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Bayesian Analyses of the Burr Type X Distribution under Doubly Type II censored samples using different Priors and Loss functions

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Abstract

The Bayesian analysis of the Burr type X distribution (Exponentiated Rayleigh) has been considered in the paper. The Gamma, Exponential, Chi-Squared and Jeffrey prior have been assumed for posterior analysis. The estimation has been made under doubly type II censored samples. The Bayes estimation has been obtained under eight different loss functions (Squared error, Quadratic, Weighted, Linear exponential, Precautionary, Entropy, De Groot and non-Linear exponential loss functions). The simulation study has been conducted to compare by mean square error (MSE) for the performance of various estimators.

Keyword: Bayesian Analyses, Exponentiated Rayleigh Distribution, Burr type X distribution, Loss function, Prior, Posterior, (Squared error, Quadratic, Weighted, Linear exponential, Precautionary, Entropy, De Groot and non-Linear exponential) loss functions.

1 – Introduction.

Burr [2] introduction twelve different forms of cumulative distribution function for modeling data. And formulas as follows: [8]

- Burr type I; $F(x) = x \qquad 0 < x < 1$
- Burr type II; $F(x) = (e^{-x} + 1)^{-\lambda} \qquad -\infty < x < \infty$
- Burr type III; $F(x) = \left(x^{-\theta} + 1\right)^{-\lambda}$
- Burr type IV; $F(x) = \left(\left((\theta - x)/x\right)^{1/\theta} + 1\right)^{-\lambda} \qquad 0 < x < \theta$
- Burr type V; $F(x) = (\theta e^{-\tan x} + 1)^{-\lambda} \qquad -\pi/2 < x < \pi/2$
- Burr type VI; $F(x) = \left(\theta e^{-\theta \sinh x} + 1\right)^{-\lambda}$
- Burr type VII; $F(x) = 2^{\lambda}\,(1 + \tanh x)^{\lambda}$
- Burr type VIII; $F(x) = \left((2/\pi)\tan^{-1} e^{x}\right)^{\lambda}$
- Burr type IX; $F(x) = 1 - 2/\left(\theta\left((1 + e^{-x})^{\lambda} - 1\right) + 2\right)$
- Burr type X; $F(x) = \left(1 - e^{-(\theta x)^2}\right)^{\lambda}$
- Burr type XI; $F(x) = \left((x - (1/2\pi)\sin 2\pi x)\right)^{\lambda} \qquad 0 < x < 1$
- Burr type XII; $F(x) = 1 - \left(1 + x^{\theta}\right)^{-\lambda}$

Among those twelve distribution functions, Burr Type X and Burr Type XII received the maximum attention. Surles and Padgett [14] observed that the Burr Type X distribution can be used quite effectively in modelling strength data and also modelling general lifetime data. Several aspects of the one parameter (scale parameter=1)

Burr Type X distribution were studies by Sartawi and Abu-Salih [10], Jaheen [7], Ahmed, Fakhry and Jaheen [1], Raqab [9], Hassan, Albadri, Ibrahim and Ameen [5], Feroze and Aslam [4] and Sindhu and Aslam [11]. The cumulative distribution function (CDF), and the probability density function (pdf) of the Burr Type X distribution with shape parameter $(\lambda > 0)$ are respectively as follows:-

$$F(x;\lambda) = \left(1 - e^{-x^2}\right)^{\lambda} \quad ; \quad x > 0, \quad \lambda > 0 \qquad \dots (1)$$

$$f(x;\lambda) = 2\lambda x e^{-x^2}\left(1 - e^{-x^2}\right)^{\lambda-1} \quad ; \quad x > 0\,,\ \lambda > 0 \qquad \dots (2)$$

The random number X has been generated by inverse function method, which is for uniform random U. $X = \left(-\ln\left(1 - U^{\frac{1}{\lambda}}\right)\right)^{\frac{1}{2}}$ … (3)

The problem of estimating the unknown parameters in statistical distributions used to study a certain phenomenon is one of the important problems facing constantly those who are interested in applied statistics. This paper considers the estimations of the unknown parameters of the Burr Type X distribution. This distribution is an important distribution in statistics and operations research. It is applied in several areas such as health, agriculture, biology, and other sciences. The main aim of this is to consider the Bayesian analysis of the unknown parameters under different priors (information and non-information) and loss functions under doubly type II censored samples.

2 – Likelihood function.

Consider a random sample of size (n) from Burr Type X distribution, and let $(x_r, \dots, x_s)$ be the ordered observation remaining when the $(r-1)$ smallest observations and the $(n-s)$ largest observation have been censored, the likelihood function for λ given the Type II doubly censored sample $\underline{x} = (x_r, \dots, x_s)$ is:[4][3]

$$L(\lambda|\underline{x}) = \left(\frac{n!}{(n-1)!(n-s)!}\right)\prod_{i=r}^{s} f(x_i,\lambda)\left(F(x_r)\right)^{r-1}\left(1 - F(x_s)\right)^{n-s}$$

Such that:

- $\prod_{i=r}^{s} f(x_i,\lambda) = \prod_{i=r}^{s}\left(2\lambda x_i e^{-x_i^2}(1 - e^{-x_i^2})^{\lambda-1}\right)$

$$= 2^{s-r+1}\lambda^{s-r+1}\prod_{i=r}^{s} x_i\, e^{-\sum_{i=r}^{s} x_i^2}\prod_{i=r}^{s}\left(1 - e^{-x_i^2}\right)^{\lambda-1}$$

$$= 2^m\lambda^m e^{\sum_{i=r}^{s}\ln x_i} e^{-\sum_{i=r}^{s} x_i^2} e^{\sum_{i=1}^{n}\ln\left(1-e^{-x_i^2}\right)^{-1}} e^{-\lambda\sum_{i=r}^{s}\ln\left(1-e^{-x_i^2}\right)^{-1}}$$

$$= 2^m\lambda^m e^{(\sum_{i=r}^{s}\ln x_i - \sum_{i=r}^{s} x_i^2 + \sum_{i=1}^{n}\ln(1-e^{-x_i^2})^{-1}} e^{-\lambda\sum_{i=r}^{s}\ln(1-e^{-x_i^2})^{-1}}$$

- $\left(F(x_r)\right)^{r-1} = \left(\left(1 - e^{-x_r^2}\right)^{\lambda}\right)^{r-1} = \left(1 - e^{-x_r^2}\right)^{\lambda(r-1)} = e^{-\lambda(r-1)\ln\left(1-e^{-x_r^2}\right)^{-1}}$

- $\left(1 - F(x_s)\right)^{n-s} = \sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j(1)^{n-s-j}\left(F(x_s)\right)^j$

$$= \sum_{j=0}^{n-s}\binom{n-s}{j}(-1)^j(1)^{n-s-j}\left(\left(1 - e^{-x_s^2}\right)^{\lambda}\right)^j = \sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\left(\left(1 - e^{-x_s^2}\right)\right)^{\lambda j}$$

$$\therefore L(\lambda|\underline{x}) = Q\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\lambda^m e^{-\lambda\left(\psi_j(x)\right)} \qquad \dots (4)$$

Where $Q = \frac{n!2^m e^{(\sum_{i=r}^{s}\ln x_i - \sum_{i=r}^{s} x_i^2 + \sum_{i=1}^{n}\ln(1-e^{-x_i^2})^{-1}}}{(n-1)!(n-s)!}$; $m = s - r + 1$

$$\psi_j(x) = \sum_{i=r}^{s}\ln(1 - e^{-x_i^2})^{-1} + (r-1)\ln\left(1 - e^{-x_r^2}\right)^{-1} + j\ln\left(1 - e^{-x_s^2}\right)^{-1}$$

3 - Bayesian Estimators under Doubly Type II censored samples using different Priors and Loss functions.

In this section Bayesian Estimators of the shape parameter for four different prior functions and under eight different loss functions has been determined.

● Types of loss function using in this paper.

If $\hat{\lambda}$ represent of estimator for the shape parameter $\hat{\lambda}$

1 - Squared error loss function(slf): the squared error loss function defined as: [13]

$$L(\hat{\lambda},\lambda) = c(\hat{\lambda}-\lambda)^2 \quad ; \quad \hat{\lambda}_{slf} = E(\lambda) \qquad \dots (5)$$

2 - Quadratic loss function(qlf): the quadratic loss function defined as: [13]

$$L(\hat{\lambda},\lambda) = \left(\frac{\hat{\lambda}-\lambda}{\lambda}\right)^2 \quad ; \quad \hat{\lambda}_{qlf} = \frac{E(\lambda^{-1})}{E(\lambda^{-2})} \qquad \dots (6)$$

3 - Weighted loss function(wlf): the weighted loss function defined as: [4]

$$L(\hat{\lambda},\lambda) = \frac{(\hat{\lambda}-\lambda)^2}{\lambda} \quad ; \quad \hat{\lambda}_{wlf} = \frac{1}{E(\lambda^{-1})} \qquad \dots (7)$$

4 - Linear exponential loss function(LINX): the (LINX) loss function defined as: [6]

$$L(\hat{\lambda},\lambda) = \left(e^{c(\hat{\lambda}-\lambda)} - c(\hat{\lambda}-\lambda) - 1\right) \quad ; \quad \hat{\lambda}_{llf} = -\frac{1}{c}\ln E\left(e^{-c\lambda}\right) \qquad \dots (8)$$

5 - Precautionary loos function(plf): the Precautionary loss function defined as: [4]

$$L(\hat{\lambda},\lambda) = \frac{(\hat{\lambda}-\lambda)^2}{\hat{\lambda}} \quad ; \quad \hat{\lambda}_{plf} = \sqrt{E(\lambda^2)} \qquad \dots (9)$$

6 - Entropy loss function(Elf): the entropy loss function defined as: [6]

$$L(\hat{\lambda},\lambda) = \left((\hat{\lambda}/\lambda)^t - t\ln(\hat{\lambda}/\lambda) - 1\right) \quad ; \quad \hat{\lambda}_{Elf} = \left(E(\lambda^{-t})\right)^{-1/t} \qquad \dots (10)$$

7 - De Groot loss function(Dlf): the De Groot loss function defined as: [6]

$$L(\hat{\lambda},\lambda) = \left(\frac{\lambda-\hat{\lambda}}{\hat{\lambda}}\right)^2 \quad ; \quad \hat{\lambda}_{Dlf} = \frac{E(\lambda^2)}{E(\lambda)} \qquad \dots (11)$$

8 - Non- Linear exponential loss functions(NLINEX): the (NLINX) loss function defined as: [12]

$$L(\hat{\lambda},\lambda) = \left(e^{c(\hat{\lambda}-\lambda)} + c(\hat{\lambda}-\lambda)^2 - c(\hat{\lambda}-\lambda) - 1\right)$$

$$\hat{\lambda}_{Nllf} = -\frac{1}{c+2}\left(\ln E\left(e^{-c\lambda}\right) - 2E(\lambda)\right) \qquad \dots (12)$$

● Posterior distributions with different prior.

The posterior density function of the shape parameter for the given random X is well known as:

$$p(\lambda|\underline{x}) = \frac{L(\lambda|\underline{x})\,.\,p(\lambda)}{\int_0^\infty L(\lambda|\underline{x})\,.\,p(\lambda)\,d\lambda} \qquad \dots (13)$$

For Bayesian estimation, we specify four different prior distributions for the shape parameter, and which can be obtained four different posterior distributions under doubly type II censored samples, as follows:

1– The Non-information prior, for any parameter λ, with pdf as: $p(\lambda) = \frac{1}{\lambda} \quad \lambda > 0$

By equation (13) the posterior distribution under the assumption Non-information prior is:

$$p_J(\lambda|\underline{x}) = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\lambda^{m-1}e^{-\lambda\left(\psi_j(x)\right)}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)\right)^m}} \qquad \text{... (14)}$$

2– The Chi-Square prior is assumed to be: $p(\lambda) = \frac{\lambda^{(d/2)-1}e^{-\frac{\lambda}{2}}}{\Gamma(d/2)\ 2^{(d/2)}};\lambda, d > 0$

By equation (13) the posterior distribution under the assumption Chi-Square prior is:

$$p_{Ch}(\lambda|\underline{x}) = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\lambda^{m+\frac{d}{2}-1}e^{-\lambda\left(\psi_j(x)+\frac{1}{2}\right)}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\left(\frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{\left(m+\frac{d}{2}\right)}}\right)} \qquad \text{... (15)}$$

3 – The Exponential prior, for any parameter λ, as: $p(\lambda) = be^{-b\lambda}$; $b > 0$

By equation (13) the posterior distribution under the assumption Exponential prior is:

$$p_E(\lambda|\underline{x}) = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\lambda^m e^{-\lambda\left(\psi_j(x)+b\right)}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\left(\frac{\Gamma(m+1)}{\left(\psi_j(x)+b\right)^{(m+1)}}\right)} \qquad \text{... (16)}$$

4 – The Gamma prior, assumed to be: $p(\lambda) = \frac{b^a\lambda^{a-1}}{\Gamma(a)}e^{-\lambda b}$; $a, b > 0$

By equation (13) the posterior distribution under the assumption Gamma prior is:

$$p_G(\lambda|\underline{x}) = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\lambda^{m+a-1}e^{-\lambda\left(\psi_j(x)+b\right)}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\left(\frac{\Gamma(m+a)}{\left(\psi_j(x)+b\right)^{(m+a)}}\right)} \qquad \text{... (17)}$$

3-1. Bayesian Estimators under Doubly Type *II* censored samples under Non-information Prior using different loss functions.

$$\hat{\lambda}_{1_S} = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+1)}{\left(\psi_j(x)\right)^{m+1}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)\right)^m}}; \qquad \hat{\lambda}_{1_Q} = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m-1)}{\left(\psi_j(x)\right)^{m-1}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m-2)}{\left(\psi_j(x)\right)^{m-2}}};$$

$$\hat{\lambda}_{1_W} = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)\right)^m}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m-1)}{\left(\psi_j(x)\right)^{m-1}}}; \qquad \hat{\lambda}_{1_L} = \frac{-1}{c}ln\left(\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)+c\right)^m}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)\right)^m}}\right);$$

$$\hat{\lambda}_{1_P} = \sqrt{\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+2)}{\left(\psi_j(x)\right)^{m+2}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)\right)^m}}}; \qquad \hat{\lambda}_{31_E} = \sqrt[t]{\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)\right)^m}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m-t)}{\left(\psi_j(x)\right)^{m-t}}}};$$

$$\hat{\lambda}_{1_N} = \left(\frac{-1}{c+2}\right)\left(\left(ln\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\left(\frac{\Gamma(m)}{\left(\left(\psi_j(x)\right)+c\right)^m}\right)}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)\right)^m}}\right) - 2\left(\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+1)}{\left(\psi_j(x)\right)^{m+1}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m)}{\left(\psi_j(x)\right)^m}}\right)\right);$$

$$\hat{\lambda}_{1_D} = \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+2)}{\left(\psi_j(x)\right)^{m+2}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+1)}{\left(\psi_j(x)\right)^{m+1}}}$$

3-2. Bayesian Estimators under Doubly Type *II* censored samples under Chi-Square Prior using different loss functions.

$$\hat{\lambda}_{2_S} = \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}+1\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}+1}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}}}} \quad ; \quad \hat{\lambda}_{2_Q} = \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}-1\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}-1}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}-2\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}-2}}};$$

$$\hat{\lambda}_{2_W} = \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}-1\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}-1}}} \quad ; \quad \hat{\lambda}_{2_L} = \frac{-1}{c} ln \left(\frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}+c\right)^{m+\frac{d}{2}}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}}}} \right);$$

$$\hat{\lambda}_{2_P} = \sqrt{\frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}+2\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}+2}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}}}}} \quad ; \quad \hat{\lambda}_{2_E} = \sqrt[t]{\frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}-t\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}-t}}}};$$

$$\hat{\lambda}_{2_N} = \left(\frac{-1}{c+2}\right)\left(\left(ln \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \left(\frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\left(\psi_j(x)+\frac{1}{2}\right)+c\right)^{m+\frac{d}{2}}} \right)}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}}}} \right) - 2\left(\frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}+1\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}+1}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}}}} \right)\right);$$

$$\hat{\lambda}_{2_D} = \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}+2\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}+2}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma\left(m+\frac{d}{2}+1\right)}{\left(\psi_j(x)+\frac{1}{2}\right)^{m+\frac{d}{2}+1}}}$$

3-3. Bayesian Estimators under Doubly Type *II* censored samples under Exponential Prior using different loss functions.

$$\hat{\lambda}_{3_S} = \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+2)}{\left(\psi_j(x)+b\right)^{m+2}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+1)}{\left(\psi_j(x)+b\right)^{m+1}}} \quad ; \quad \hat{\lambda}_{3_Q} = \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m)}{\left(\psi_j(x)+b\right)^{m}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m-1)}{\left(\psi_j(x)+b\right)^{m-1}}};$$

$$\hat{\lambda}_{3_W} = \frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+1)}{\left(\psi_j(x)+b\right)^{m+1}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m)}{\left(\psi_j(x)+b\right)^{m}}} \quad ; \quad \hat{\lambda}_{3_L} = \frac{-1}{c} ln \left(\frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+1)}{\left(\psi_j(x)+b+c\right)^{m+1}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+1)}{\left(\psi_j(x)+b\right)^{m+1}}} \right);$$

$$\hat{\lambda}_{3_P} = \sqrt{\frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+3)}{\left(\psi_j(x)+b\right)^{m+3}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+1)}{\left(\psi_j(x)+b\right)^{m+1}}}} \quad ; \quad \hat{\lambda}_{3_E} = \sqrt[t]{\frac{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+1)}{\left(\psi_j(x)+b\right)^{m+1}}}{\sum_{j=0}^{n-s}(-1)^j \binom{n-s}{j} \frac{\Gamma(m+1-t)}{\left(\psi_j(x)+b\right)^{m+1-t}}}};$$

$$\hat{\lambda}_{3_N} = \left(\frac{-1}{c+2}\right)\left(\left(ln\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\left(\frac{\Gamma(m+1)}{\left(\left(\psi_j(x)+b\right)+c\right)^{m+1}}\right)}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+1)}{\left(\psi_j(x)+b\right)^{m+1}}}\right) - 2\left(\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+2)}{\left(\psi_j(x)+b\right)^{m+2}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+1)}{\left(\psi_j(x)+b\right)^{m+1}}}\right)\right);$$

$$\hat{\lambda}_{3_D} = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+3)}{\left(\psi_j(x)+b\right)^{m+3}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+2)}{\left(\psi_j(x)+b\right)^{m+2}}}$$

3-4. Bayesian Estimators under Doubly Type *II* censored samples under Gamma Prior using different loss functions.

$$\hat{\lambda}_{4_S} = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a+1)}{\left(\psi_j(x)+b\right)^{m+a+1}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a)}{\left(\psi_j(x)+b\right)^{m+a}}} \quad ; \quad \hat{\lambda}_{4_Q} = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a-1)}{\left(\psi_j(x)+b\right)^{m+a-1}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a-2)}{\left(\psi_j(x)+b\right)^{m+a-2}}};$$

$$\hat{\lambda}_{4_W} = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a)}{\left(\psi_j(x)+b\right)^{m+a}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a-1)}{\left(\psi_j(x)+b\right)^{m+a-1}}} \quad ; \quad \hat{\lambda}_{4_L} = \frac{-1}{c}\,ln\left(\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a)}{\left(\psi_j(x)+b+c\right)^{m+a}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a)}{\left(\psi_j(x)+b\right)^{m+a}}}\right);$$

$$\hat{\lambda}_{4_P} = \sqrt{\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a+2)}{\left(\psi_j(x)+b\right)^{m+a+2}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a)}{\left(\psi_j(x)+b\right)^{m+a}}}} \quad ; \quad \hat{\lambda}_{4_E} = \sqrt[t]{\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a)}{\left(\psi_j(x)+b\right)^{m+a}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a-t)}{\left(\psi_j(x)+b\right)^{m+a-t}}}};$$

$$\hat{\lambda}_{4_N} = \left(\frac{-1}{c+2}\right)\left(\left(ln\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\left(\frac{\Gamma(m+a)}{\left(\left(\psi_j(x)+b\right)+c\right)^{m+a}}\right)}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a)}{\left(\psi_j(x)+b\right)^{m+a}}}\right) - 2\left(\frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a+1)}{\left(\psi_j(x)+b\right)^{m+a+1}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a)}{\left(\psi_j(x)+b\right)^{m+a}}}\right)\right);$$

$$\hat{\lambda}_{4_D} = \frac{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a+2)}{\left(\psi_j(x)+b\right)^{m+a+2}}}{\sum_{j=0}^{n-s}(-1)^j\binom{n-s}{j}\frac{\Gamma(m+a+1)}{\left(\psi_j(x)+b\right)^{m+a+1}}}$$

4 – Simulation results and Conclusions.

In this section, the results presented of some of numerical experiments to compare the performance of the Bayes estimators for shape parameter under four prior distributions and eight loss functions proposed in the previous sections, applying Monte Carlo simulations to comer the performance of different estimators, mainly with respect to their mean squared error (MSE) for different sample size ($n = 10, 20, 30, 50, 75$ and 100) and two values of the shape parameters(1.5 and 3). The results of (MSE) are computed over (1000) replications for two different cases $\{$case I: $(a = 3,\ b = 0.8,\ d = 3,\ c = 1,\ t = 2)$; case II: $(a = 4, b = 1.2, d = 4, c = 1, t = 2)\}$ and recorded in sections $(3-1), (3-2), (3-3), (3-4)$. And for the purpose of conducting was determined fixed values (r, s) and by the size of each sample and Table (1) shows that.

Table (1)

n	10	20	30	50	75	100
r	3	6	10	30	40	70
s	7	14	20	42	72	93

Table (2). The value of (MSE) for Bayesian est. when $\lambda = 1.5$

1Xn	Jeffery								Best
	BS	BQ	BW	BP	BD	BL	BE	BNL	
10	0.4725	0.2881	0.3272	0.5815	0.7246	0.2694	0.2937	0.3934	L
20	0.1949	0.1544	0.1624	0.2199	0.2519	0.1441	0.1553	0.1757	L
30	0.1341	0.1131	0.1176	0.1466	0.1624	0.1108	0.1139	0.1254	L
50	0.1375	0.1141	0.1199	0.1506	0.1669	0.1127	0.1155	0.1283	L
75	0.0722	0.0647	0.0665	0.0764	0.0816	0.0645	0.0651	0.0693	L
100	0.0568	0.0517	0.0528	0.0599	0.0637	0.0514	0.0519	0.0548	L
	Chi-Square priors $(d = 3)$								
10	0.4666	0.2341	0.3093	0.5732	0.7063	0.2680	0.2603	0.3913	Q
20	0.1978	0.1352	0.1557	0.2265	0.2617	0.1430	0.1426	0.1775	Q
30	0.1400	0.1060	0.1175	0.1552	0.1735	0.1122	0.1103	0.1298	Q
50	0.1434	0.1073	0.1199	0.1591	0.1779	0.1143	0.1122	0.1328	Q
75	0.0750	0.0627	0.0671	0.0803	0.0865	0.0656	0.0644	0.0716	Q
100	0.0591	0.0502	0.0532	0.0630	0.0676	0.0523	0.0514	0.0566	Q
	Exponential priors $(b = 0.8)$								
10	0.2679	0.2144	0.2055	0.3242	0.4019	0.1767	0.2012	0.2315	L
20	0.1438	0.1266	0.1251	0.1605	0.1828	0.1133	0.1233	0.1320	L
30	0.1123	0.1006	0.1012	0.1216	0.1338	0.0954	0.0996	0.1059	L
50	0.1144	0.1007	0.1023	0.1242	0.1368	0.0963	0.1002	0.1075	L
75	0.0655	0.0604	0.0612	0.0689	0.0733	0.0592	0.0603	0.0631	L
100	0.0522	0.0487	0.0491	0.0547	0.0579	0.0477	0.0486	0.0505	L
	Gamma priors $(a = 3 \ , b = 0.8)$								
10	0.6078	0.2679	0.4019	0.7348	0.8858	0.3556	0.3242	0.5146	Q
20	0.2421	0.1438	0.1828	0.2789	0.3218	0.1716	0.1605	0.2166	Q
30	0.1658	0.1123	0.1338	0.1856	0.2084	0.1293	0.1216	0.1527	Q
50	0.1697	0.1144	0.1368	0.1899	0.2131	0.1320	0.1242	0.1562	Q
75	0.0846	0.0655	0.0733	0.0915	0.0994	0.0723	0.0689	0.0802	Q
100	0.0664	0.0522	0.0579	0.0716	0.0775	0.0574	0.0547	0.0632	Q
	Chi-Square priors $(d = 4)$								
10	0.5761	0.2614	0.3776	0.7030	0.8573	0.3277	0.3076	0.4832	Q
20	0.2270	0.1427	0.1740	0.2611	0.3018	0.1610	0.1554	0.2028	Q
30	0.1554	0.1104	0.1274	0.1733	0.1944	0.1223	0.1174	0.1434	Q
50	0.1593	0.1122	0.1303	0.1776	0.1992	0.1249	0.1198	0.1468	Q
75	0.0803	0.0644	0.0706	0.0865	0.0936	0.0693	0.0670	0.0764	Q
100	0.0630	0.0514	0.0558	0.0676	0.0730	0.0550	0.0532	0.0601	Q
	Exponential priors $(b = 1.2)$								
10	0.1861	0.2263	0.1760	0.2129	0.2568	0.1481	0.1945	0.1692	L
20	0.1173	0.1266	0.1127	0.1264	0.1405	0.1010	0.1175	0.1105	L
30	0.0978	0.0998	0.0939	0.1034	0.1116	0.0876	0.0956	0.0937	L
50	0.0990	0.0990	0.0941	0.1050	0.1137	0.0877	0.0953	0.0945	L
75	0.0602	0.0596	0.0582	0.0625	0.0656	0.0560	0.0585	0.0586	L
100	0.0484	0.0484	0.0471	0.0501	0.0524	0.0455	0.0474	0.0473	L
	Gamma priors $(a = 4 \ , b = 1.2)$								
10	0.5805	0.2568	0.3881	0.6971	0.8340	0.3506	0.3133	0.4963	Q
20	0.2427	0.1405	0.1822	0.2794	0.3218	0.1731	0.1587	0.2176	Q
30	0.1688	0.1116	0.1353	0.1891	0.2122	0.1314	0.1221	0.1555	Q
50	0.1725	0.1137	0.1382	0.1931	0.2166	0.1340	0.1246	0.1588	Q
75	0.0866	0.0656	0.0744	0.0939	0.1022	0.0736	0.0696	0.0820	Q

100	0.0681	0.0524	0.0589	0.0737	0.0800	0.0586	0.0553	0.0647	Q

Table (3). The value of (MSE) for Bayesian est. when λ =3

1Xn	Jeffery								**Best**
	BS	BQ	BW	BP	BD	BL	BE	BNL	
10	1.8899	1.1524	1.3087	2.3260	2.8986	0.7885	1.1746	1.3903	L
20	0.7795	0.6177	0.6497	0.8794	1.0075	0.4807	0.6212	0.6498	L
30	0.5363	0.4524	0.4706	0.5863	0.6496	0.3913	0.4554	0.4745	L
50	0.5499	0.4565	0.4795	0.6022	0.6678	0.3940	0.4619	0.4843	L
75	0.2887	0.2588	0.2662	0.3055	0.3263	0.2387	0.2605	0.2679	L
100	0.2271	0.2067	0.2112	0.2394	0.2546	0.1922	0.2075	0.2124	L
	Chi-Square priors $(d = 3)$								
10	1.0452	0.7733	0.7781	1.2708	1.5756	0.5515	0.7429	0.8101	L
20	0.5670	0.4763	0.4829	0.6370	0.7288	0.3840	0.4698	0.4845	L
30	0.4472	0.3881	0.3974	0.4869	0.5377	0.3379	0.3877	0.3997	L
50	0.4554	0.3885	0.4018	0.4969	0.5495	0.3381	0.3900	0.4051	L
75	0.2619	0.2369	0.2425	0.2767	0.2951	0.2190	0.2380	0.2439	L
100	0.2289	0.2158	0.2170	0.2386	0.2512	0.1999	0.2151	0.2164	L
	Exponential priors $(b = 0.8)$								
10	0.6031	1.0285	0.7120	0.6253	0.7022	0.6595	0.8496	0.5820	NL
20	0.4110	0.5368	0.4398	0.4220	0.4507	0.4067	0.4806	0.3941	NL
30	0.3556	0.4111	0.3649	0.3647	0.3834	0.3399	0.3837	0.3416	L
50	0.3568	0.4051	0.3625	0.3676	0.3880	0.3349	0.3795	0.3406	L
75	0.2265	0.2408	0.2271	0.2311	0.2390	0.2152	0.2323	0.2195	L
100	0.2056	0.2221	0.2088	0.2077	0.2124	0.2001	0.2001	0.2013	L ,E
	Gamma priors $(a = 3 \ , b = 0.8)$								
10	1.0101	0.6031	0.7022	1.2369	1.5273	0.4725	0.6253	0.7710	L
20	0.5588	0.4110	0.4507	0.6375	0.7357	0.3487	0.4220	0.4692	L
30	0.4483	0.3556	0.3834	0.4942	0.5504	0.3187	0.3647	0.3946	L
50	0.4563	0.3568	0.3880	0.5038	0.5615	0.3197	0.3676	0.4002	L
75	0.2647	0.2265	0.2390	0.2823	0.3035	0.2128	0.2311	0.2438	L
100	0.2293	0.2056	0.2124	0.2414	0.2562	0.1931	0.2077	0.2145	L
	Chi-Square priors $(d = 4)$								
10	1.2774	0.7429	0.8788	1.5677	1.9398	0.5655	0.7761	0.9619	L
20	0.6382	0.4699	0.5152	0.7274	0.8389	0.3913	0.4825	0.5330	L
30	0.4874	0.3877	0.4173	0.5371	0.5982	0.3447	0.3972	0.4282	L
50	0.4974	0.3901	0.4236	0.5489	0.6117	0.3467	0.4016	0.4355	L
75	0.2768	0.2380	0.2505	0.2950	0.3170	0.2228	0.2425	0.2550	L
100	0.2198	0.1917	0.2005	0.2334	0.2499	0.1807	0.1947	0.2039	L
	Exponential priors $(b = 1.2)$								
10	0.6242	1.3800	0.9226	0.5365	0.4851	0.8985	1.1391	0.6920	D
20	0.3954	0.6576	0.4972	0.3668	0.3522	0.4919	0.5715	0.4163	D
30	0.3358	0.4711	0.3869	0.3228	0.3178	0.3806	0.4254	0.3438	D
50	0.3318	0.4606	0.3797	0.3203	0.3168	0.3724	0.4165	0.3382	D
75	0.2148	0.2592	0.2308	0.2114	0.2110	0.2267	0.2436	0.2159	D
100	0.1761	0.2120	0.1893	0.1731	0.1725	0.1869	0.1995	0.1774	D
	Gamma priors $(a = 4 \ , b = 1.2)$								
10	0.6864	0.4851	0.5057	0.8336	1.0276	0.3742	0.4752	0.5401	L
20	0.4420	0.3522	0.3677	0.5003	0.5751	0.2978	0.3525	0.3779	L
30	0.3811	0.3178	0.3328	0.4172	0.4625	0.2834	0.3211	0.3394	L
50	0.3862	0.3168	0.3350	0.4237	0.4704	0.2829	0.3217	0.3426	L

75	0.2404	0.2110	0.2195	0.2553	0.2735	0.1975	0.2137	0.2228	L
100	0.1943	0.1725	0.1786	0.2056	0.2196	0.1625	0.1743	0.1811	L

We have presented the simulation results using MATLAB program. A simulation results are conducted to examine and compare the performance of the estimates for shape parameter respecting to their MSE. The best estimator has the smallest value of MSE. When we have as well as doubly type II censored sample are summarized tables (2) and (3) in tables (4).

Table (4) the best performances for loss function and prior dist.

Test	prior	MSE					
		10	20	30	50	75	100
1	Jeffery	BL	BL	BL	BL	BL	BL
	Chi-Sq.	BQ	BQ	BQ	BQ	BQ	BQ
	Exp.	BL	BL	BL	BL	BL	BL
	Gamma	BQ	BQ	BQ	BQ	BQ	BQ
	Best prior	Exp.	Exp.	Exp.	Exp.	Exp.	Exp.
2	Jeffery	BL	BL	BL	BL	BL	BL
	Chi-Sq.	BQ	BQ	BQ	BQ	BQ	BQ
	Exp.	BL	BL	BL	BL	BL	BL
	Gamma	BQ	BQ	BQ	BQ	BQ	BQ
	Best prior	**Exp.**	**Exp.**	**Exp.**	**Exp.**	**Exp.**	**Exp.**
3	Jeffery	BL	BL	BL	BL	BL	BL
	Chi-Sq.	BL	BL	BL	BL	BL	BL
	Exp.	BNL	BNL	BL	BL	BL	BL, BE
	Gamma	BL	BL	BL	BL	BL	BL
	Best prior	Gamma	Gamma	Gamma	Gamma	Gamma	Jeffery
4	Jeffery	BL	BL	BL	BL	BL	BL
	Chi-Sq.	BL	BL	BL	BL	BL	BL
	Exp.	BD	BD	BD	BD	BD	BD
	Gamma	BL	BL	BL	BL	BL	BL
	Best prior	**Gamma**	**Gamma**	**Gamma**	**Gamma**	**Gamma**	**Gamma**

From the table (4) it can be assessed that the rate of converges of the estimates towards the true value of the shape parameter increases with increase of sample size. However, the rate of convergence is random with respect to loss function and prior distribution. The estimates with $(\lambda = 1.5)$ under Exponential prior $(b = 1.2)$ with linear Exponential loss function the Best than those under Jeffery, Chi-Square and Gamma prior using other loss function. And the estimates with $(\lambda = 3)$ under Gamma prior $(a = 4, b = 1.2)$ with linear Exponential loss function the Best than those under Jeffery, Chi-Square and Exponential prior using other loss function for the doubly type II censoring Similarly, the estimates based on linear Exponential loss function are associated with the minimum MSE.

5 - Conclusions:

The above study suggests that in order to estimate the parameter of Burr type X distribution under a Bayesian framework, When $(\lambda = 1.5)$, the performance of Bayes estimator under linear loss function with Exponential $(b = 1.2)$ prior, records full appearance "for all sample sizes", as the best loss function and prior distribution. And when $(\lambda = 3)$, the performance of Bayes estimator under linear loss function with Gamma $(a = 4, b = 1.2)$prior, records full appearance "for all sample sizes", as the best loss function and prior distribution. Can be preferred for the doubly type II censored sample.

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