it is in exactly one, as the paths in P are pair-wise internally-disjoint. Denote the path containing v as p. Within p, for every edge $\overline{uv}$ entering v there is another path $\overline{vw}$ leaving v, with $f(u, v) = f(v, w) = 1$. Thus, for any vertex that is not a source or sink, the flow entering that vertex is equal to the flow leaving that vertex, satisfying the conversion-of-flow restraint.

Now we have a flow from s to t. Our next step will be to show that this flow is maximal. Suppose there is an augmenting path q from s to t. All edges are at full capacity, so any augmenting path cannot share any edge with any of the existing paths. Furthermore, q cannot pass through a non-source non-sink vertex belonging to a path in P, as we erased these edges in our construction of f. Therefore q is a path from s to t with no internal vertices in common with any member of P, and q is not in P. However, P was constructed to be a maximal set of internally-disjoint paths from s to t, a contradiction. Therefore f is maximal. Since f was constructed to have flow 1 along each of its n pair-wise internally-disjoint paths from x to y, the net flow of F is simply n. By the Max-Flow Min-Cut Theorem, the maximum flow from x to y is equal to the size of the minimal vertex cut of x and y, so the minimal vertex cut of x and y must be of size n. Thus the number of pair-wise-internally disjoint paths is equal to the size of the minimum vertex cut, proving the result.

Theorem 3.2: For any finite bipartite graph G, the number of edges in a maximal matching equals the number of vertices in a minimal vertex cover.

Proof: We will first extend G to a network, adding a source and a sink. We will then see that, in our new network, a maximal flow corresponds to a maximal matching and a minimum cut corresponds to a minimum cover. From here, the min-cut max-flow theorem implies the desired result.

Let X and Y be a bipartite separation of the vertices of G. Starting with this graph, construct a digraph G_0 = (V_0, E_0) where V_0 has all the vertices of V as well as a source s and a sink t. E_0 consists of all the edges in E, as well as new edges leading from s to every vertex in X, and also edges leading from each vertex in Y into t. Assign capacity values to edges as follows: give infinite capacity to each edge in E_0 that was originally in E (i.e., each edge from X to). To each of the newly added edges, give capacity 1.

Given a matching of cardinality k, it is easy to find a flow of value k. Simply push a flow of value 1 along the paths$\overline{sx}$, $\overline{xy}$,$\overline{yt}$ where (x, y) is one of the matched pairs. Likewise, any flow f must have a corresponding matching with cardinality equal to the flow's value. Thus, a maximal flow in G_0 corresponds to a maximal cardinality matching in G.

Let W be a covering in G with r vertices, and let $W(X)$ and $W(Y\ )$ be subsets of W consisting of the vertices of W in X and Y , respectively. Next, let X_0 be $X\ -\ W(X)$ and Y_0 be $Y\ -\ W(Y)$. The fact that W covers G implies that there is no edge from X_0 to Y_0 Let S be the union of s, $W(Y)$, and Y_0 and let T be the union of t, $W(X)$, and Y_O. Consider the cut (S, T) of G_0; its cardinality is r, the cardinality of W. Thus, any vertex covering defines a vertex cut with equal value. Likewise, consider any vertex cut (S, T) in G_0 with finite value r. Edges from X to Y have infinite capacity, so each edge from S to T must either go from s to X or from Y to t, which have capacity 1. Since the cut had value r, (S, T) have r arcs.

Let W be the union of the set of vertices x in X such that $\overline{sx}$ is in (S, T) and the set of vertices y in Y such that $\overline{yt}$ is in (S, T); clearly, W has r vertices. For every covering in G there is a corresponding cut in G_0. So a minimum cut corresponds to a minimum covering in G.

Thus far, we have shown that a maximal flow corresponds to a maximum-cardinality matching, and a minimum cut corresponds to a minimum vertex cover. By the max-flow min-cut theorem, a minimum cut is equal in value to a maximal flow. Therefore, by transitivity, the cardinality of a maximal matching in G is equal in value to that of a minimum covering.

Theorem 3.3: Suppose that G is a bipartite graph (V_1, V_2, E), with $|V_1| = |V_2|$ Then G has a perfect matching if the following condition holds:

$$\forall\ S \subseteq V_1, |S| \leq |N(S)|.$$

Proof: We first notice that the condition above is trivially necessary for a perfect matching to exist; indeed, if we had a subset $S \subseteq V_1$ with $|S| > |N(S)|$, then there is no way to match up all of the $|S|$ elements of S along edges with elements in V_2 without using some elements more than once.

We now prove that our condition is sufficient, via the Max-Flow Min-Cut theorem. Take our graph G, orient all of its edges so that they go from V_1 to V_2 add a source vertex s, a sink vertex t, edges from s to all of V_1 from all of V_2 to t, and let c be a capacity function that's identically 1 on all of the edges $s \to V_1, t \to V_2$, and ∞ on all of the original edges in G. By Ford-Fulkerson, **[6]** there is a minimal cut on this graph: call it S. We trivially know that $c(S, \bar{S}) \leq n$, as there is a $n-$cut given by simply setting $S = \{s\}$, we seek to show that $c(S, \bar{S})$ is in fact equal to n.

Let $X = S \cap V_1$ because the capacity of all of the edges originally in G is infinite, we know that any minimal cut cannot contain half of any such edges; therefore, we have $N(X) \subseteq S \cap V_2$

But this means that

$$c(S, \bar{S}) = \sum_{x \in S, y \in \bar{S}} c(x, y) = \sum_{x \in (S \cap \{s\}, y \in (\bar{S} \cap V_1)} c(x, y) + \sum_{x \in (S \cap V_2), y \in (\bar{S} \cap \{t\}} c(x, y)$$

$$\leq n - |X| + |N(X)|$$

$$\leq n - |X| + |X|$$

$$= n$$

4. Conclusion:

In this paper we have shown that the maximum flow from x to y is equal to the size of the minimal vertex cut of x and y, so the minimal vertex cut of x and y must be of size n. Thus the number of pair-wise-internally disjoint paths is equal to the size of the minimum vertex cut. So any minimal cut has capacity n, therefore, there is a flow with value n. Such a flow sends one unit to each vertex in V_1 and sends one unit from each vertex in V_2 to t such a flow form a perfect matching of $G's$ vertices. Thus the maximal flow corresponds to a maximum-cardinality matching, and a minimum cut corresponds to a minimum vertex cover.

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