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Weighted Composition Operators from Hardy Spaces to Weighted-Type Spaces on the Upper Half-Plane

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Abstract

Let the holomorphic mapping ψ, and the holomorphic self-map φ are on the upper half-plane. We characterize bounded weighted composition operators between the Hardy space and the weighted-type space on the upper half-plane, and we study the special cases when $\alpha = 1$ which is the Hilbert space. Under a mild condition on ψ; we also show the compactness of these operators and there special cases.

Keywords: Weighted composition operators, Hardy spaces, weighted type spaces, upper half plane.

1. Introduction

Let $\Pi^+ = \{z \in C : \mathfrak{J}z > 0\}$ be the upper half-plane, Ω a domain in C or C^n, and $H(\Omega)$ the space of all holomorphic functions on Ω. Let $\psi \in H(\Omega)$, and let φ be a holomorphic self-map of Ω. Then by

$$W_{\phi,\psi}(f)(z) = \psi(fo\phi)(z)\,, z \in \Omega \tag{1}$$

is defined a linear operator on $H(\Omega)$ which is called weighted composition operator. If $\psi(z) = 1$ then $W_{\phi,\psi}$ becomes composition operator and is denoted by C_φ, and if $\varphi(z) = z$ then $W_{\phi,\psi}$ becomes multiplication operators and is denoted by M_ψ .

During the past few decades, composition operators and weighted composition operators have been studied extensively on spaces of holomorphic functions on various domains in C or C^n (see, e.g. [1− 21]. For some other operators related to weighted composition operators, see [22-29].

While there is a vast literature on composition and weighted composition operators between spaces of holomorphic functions on the unit disk $\mathbb{D}$, there are few papers on these and related operators on spaces of functions holomorphic in the upper half-plane (see, e.g., [5,7,8,9,16,17,18,30]. For related results in the setting of the complex plane see also papers[19− 21].

The behavior of composition operators on spaces of functions holomorphic in the upper half-plane is considerably different from the behavior of composition operators on spaces of functions holomorphic in the unit disk $\mathbb{D}$. For example, there are holomorphic self-maps of Π^+ which do not induce composition operators on Hardy and Bergman spaces on the upper half-plane, where as it is a well-known consequence of the Little wood subordination principle that every holomorphic self-map φ of $\mathbb{D}$ induces a bounded composition operator on the Hardy and weighted Bergman spaces on $\mathbb{D}$. Also, Hardy and Bergman spaces on the upper half-plane do not support compact composition operators (see[5]).

For $0 < p < \infty$ and $\alpha \in (-1, \infty)$, let $L^p(\Pi^+, dA_\alpha)$ denote the collection of all Lebesgue *p*-integrable functions $f : \Pi^+ \to c$ such that :

$$\int_{\Pi^+} |f(z)|^p \, dA_\alpha(z) < \infty \tag{2}$$

where

$$dA_\alpha(z) = \frac{1}{\pi}(\alpha + 1)(2\mathfrak{J}z)^\alpha dA(z) \tag{3}$$

Let $A^p_\alpha(\Pi^+) = L^p(\Pi^+, dA_\alpha) \cap H(\Pi^+)$. For $1 \le P < \infty$, $A^p_\alpha(\Pi^+)$ is a Banach space with the norm defined by

$$\|f\|_{A^p_\alpha(\Pi^+)} = \left(\int_{\Pi^+} |f(z)|^p \, dA_\alpha(z) \right)^{1/p} < \infty \tag{4}$$

with this norm $A^p_\alpha(\Pi^+)$ becomes a Banach space when $p \geq 1$ while for $p \in (0,1)$ it is a Frechet space with the translation invariant metric

$$d(f,g) = \|f-g\|^p_{A^p_\alpha(\Pi^+)}, \quad f,g \in A^p_\alpha(\Pi^+). \tag{5}$$

Recall that for every $f \in H^p(\Pi^+)$ the following estimate holds:

$$|f(x+iy)|^p \leq C \frac{\|f\|^p_{A^p_\alpha(\Pi^+)}}{y^{\alpha+2}}, \tag{6}$$

where C is a positive constant independent of f.

Let $\alpha > 0$. The weighted-type space (or growth space) on the upper half-plane $A^p_\alpha(\Pi^+)$ consists of all $f \in H(\Pi^+)$ such that

$$\|f\|_{A^p_\alpha(\Pi^+)} = \sup_{z\in\Pi^+} (\mathcal{J}z)^\alpha |f(z)| < \infty. \tag{7}$$

For weighted type spaces on the unit disk, polydisk, or the unit ball see, for example, paper[30]. Given two Banach spaces Y and Z, we recall that a linear map $T: Y \to Z$ is bounded if $T(E) \subset Z$ is bounded for every bounded subset E of Y. In addition, we say that T is compact if $T(E) \subset Z$ is relatively compact for every bounded set $E \subset$ Y.

In this paper we follow the same Literature and methods of Stevo Stevic, Ajay K. Sharma, and S.D. Sharma [29] with a little change. We consider the boundedness and compactness of weighted composition operators acting from H^p to the weighted-type space $\mathcal{A}^\infty_\alpha(\Pi^+)$. Throughout this paper, constants are denoted by C; they are positive and may different from one occurrence to the other.

2. Main Results

The boundedness and compactness of the weighted composition operators from the Hardy space to the weighted type space on the upper half plane $w_{\varphi,\psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$, and the special cases when $\alpha = 1$ which is a Hilbert space are characterize in this section.

Theorem 2.1. Let $1 \leq p < \infty$, $\alpha > 0$, $\psi \in H(\Pi^+)$, and let φ be a holomorphic self-map of Π^+. Then $w_{\varphi,\psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded if and only if

$$M := sup_{z\in\Pi^+} \frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^{2/p}} |\varphi(z)| < \infty \tag{8}$$

Moreover, if the operator $w_{\varphi,\psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded then the following asymptotic relationship holds

$$\|w_{\varphi,\psi}\|_{H^p(\Pi^+)\to\mathcal{A}^\infty_\alpha(\Pi^+)} \approx M \tag{9}$$

Proof. First suppose that (8) holds. Then for any $z \in \Pi^+$ and $f \in H^p(\Pi^+)$, by(6) we have

$$(\mathcal{J}z)^\alpha |(w_{\varphi,\psi}f)(z)| = (\mathcal{J}z)^\alpha |\Psi(z)||f(\varphi(z))| \leq \frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^{2/p}} |\Psi(z)| \|f\|_{H^p(\Pi^+)}, \tag{10}$$

and so by (8) $w_{\varphi,\psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded and moreover

$$\|w_{\varphi,\psi}\|_{H^p(\Pi^+)\to\mathcal{A}^\infty_\alpha(\Pi^+)} \leq M \tag{11}$$

Conversely suppose $w_{\varphi,\psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded. Consider the function

$$f_\omega(z) = \frac{(\mathcal{J}\omega)^{2/p}}{(z-\overline{\omega})^{4/p}}, \qquad \omega \in \Pi^+ \tag{12}$$

Then $f_w \in H^p(\Pi^+)$, and moreover $sup_{\omega\in\Pi^+}\|f_w\|_{H^p(\Pi^+)} \leq 1$ (see, e.g., Lemma1 in [18]). Thus the boundedness of $w_{\varphi,\Psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ implies that

$$(\mathcal{J}z)^\alpha|\Psi(z)||f_\omega(\varphi(z))| \leq \|w_{\varphi,\Psi}f_w\|_{\mathcal{A}^\infty_\alpha(\Pi^+)} \leq \|w_{\varphi,\Psi}\|_{H^p(\Pi^+)\to\mathcal{A}^\infty_\alpha(\Pi^+)} \tag{13}$$

for every $z, w \in \Pi^+$. In particular, if $z \in \Pi^+$ is fixed then for $w = \varphi(z)$, we get

$$\frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^{2/p}}|\Psi(z)| \leq \|w_{\varphi,\Psi}\|_{H^p(\Pi^+)\to\mathcal{A}^\infty_\alpha(\Pi^+)}. \tag{14}$$

Since $w \in \Pi^+$ is arbitrary,(8) follows and moreover

$$M \leq \|w_{\varphi,\Psi}\|_{H^p(\Pi^+)\to\mathcal{A}^\infty_\alpha(\Pi^+)} \tag{15}$$

If $w_{\varphi,\Psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded then from (11) and (15) asymptotic relationship (9) follows. □

Corollary 2.2. Let $1 \leq p < \infty,\ and\ \alpha > 0$ be such that $\alpha p \geq 2$ and $\psi \in H(\Pi^+)$. Then $M_\Psi: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded if and only if $\Psi \in X$, where

$$X = \begin{cases} \mathcal{A}^\infty_{\alpha-(2/p)}(\Pi^+) & if \quad \alpha p > 2, \\ H^\infty(\Pi^+) & if \quad \alpha p = 2 \end{cases} \tag{16}$$

Example 2.3. Let $1 \leq p < \infty$, and $\alpha > 0$ be such that $\alpha p \geq 2$ and $w \in \Pi^+$. Let Ψ_ω be a holomorphic map of Π^+ defined as

$$\Psi_\omega(z) = \begin{cases} \dfrac{1}{(z-\overline{\omega})^{\alpha-(2/p)}} & if \quad \alpha p > 2 \\ \dfrac{\mathcal{J}\omega}{z-\overline{\omega}} & if \quad \alpha p = 2 \end{cases} \tag{17}$$

For $z = x + iy$ and $w = u + iv$ in Π^+, we have

$$sup_{z\in\Pi^+}(\mathcal{J}z)^{\alpha-(2/p)}|\Psi_\omega(z)| = sup_{z=x+iy\in\Pi^+}\frac{y^{\alpha-\frac{2}{p}}}{((x-u)^2+(y+v)^2)^{\frac{\alpha p-2}{2p}}} \leq sup_{z=x+iy\in\Pi^+}\frac{y^{\alpha-\frac{2}{p}}}{(y+v)^{\alpha-\frac{2}{p}}} \tag{18}$$

Thus $\Psi_\omega \in \mathcal{A}^\infty_{\alpha-(2/p)}(\Pi^+)$ if $\alpha p > 2$ Similarly $\Psi_\omega \in H(\Pi^+)$ if $\alpha p = 2$. By Corollary 2.2 it follows that $M_\Psi: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded.

Corollary 2.4. Let $1 \leq p < \infty, \alpha > 0,$ and let φ be a holomorphic self-map of Π^+. Then $C_\varphi : H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded if and only if

$$sup_{z\in\Pi^+}\frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^{2/p}} < \infty \tag{19}$$

Corollary 2.5. Let φ be the linear fractional map

$$\varphi(z) = \frac{az+p}{cz+d}, \quad a,b,c,d \in \mathbb{R}, ad-bc > 0. \tag{20}$$

Then necessary and sufficient condition that $C_\varphi : H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded is that $c = 0$ and $\alpha p = 2$.

Proof. Assume that $C_\varphi : H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded. Then

$$sup_{z\in\Pi^+} \frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^{2/p}} = sup_{z=x+iy\in\Pi^+} \frac{((cx+d)^2+c^2y^2)^{\frac{2}{p}} y^\alpha}{(ad-bc)^{\frac{2}{p}} y^{\frac{2}{p}}} \tag{21}$$

which is finite only if $c = 0$ and $\beta p = 2$.
Conversely, if $c = 0$ and $\alpha p = 2$, then from(20) we get $a \neq 0$, and by some calculation

$$sup_{z\in\Pi^+} \frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^{2/p}} = \left(\frac{d}{a}\right)^\alpha < \infty \tag{22}$$

Hence that $C_\varphi : H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded.
□

Corollary 2.6. Let $1 \le p < \infty$, and $\alpha > 0$ be such that $\alpha p = 2$. Let φ be a holomorphic self-map of Π^+ and $\Psi = (\acute{\phi})^\alpha$. Then the weighted composition operator $W_{\phi,\psi}$ acts boundedly from $H^p(\Pi^+)$ to $\mathcal{A}^\infty_\alpha(\Pi^+)$.
Proof. By Theorem 2.1 $w_{\varphi,\Psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded if and only if

$$sup_{z\in\Pi^+} \frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^\alpha} |\varphi'(z)|^\alpha < \infty. \tag{23}$$

By the Schwarz-Pick theorem on the upper half-plane we have that for every holomorphic self-map φ of Π^+ and all $z\in\Pi^+$

$$\frac{|\varphi'(z)|}{\mathcal{J}\varphi(z)} \le \frac{1}{\mathcal{J}z} \tag{24}$$

where the equality holds when φ is a Mobius transformation given by(20). From (24) , condition (23) follows and consequently the boundedness of the operator $w_{\varphi,\Psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$

Corollary 2.6. Enables us to show that there exist $1 \le p < \infty$, $\alpha > 0$, and holomorphic maps φ and ψ of the upper half-plane Π^+such that neither $C_\varphi : H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ nor $M_\Psi: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded, but $w_{\varphi,\Psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ bounded.
□

Example 2.7. Let $1 \le p < \infty$, and $\alpha > 0$ be such that $\alpha p = 2$. Let $\varphi(z) = \frac{az+b}{cz+d}$, $a, b, c, d \in R$, $ad-bc > 0$, and $c \neq 0$. Then by Corollary 2.5 , $C_\varphi : H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is not bounded. On the other hand, if

$$\varphi(z) = (\varphi'(z))^\alpha = \left(\frac{ad-bc}{(cz+d)^2}\right)^\alpha, \tag{25}$$

then $\Psi \notin H^\infty(\Pi^+)$ and so by Corollary 2.2, $M_\Psi: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is not bounded. However, by Corollary 2.6, we have that $w_{\varphi,(\Psi)^\alpha}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is bounded.

The next Schwartz-type lemma characterizes compact weighted composition operators $w_{\varphi,\Psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ and it follows from standard arguments([4]).

Lemma 2.8. Let $1 \le p < \infty$, $\alpha > 0$, $\psi \in H(\Pi^+)$, and let φ be a holomorphic self-map of Π^+. Then $w_{\varphi,\Psi}: H^p(\Pi^+) \to \mathcal{A}^\infty_\alpha(\Pi^+)$ is compact if and only if, for any bounded sequence $(fn)_{n\in N} \subset H^p(\Pi^+)$ converging to zero on compacts of Π^+ , one has

$$\lim_{n\to\infty} \|w_{\varphi,\Psi} f_n\|_{H^p(\Pi^+)} = 0 \tag{26}$$

Remark 2.9. Let $p \ge 2$, $\psi \in H(\Pi^+)$, and let φ be a holomorphic self-map of Π^+. Then
(i) $M_\Psi: H^p(\Pi^+) \to \mathcal{A}^\infty_1(\Pi^+)$ is bounded if and only if $\Psi \in X$, where

$$X = \begin{cases} \mathcal{A}^{\infty}_{1-(2/p)}(\Pi^{+}) & if \quad p > 2, \\ H^{\infty}(\Pi^{+}) & if \quad p = 2 \end{cases} \tag{27}$$

(ii) $C_{\varphi} : H^{p}(\Pi^{+}) \to \mathcal{A}^{\infty}_{1}(\Pi^{+})$ is bounded if and only if

$$sup_{z \in \Pi^{+}} \frac{\mathcal{J}z}{(\mathcal{J}\varphi(z))^{2/p}} < \infty. \tag{28}$$

(iii) The weighted composition operator $W_{\phi,\psi}$ acts boundedly from $H^{2}(\Pi^{+})$ to $\mathcal{A}^{\infty}_{\alpha}(\Pi^{+})$ where $\Psi = \acute{\varphi}$.

(iv) $w_{\varphi,\psi}: H^{p}(\Pi^{+}) \to \mathcal{A}^{\infty}_{1}(\Pi^{+})$ is compact if and only if, for any bounded sequence $(fn)_{n \in N} \subset H^{p}(\Pi^{+})$ converging to zero on compacts of Π^{+} , one has

$$\lim_{n\to\infty} \left\| w_{\varphi,\psi} f_n \right\|_{H^{p}(\Pi^{+})} = 0 \tag{29}$$

Theorem 2.10 . Let $1 \le p < \infty, \alpha > 0, \psi \in H(\Pi^{+})$ and φ be a holomorphic self-map of Π^{+} if $w_{\varphi,\psi}: H^{p}(\Pi^{+}) \to \mathcal{A}^{\infty}_{\alpha}(\Pi^{+})$ is compact, then

$$\lim_{r\to 0} \; sup_{\mathcal{J}\varphi(z)<r} \frac{(\mathcal{J}z)^{\alpha}}{(\mathcal{J}\varphi(z))^{2/p}} |\varphi(z)| = 0. \tag{30}$$

Proof. Suppose $w_{\varphi,\psi}: H^{p}(\Pi^{+}) \to \mathcal{A}^{\infty}_{\alpha}(\Pi^{+})$ is compact and (30) does not hold. Then there is $a\ \delta > 0$ and a sequence $(zn)_{n \in N} \subset \Pi^{+}$ such that $\mathcal{J}\varphi(z_n) \to 0$ and

$$\frac{(\mathcal{J}z_n)^{\alpha}}{(\mathcal{J}\varphi(z_n))^{2/p}} |(\varphi z_n)| > \delta \tag{31}$$

for all $n \in \mathrm{N}$. Let $w_n = \varphi(z_n), n \in N$, and

$$f_n(z) = \frac{(\mathcal{J}\omega_n)^{2/p}}{(z - \bar{\omega}_n)^{4/\acute{p}}}, n \in N \tag{32}$$

Then f_n is a norm bounded sequence and $f_n \to 0$ on compacts of Π^{+} as $\mathcal{J}_{\varphi}(z_n) \to 0$. By Lemma 2.8 it follows that

$$\lim_{n\to\infty} \left\| w_{\varphi,\psi} f_n \right\|_{\mathcal{A}^{\infty}_{\alpha}(\Pi^{+})} = 0 \tag{33}$$

On the other hand,

$$\begin{aligned} \left\| w_{\varphi,\psi} f_n \right\|_{\mathcal{A}^{\infty}_{\alpha}(\Pi^{+})} &\ge (\mathcal{J}z_n)^{\alpha} \left| (w_{\varphi,\psi} f_n)(z_n) \right| \\ &= (\mathcal{J}z_n)^{\alpha} |\Psi(z_n)| |f_n \varphi(z_n)| \\ &= \frac{(\mathcal{J}z_n)^{\alpha}}{2^{4/p} (\mathcal{J}\varphi(z_n))^{2/p}} |\Psi(z_n)| > \frac{\delta}{2^{4/p}} \end{aligned} \tag{34}$$

which is a contradiction. Hence (30) must hold, as claimed. □

Before we formulate and prove a converse of Theorem 2.10, we define, for every $a, b \in (0, \infty)$ such that $a < b$, the following subset of Π^{+} :

$$\Gamma_{a,b} = \{ z \in \Pi^{+} : a \le \mathcal{J}z \le b \}. \tag{35}$$

Theorem 2.11. Let $1 \le p < \infty, \alpha > 0$, $\psi \in H(\Pi^{+})$, and let φ be a holomorphic self-map of Π^{+} and $w_{\varphi,\psi}: H^{p}(\Pi^{+}) \to \mathcal{A}^{\infty}_{\alpha}(\Pi^{+})$ be bounded. Suppose that $\psi \in \mathcal{A}^{\infty}_{\alpha}(\Pi^{+})$ and $(\mathcal{J}z)^{\alpha} |\Psi(z)| \to 0$ as $|\Re\varphi(z)| \to \infty$ within $\Gamma_{a,b}$ for all *a* and *b*, $0 < a < b < \infty$. Then $w_{\varphi,\psi}: H^{p}(\Pi^{+}) \to \mathcal{A}^{\infty}_{\alpha}(\Pi^{+})$ is compact if condition (30) holds.

Proof. Assume(30) holds. Then for each $\varepsilon > 0$, there is an $M_1 > 0$ such that

$$\frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^{2/p}}|\varphi(z)| < \varepsilon, \text{ whenever } \mathcal{J}\varphi(z) < M_1. \quad (36)$$

Let$(f_n)_{n\in N}$ be a sequence in $H^p(\Pi^+)$ such that $sup_{n\in N}\|f_n\|_{H^p(\Pi^+)} \leq M$ and $f_n \to 0$ uniformly on compact subsets of Π^+ as $n\to\infty$. Thus for $z\in\Pi^+$ such that $\mathcal{J}\varphi(z) < M_1$ and each $n\in N$, we have

$$(\mathcal{J}z)^\alpha|\Psi(z)||f_n(\varphi(z))| \leq \frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^{2/p}}|\varphi(z)|\|f_n\|_{H^p(\Pi^+)} < \varepsilon M. \quad (37)$$

From estimate(16) we have

$$|f_n(z)| \leq \frac{\|f_n\|_{H^p(\Pi^+)}}{(\mathcal{J}z)^{2/p}} \leq \frac{M}{(\mathcal{J}z)^{2/p}} \quad (38)$$

Thus there is an $M_2 > M_1$ such that

$$|f_n(\varphi(z))| < \varepsilon \quad (39)$$

whenever $\mathcal{J}\varphi(z) < M_2$. Hence for $z\in\Pi^+$ such that $\mathcal{J}\varphi(z) < M_2$ and each $n\in N$ we have

$$(\mathcal{J}z)^\alpha|\Psi(z)||f_n(\varphi(z))| < \varepsilon\|\Psi\|_{\mathcal{A}_\alpha^\infty(\Pi^+)}. \quad (40)$$

If $M_1 \leq \mathcal{J}\varphi(z) \leq M_2$, then by the assumption there is an $M_3 > 0$ such that $(\mathcal{J}z)^\alpha|\Psi(z)| < \varepsilon$, whenever $|\Re\varphi(z)| > M_3$. Therefore, for each $n\in N$ we have

$$(\mathcal{J}z)^\alpha|\Psi(z)||f_n(\varphi(z))| < \varepsilon\frac{\|f_n\|_{H^p(\Pi^+)}}{(\mathcal{J}\varphi(z))^{2/p}} \leq \frac{\varepsilon M}{M_1^{2/p}} \quad (41)$$

whenever $M_1 \leq \mathcal{J}\varphi(z) \leq M_2$ and $|\Re\varphi(z)| > M_3$.
If $M_1 \leq \mathcal{J}\varphi(z) \leq M_2$ and $|\Re\varphi(z)| \leq M_3$, then there exists some $n_0 \in N$ such that $|f_n\varphi(z)| < \varepsilon$ for all $n \geq n_0$, and so

$$(\mathcal{J}z)^\alpha|\Psi(z)||f_n(\varphi(z))| < \varepsilon\|\Psi\|_{\mathcal{A}_\alpha^\infty(\Pi^+)}. \quad (42)$$

Combining(37)-(42), we have that

$$\|w_{\varphi,\psi}f_n\|_{\mathcal{A}_\alpha^\infty(\Pi^+)} < \varepsilon c, \quad (43)$$

for $n \geq n_0$ and some $C > 0$ independent of n. Since ε is an arbitrary positive number, by Lemma 2.8, it follows that $w_{\varphi,\psi}: H^p(\Pi^+) \to \mathcal{A}_\alpha^\infty(\Pi^+)$ is compact.
□

Example 2.12. Let $1 \leq p < \infty$, and $\alpha > 0$ be such that $\alpha p = 2$.
Let $\varphi(z) = z + i$ and $\Psi(z) = \frac{1}{(z+i)^\alpha}$, then $\Re\varphi(z) = x$ and $\mathcal{J}\varphi(z) = y + 1$. It is easy to see that $\Psi \in \mathcal{A}_\alpha^\infty(\Pi^+)$. Beside this, for $z\in\Gamma_{a,b}$ we have

$$(\mathcal{J}z)^\alpha|\Psi(z)| = \frac{y^\alpha}{(x^2+(y+1)^2)^{\alpha/2}} \leq \frac{b^\alpha}{(x^2+a^2)^{\alpha/2}} \to 0 \text{ as } \Re\varphi(z) = x \to \infty. \quad (44)$$

Also

$$sup_{z\in\Pi^+}\frac{(\mathcal{J}z)^\alpha}{(\mathcal{J}\varphi(z))^\alpha}|\varphi'(z)|^\alpha = sup_{z\in\Pi^+}\frac{y^\alpha}{(y+1)^\alpha}\frac{1}{(x^2+(y+1)^2)^{\alpha/2}} \leq 1 \leq \infty, \quad (45)$$

and the set $\{z : \mathcal{J}\varphi(z) < 1\}$ is empty. Thus φ and ψ satisfy all the assumptions of Theorem 2.11, and so $w_{\varphi,\psi}: H^p(\Pi^+) \to \mathcal{A}_\alpha^\infty(\Pi^+)$ is compact.

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