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THE SYLOW THEOREM AND ITS CONSEQUENCES

Dr Garima Singh

Co-ordinator PG Course,

Barkatuallah University institute of Technology Bhopal (MP) India.

Ajjaz Maqbool Dar

Research scholar, Barkatuallah University institute of Technology Bhopal (MP) India..

Ajjazmaqbool013@gmail.com

Abstract

The aim of the paper is to present some problems and also some partial results mainly on p −groups and converse of langrage's theorem with the help of Sylow theorems.in this paper we find different p −sylow sub-groups and deduce the normalizer of p −sylow subgroups.

Key words: Finite group, the number of subgroups of prime power, index in a group G of the normalizer of any Sylow p −subgroups.

1. Introduction

The converse of Lagrange's theorem is false: if G is a finite group and d|#G, then there may not be a subgroup of G with order d. The simplest example of this is the group A_4, of order 12, which has no subgroup of order 6. The Norwegian mathematician Peter Ludwig Sylow [1] discovered that a converse result is true when d is a prime power. if p is a prime number and P^k |#G.Then G must contain a subgroup of order P^k. Sylow also discovered important relationships among the subgroups whose order is the largest power of p dividing #G, such as the fact that all subgroups of that order are conjugate to each other.

For example,This will be illustrated by an example and the general proof will not be given. Consider the symmetric P_4 of permutation of degree 4.Then $o(P_4)$ =4! = 24 Let A_4 be the alternating group of permutation (i.e. group of even permutation) or degree 4. Then $o(A_4) = \frac{24}{2} = 12$.There exists no subgroup $H$of A_4such that o (H) = 6, though 6 is a divisor of 12.the Lagrange's theorem has important application in group theory. If G is a group of order 8, then there will not exists subgroup of G of order 3,5,6,7. The only subgroup of G may be of order 2, 4. Since 2 and 4 both are divisors of 8.

The converse of Lagrange's theorem is not true, as shown by the alternating group A_4 , which has order 12, but has no subgroup of order 6. The Sylow theorems give the best attempt at a converse, showing that if p^{α} is a prime power that divides o (G), then G has a subgroup of order p^{α}.

For example, a group of order $100 = 2^2 . 5^2$ must contain subgroups of order $1,2,3,4,5, and\ 25$ the subgroups of order 4 are conjugate to each other, and the subgroups of order 25 are conjugate to each other. It is not necessarily the case that the subgroups of order 2 are conjugate or that the subgroups of order 5 are conjugate.

By definition of p-sylow subgroup, In a group of order 100, a 2-Sylow subgroup has order 4 and a 5-Sylow subgroup has order 25. For $p \neq 2\ or\ 5$, a p-Sylow subgroup of a group of order 100 is trivial. In a group of order 12, a 2-Sylow subgroup has order 4 and a 3-Sylow subgroup has order 3. Let's look at a few examples of p-Sylow subgroups in groups of order 12.

Defnition.1.1 Let G be a finite group and p be a prime. Any subgroup of G whose order is the highest power of p dividing #G is called a p-Sylow subgroup of G. In a group of order 100, a 2-Sylow subgroup has order 4 and a 5-Sylow subgroup has order 25. For $p \neq 2$ or 5, a p-Sylow subgroup of a group of order 100 is trivial. In a group of order 12, a 2-Sylow subgroup has order 4 and a 3-Sylow subgroup has order 3. Let's look at a few examples of p-Sylow subgroups in groups of order 12.[2]

Example 1.2. In Z|(12), the only 2-Sylow subgroup is $< 3 >= \{0,3,6,9\}$ and the only 3-Sylow subgroup is $< 4 > = \{0,4,8\}$.

Example 1.3. In A_4, the only 2-Sylow subgroup is and there are four 3-Sylow subgroups:

{(1) (12) (34), (13) (24), (14) (43)}

And there are four 3-Sylow subgroups:

{(1), (123), (132)}, {(1), (124), (142)}, {(1), (134), (143)} {(1), (234), (243)}.

Example 1.4. In D_6 there are three 2-Sylow subgroups:

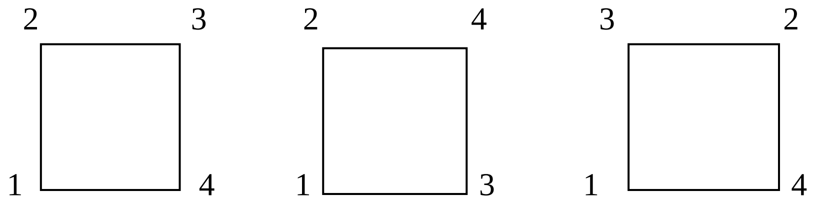
$\{1, r^3,\ s,\ r^3 s\}, \{1, r^3,\ rs,\ r^4 s\}, \{1,\ r^3,\ r^2 s,\ r^5 s\}$

The only 3-Sylow subgroup of D_6 is $\{1, r^2\ ,\ r^4\}$

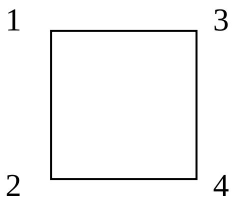
In a group of order24, a 2-Sylow subgroup has order 8 and a 3-Sylow subgroup has order 3.

Example 1.5. In S_4, the 3-Sylow subgroups are the 3-Sylow subgroups of A_4 (an element of 3-power order in S_4 must be a 3-cycle, which all lie in A_4). We determined the 3-Sylow subgroups of A_4 in Example 1.6. There are four of them.

There are three 2-Sylow subgroups ofS_4, and they are interesting to work out since they can be understood as copies of D_4 inside S_4. The number of ways to label the four vertices of a square as 1, 2, 3, and 4 is 4! = 24, but up to rotations and reactions of the square there are really just three different ways of carrying out the labeling, as follows.



Any other labeling of the square is a rotated or reflected version of one of these three labeled squares. For example, the square below is obtained from the middle square above by reflecting across a horizontal line through the middle of the square.



When D_4 acts on a square with labeled vertices, each motion of D_4 creates a permutation of the four vertices, and using the vertex labels this becomes an element of S_4. For example, a 90 degree rotation of the square is a 4-cycle on the vertices. In this way we obtain a copy of D_4 inside S_4. The three essentially different labeling of the vertices of the square above embed D_4 into D_4 as three different subgroups of order 8.

$$\{1,\ (1234),\ (1432)\ (12)(34),\ (13)(24),\ (14)(23),\ (13),(24)\},$$
$$\{1,\ (1243),\ (1342),\ (12)(34),\ (13)(24),\ (14)(23),\ (14),(23)\},$$
$$\{1,\ (1324),\ (1423),\ (12)(34),\ (13)(24),\ (14)(23),\ (12),(34)\}.$$

These are the three 2-Sylow subgroups of S_4, and they are conjugate to each other by a Permutation in S_4 that carries one of the vertexes labeling to another.

Here are the Sylow theorems. They are often given in three parts. The result we call Sylow III* is not always stated explicitly as part of the Sylow theorems.

Theorem 1.6 (Sylow I). A finite group G has a p-Sylow subgroup for every prime p and any p-subgroup of G lies in a p-Sylow subgroup of G.

Theorem 1.7 (Sylow II). For each prime p, the p-Sylow subgroups of G are conjugate.

Theorem 1.8 (Sylow III). For each prime p, let n_p be the number of p-Sylow subgroups of

G. Write $\#G = P^k m$, where p doesn't divide m. Then

$n_p \equiv 1 \; mod \; p$ and $n_p | m$.

2. Main Results

Theorem 2.1. For each prime p, let n_p be the number of p-Sylow subgroups

Of G. Then $n_p = [G : N(\mathrm{P})]$, where P is any p-Sylow subgroup and $N(P)$ is its normalizer.

Let's see what Sylow III tells us about the number of 2-Sylow and 3-Sylow subgroups of a group of order 12. For $p = 2$ and $p = 3$ in Sylow III, the divisibility conditions are $n_2 \,/3 \; and \; n_3 \,/4$ and the congruence conditions are $n_2 \equiv 1 \; mod2 \;\; and \;\; n_3 \equiv 1 \; mod3.$. The divisibility conditions imply n_2 is 1 or 3 and n_3 is $1,2, or \; 4$. The congruence on n_2 tells us nothing new (1 and 3 are both odd), but the congruence on n_3 rules out the option $n_3 = 2$. Therefore n_2 is 1 or 3 and n_3 is 1 or 4. If

$(G) = 24$ the same conclusions follow. (For instance, from $n_3 \,/\, 8$ and $n_3 \equiv 1 \; mod3$ the only choices are$n_2 = 1$ and $n_3 = 4$.) The table below shows the values of n_2 and n_3 in the examples above.

Table.

Group	Size	n_2	n_3
$Z \,/\, (12)$	12	1	1
A_4	12	1	4
D_6	12	3	1
S_4	24	3	4

By above result $[G : N(\mathrm{P})]$, where P is any p-Sylow subgroup and $N(P)$ is its normalizer.

But the other result which distinguish p −sylow subgroups of G such that intersection of both the sub groups is of index is less than or equal to some prime power.

Theorem 2.2. If $n_p \not\equiv 1 \; mod \; p^k$ there exist distinct Sylow p-subgroups P and R of G such that P ∩ R is of index $\leq p^{k-1}$ in both P and R.

We have shown that if G has a subgroup H with index 5 then the left multiplication action of G on the coset space $G|H$gives an isomorphism of G with A_5. The rest of the proof is devoted to showing G has a subgroup with index 5.

G has a subgroup with index 5.

We use Sylow III for the primes 2, 3, and 5. They tell us that
$n_2 \in \{1, 3, 5, 15\}$, $n_3 \in \{1,4,10\}$, $n_5 \in \{1, 6\}$.

Since G is simple, the nontrivial Sylow subgroups are not normal, so $n_2 \mathrm{l} n_3$, and $n_5$exceed 1. Moreover, because Sylow III* says each n_p is the index of a subgroup of G, n_2, n_2 and $n_5 \geq 5$. Therefore

$$n_2 \in \{5,15\}, \qquad n_3 = 10, \qquad n_5 = 15$$

If $n_2 = 5$ then Sylow III* says there is a subgroup of G with index 5 and we're done. What Should we do now: show the only other possibility, that n_2 = 15, leads to a contradiction? Instead we will show that if n_2 = 15 then there is a second way to show G has a subgroup With index 5.

Assume n_2 = 15. G has n_3 .2 = 20 elements of order 3 and n_5 .4 = 24 elements of order 5. This is a total of 44 elements, which leaves at most 60 - 44 = 16 elements that can lie in the 2-Sylow subgroups of G. Each 2-Sylow subgroup of G has size 4 (and thus is abelian), so if n_2 = 15 then we have 15 different subgroups of size 4 squeezed into a 16-element subset of G. These 2-Sylow subgroups can't all pairwise intersect trivially (otherwise there would be 3.15 = 45 non-identity elements among them). Pick two different 2-Sylows, say P and Q, which intersect nontrivially. Let I = P ∩Q. Both P and Q are abelian (they have size 4), so I is normal in each. Therefore the normalizer of I in G contains both P and Q, so it has size properly divisible by 4. The normalizer of I is not all of G since G has no proper nontrivial normal subgroups. Since proper subgroups of G have size 1, 2, 3, 4, 6, or 12, the normalizer of I has size 12 and thus [G : I] = 5.

Since n_2(A5) = 5, we know after the proof that the assumption n_2 = 15 in the last paragraph does not actually occur.

conclusion

Sylow's proof of his theorems appeared in [3]. Here is what he showed (of course, without using the label "Sylow subgroup").

1) There exist p-Sylow subgroups. Moreover, [G : N(P)] ≡ 1 mod p for any p-Sylow subgroup P

2) Let P be a p-Sylow subgroup. The number of p-Sylow subgroups is [G : N(P)]. All p-Sylow subgroups are conjugate.

3) Any finite p-group G contains an increasing chain of subgroups

{e}= $G_0 \subset G_1 \subset G_2 \subset \_\_\_\_ \subset G_n \subset G,$

where each subgroup has index p in the next one. In particular, $\#G_i = p^i$ for all i..

If p^a is the largest power of the prime p which divides the size of the group G,p this group contains a subgroup H of order p^a if moreover $p^a v$ is the size of the largest subgroup of G that normalizes H, the size of G is of the form $p^v(pm+1)$.

Sylow did not have the abstract concept of a group: all groups for him arise as subgroups

of symmetric groups, so groups are always groupes de substitutions." The condition that an element is permutable" with a subgroup$H$ means xH = Hx, or in other words $x \in$ N(H). The end of the first part of his theorem says the normalizer of a Sylow subgroup has index (pm+ 1) for some m which means the index is ≡1 mod p.

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