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Homotopy Analysis Method for Solving Delay Differential Equations of Fractional Order

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Abstract: *In this paper, we implement the Homotopy Analysis method for solving numerically non-linear delay differential equations of fractional order. The fractional derivative will be in the Caputo sense. In this approach, the solutions are found in the form of a convergent power series with easily computed components. Some numerical examples are presented to illustrate the accuracy and ability of the proposed method.*

Keywords: *Homotopy Analysis method, delay differential equations, fractional calculus, fractional delay differential equations.*

1. Introduction

The subject of fractional calculus (that is calculus of integrals and derivatives of any arbitrary real or complex order) has gained considerable popularity and importance during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. It does indeed provide several potentially useful tools for solving differential and integral equations, and various other problem involving special functions of mathematical physics as well as their extensions and generalizations in one and more variables (Kilbas,2006).

In real world systems, delays can be recognized everywhere and there has been widespread interest in the study of delay differential equations for many years.

However, fractional delay differential equations($FDDE_S$) are a very recent topic.Although it seems natural to model certain processes and systems in engineering and other sciences (with memory and heritage properties) with this kind of equations, only in the last few years has the attention of the scientific community been devoted to them (Moragdo,2013).

Concerning the existence of solutions of ($FDDE_S$) we refer (Lakshmikantham,2007), (Ye and Gao,2007), (Liao and Ye ,2009), . In (Lakshmikantham,2007) Lakshmikantham provides sufficient conditions for the existence of solutions to initial value problems to single term nonlinear delay fractional differential equations, with the fractional derivative defined in the Riemann-Liouville sense. In (Ye and Gao,2007),Ye et al. investigate the existence of positive solutions for a class of single term delay fractional differential equations. Later in (Liao and Ye ,2009), for the same class of equations, sufficient condition for the uniqueness of the solution are reported.

For the stability issues of the ($FDDE_S$) we refer the references (Chen Moore,2002),(Mihailo and Aleksandar,2009), (Krolk,2011),(Deng and Lu,2007).

In this paper we shall use the Homotopy Analysis method to find the approximate solution of the ($FDDE_S$) with variable delays .

The structure of this paper is organized as follows:

In section 2, we recall the definitions of fractional derivatives and fractional integration in section 3 the basic concept of the Homotopy Analysis method will be given in section 4 we present our approach to solve the variable delay differential equation of fractional order in section 5 numerical examples are given followed by conclusions in section 6.

2. Fractional Derivatives and Fractional Integration

Definition(1): The Riemann-Liouville fractional integral operator of order α>0 is defined as:

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx, \alpha > 0, x > 0$$
$$I_t^0 f(t) = f(t)$$

Definition(2): The Riemann-Liouville fractional derivative operator of order α>0 is defined as:

$$D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-x)^{n-\alpha-1} f(x)dx$$

where n is an integer and $n-1<\alpha\le n$.

Definition(3): The Caputo fractional derivative operator of order α is defined as:

$$ {}^{c}D_t^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-x)^{n-\alpha-1}\frac{d^n}{dx^n}f(x)dx $$

where n is an integer and $n-1<\alpha\le n$.

Caputo fractional derivative has a useful property:

$$ I_t^{\alpha}\ {}^{c}D_t^{\alpha}f(t)=f(t)-\sum_{k=0}^{n-1}f^{(k)}(0^+)\frac{t^k}{k!} $$

where n is an integer and $n-1<\alpha\le n$.
And similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$ {}^{c}D_t^{\alpha}\big(\lambda f(t)+\mu g(t)\big)=\lambda\ {}^{c}D_t^{\alpha}f(t)+\mu\ {}^{c}D_x^{\alpha}g(t) $$

whereλ and μ are constants. For the Caputo's derivative, also we have:

$$ {}^{c}D_t^{\alpha}C=0, \text{C is constant} $$

$$ {}^{c}D_t^{\alpha}t^n=\begin{cases} 0, \text{for } n\in N_0 \text{ and } n\ge[\alpha] \\ \dfrac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}t^{n-\alpha} \text{ for } n\in N \text{ and } n\ge[\alpha] \end{cases} $$

We use the ceiling function $[\alpha]$ to denote the smallest integer greater than or equal to α and N={0,1,2,3,…}.

3. Homotopy Analysis Method (HAM)

In this section the basic ideas of the HAM are introduced . Here a description of the method (Liao,2003), is given to solve a general nonlinear problem

$$ A[y(t)]=0 \quad ,t>0 \qquad (1) $$

Where A is a nonlinear operator and $y(t)$ is an unknown function of the independent variable t .

3.1 Zero – order deformation equation

Let $y_0(t)$ denote an initial guess of *the exact solution of equation (1) , $h\neq 0$* an auxiliary parameter , $H(t)\neq 0$ an auxiliary function and L an auxiliary linear operator with the property

$$ L[f(t)]=0 \quad , when\ f(t)=0. \qquad (2) $$

[Liao,2003] constract using $q\in[0,1]$ as an embedding parameter , the so-called zero – order deformation equation

$$ (1-q)L[\Phi(t;q)-y_0(t)]-qhHA[\Phi(t,q)]=0 \ , \qquad (3) $$

Where Φ(t,q) is the solution which depends on h , H(t) , L , $y_0(t)$and q . when q = 0 , the zero – order deformation equation(3) becomes

$$ \Phi(t;0)=y_0(t) \qquad (4) $$

And when q = 1 , since $h\neq 0$ and $H(t)\neq 0$, then the zero – order deformation equation (3) reduces to

$$ A[\Phi(t,1)]=0 \qquad (5) $$

So , Φ (t,1) is exactly the solution of the nonlinear equation (1) .

Define the so-called m th order deformation derivatives

$$y_m = \frac{1}{m!} \frac{\partial^m \Phi(t,q)}{\partial q^m}\bigg|_{q=0} \qquad (6)$$

If the power series (6) of Φ(t,q) converges at q = 1 , then we get the following series solution

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \qquad (7)$$

The above expression provides us with a relationship between the initial guess $y_0(t)$ and the exact solution y(t) by means of the terms $y_m(t)$ $(m = 1,2,3, \dots)$ which are unknown up to now.

3.2 High – Order deformation equation

Define the vector

$$\vec{y}_m = \{y_0(t), y_1(t), \dots, y_m(t)\} \qquad (8)$$

Differentiating equation (3) m times with respect to the embedding parameter q and then setting q = 0 and finally dividing by $m!$ We have the so- called m^{th} – order deformation equation

$$L[y_m(t) - X_m y_{m-1}(t)] = hH(t) R_{y_m}(\vec{y}_{m-1}, t) \qquad (9)$$

Where

$$X_m = \begin{cases} 0 & , \ m \le 1 \\ 1 & , \ m > 1 \end{cases} \qquad (10)$$

And

$$R_{y_m}(\vec{y}_{m-1}, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} A[\Phi(t,q)]}{\partial q^{m-1}}\bigg|_{q=0} \qquad (11)$$

Thus , we can get $y_0(t), y_1(t), y_2(t), \dots$ by means of solving the linear high-order deformation equation (9) One after One in order .

The m^{th} – order approximation of y(t) is given by

$$y(t) \approx \sum_{k=0}^{m} y_k(t) \qquad (12)$$

4. The Approach

In this section the basic ideas of the HAM are introduced in order to solve the following problem :

$$ {}^{c}D_t^{\alpha} y(t) = f(t, y(t), y(\Phi(t))) \quad , n-1 < \alpha \le n \qquad (13)$$

$$y(t) = \psi(t). \qquad -\tau \le t \le 0 \qquad (14)$$

$$y^{(i)}(0) = y_0^i \quad i = 0,1,..,n-1 \qquad (15)$$

Where ${}^{c}D_t^{\alpha}$ is the fractional derivative in the Caputo sense and $\psi(t)$ is a continuous function , f is a nonlinear operator and y_0^i are prescribed constants .

4.1 Zero – Order Deformation Equation

In HAM equation (13) is first written in the form

$$N[t, y(t), y(\Phi(t))] = 0 \tag{16}$$

Where N is a nonlinear operator given by

$$N[t, y(t), y(\Phi(t))] = {}^{c}D_t^{\alpha} y(t) - f(t, y(t), y(\Phi(t))) \tag{17}$$

t denote the in depended variable , y(t) is the unknown function and Φ(t) is the delay function .

let $y_0(t)$ denote the initial guess of the exact solution y(t) , h ≠ 0 an auxiliary parameter and let L to be an auxiliary operator defined by

$$L = {}^{c}D_t^{\alpha}$$

Then using $q \in [0,1]$ as an embedding parameter , in view of Liao [Liao,2003] we construct such a homotopy

$$H[\tilde{y}(t,q), y_0, h, q] = (1-q)L[\tilde{y}(t,q) - y_0] - qhHN[t, \tilde{y}(t), \tilde{y}(\phi(t)), q] \tag{18}$$

For the $FDDE_s$ (16) .

Enforcing the homotopy (18) to be zero , we have the so called zeroth-order deformation equation as

$$(1-q)L[\tilde{y}(t,q) - y_0] = qhHN[t, \tilde{y}(t), \tilde{y}(\phi(t)), q] \tag{19}$$

Where $\tilde{y}(t,q)$ is the solution which depends on the initial guess $y_0(t)$, the auxiliary linear operator L , the nonzero auxiliary parameter h , the auxiliary function H and the embedding parameter $q \in [0,1]$.

Obviously , when q = 0 and q = 1 , both

$$\tilde{y}(t,0) = y_0(t) \ , \ \tilde{y}(t,1) = y(t) \tag{20}$$

Respectively hold .

Thus, according to above equation , as the embedding parameter q increases from 0 to 1 , $\tilde{y}(t,q)$ varies continuously from the initial approximate $y_0(t)$ to the exact solution y(t) of the original equation (13) .

The zero-order deformation equation (19) defines a family of homotopies between the initial approximation $y_0(t)$ and the exact solution y(t) via auxiliary parameter h .

The mapping to the exact solution is implemented through a successive approximation with the initial approximation as the first term .

To this end , the mapping function $\tilde{y}(t,q)$ are expanded in Taylor series about q = 0 as

$$\tilde{y}(t,q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) q^m \tag{21}$$

Where

$$y_m = \frac{1}{m!}\frac{\partial^m \tilde{y}(t,q)}{\partial q^m}\bigg|_{q=0} \qquad (22)$$

Assume that the auxiliary parameter h , the auxiliary function H , the initial guess $y_0(t)$ and the auxiliary linear operator L are so properly chosen that the series (21) converges at q = 1 .

Then at q = 1 , the series (21) becomes

$$\tilde{y}(t,1) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \qquad (23)$$

Therefore , using equation (20) , we have

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t) \qquad (24)$$

Where the terms $y_m(t)$ can be determined by the so-called high-order deformation equation described below .

4.2 High – Order Deformation Equation

Define the vector

$$\vec{y}_m = \{y_0(t), y_1(t), \dots, y_m(t)\}$$

According to the definition (22) the governing equation of $y_m(t)$ can be derived from the zeroth-order deformation equation(19) .

Differentiating zeroth-order deformation equation (19) m times with respect to the embedding parameter q and then setting q = 0 and finally dividing by $m!$ We have the so called m^{th} – order deformation equation

$$L[y_m(t) - X_m y_{m-1}(t)] = hH(t)R_{y_m}(\vec{y}_{m-1}, t) \qquad (25)$$

Where

$$R_{y_m}(\vec{y}_{m-1}, t) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[t, \tilde{y}(t), \tilde{y}(\phi(t))]}{\partial q^{m-1}}\bigg|_{q=0} \qquad (26)$$

And

$$X_m = \begin{cases} 0 \ , & m \le 1 \\ 1 \ , & m > 1 \end{cases} \qquad (27)$$

Notice that $R_{y_m}(\vec{y}_{m-1}, t)$ given by the above expression is only dependent upon y_0 (t),y_1 (t),y_2 (t),…,y_(m-1) (t) which are known when solving the mth – order deformation equation (25) .

The m^{th} – order approximation of y(t) is given by equ.(12)

5. Numerical Examples

In this section we shall use the HAM to solve the non-linear delay differential equations of fractional order and the results obtained using this scheme will be compare with the analytical solution

Example (1):-

Consider the FDDE$_S$

$$ {}^{c}D_t^{\alpha} y(t) = \frac{1}{2} e^{\frac{t}{2}} y\left(\frac{t}{2}\right) + \frac{1}{2} y(t) \quad , \quad 0 \le t \le 1 \; , 0 < \alpha \le 1 \qquad (28)$$

$$y(0) = 1$$

Where the exact solution was given in (Evans,2004), as $y(t) = e^t$.

And in order to solve equation (29) using HAM.

First we choose the initial approximation $y_0(t)$ to be

$$y_0(t) = 1 \qquad (29)$$

And according to equation (17) then

$$N\left[t, y(t), y\left(\frac{t}{2}\right)\right] = {}^{c}D_t^{\alpha} y(t) - \frac{1}{2} e^{\frac{t}{2}} y\left(\frac{t}{2}\right) - \frac{1}{2} y(t) \qquad (30)$$

Set $L = {}^{c}D_t^{\alpha}$, $h = -1 \quad and \quad H = 1$ hence according to equation (25) we have

$$ {}^{c}D_t^{\alpha} [y_m(t) - X_m y_{m-1}(t)] = -R_{y_m}(\vec{y}_{m-1}, t) \qquad (31)$$

where

$$R_{y_m}(\vec{y}_{m-1}, t) = {}^{c}D_t^{\alpha} y_{m-1}(t) - \frac{1}{2} e^{\frac{t}{2}} y_{m-1}\left(\frac{t}{2}\right) - \frac{1}{2} y_{m-1}(t) \qquad (32)$$

Operating I_t^{α} to the both sides of equation (31) and using equation (29) therefore one can get the functions $y_1, y_2, \ldots$ one after one in order by solving the linear high – order deformation (31) .

Following table(1) represent the approximate solution of example (1) using HAM up to three terms for different values of α and with a comparison with the exact solution when α = 1. It is noticed that we are made an approximation to $e^{\frac{t}{2}}$ using Maclurian series expansion up to three terms .

Table (1)The approximate solution of example (1) using different values of α with a comparison with the exact solution when α = 1

t	HAM α= 0.5	HAM α=0.75	HAM α=1	Exact
0	1	1	1	1
0.1	1.45	1.216	1.105	1.105
0.2	1.701	1.391	1.221	1.221
0.3	1.926	1.565	1.347	1.35
0.4	2.142	1.745	1.485	1.492
0.5	2.356	1.932	1.636	1.649
0.6	2.573	2.128	1.799	1.822

0.7	2.794	2.334	1.976	2.014
0.8	3.022	2.553	2.167	2.226
0.9	3.26	2.784	2.374	2.46
1	3.508	3.029	2.598	2.718

Examle (2):-

Consider the FDDE_S

$$ {}^{c}D_t^{\alpha} y(t) = \frac{3}{4} y(t) + y\left(\frac{t}{2}\right) - t^2 + 2 \ , \quad 0 \le t \le 1, \quad 1 < \alpha \le 2 \tag{33} $$

$$ y(0) = 0 $$

Where the exact solution was given in (Evans,2004), as $y(t) = t^2$.

And in order to solve equation (33) using HAM

First we choose the initial approximation $y_0(t)$ to be

$$ y_0(t) = 0 \tag{34} $$

And according to equation (17) then

$$ N\left[t, y(t), y\left(\frac{t}{2}\right)\right] = {}^{c}D_t^{\alpha} y(t) - \frac{3}{4} y(t) - y\left(\frac{t}{2}\right) + g(t;q) \tag{35} $$

$where\ g(t;q) = 2 - t^2 q$

Set $L = {}^{c}D_t^{\alpha}$, $h = -1 \quad and \quad H = 1$ hence according to equation (25) we have

$$ {}^{c}D_t^{\alpha}[y_m(t) - X_m y_{m-1}(t)] = -R_{y_m}(\vec{y}_{m-1}, t) \tag{36} $$

Where

$$ R_{y_m}(\vec{y}_{m-1}, t) = {}^{c}D_t^{\alpha} y_{m-1}(t) - \frac{3}{4} y_{m-1}(t) - y_{m-1}\left(\frac{t}{2}\right) - W_n(t) \tag{37} $$

$$ where\ W_1 = 2 \ \ and \ \ W_2 = -t^2 \ , \qquad and \ \ W_n = 0 \ , \qquad n = 3,4, \dots $$

Operating I_t^{α} to the both sides of equation (36) and using equation (34) therefore one can get the functions $y_1, y_2, \dots$ one after one in order by solving the linear high – order deformation (36).

Following table(2) represent the approximate solution of example (2) using HAM up to three terms for different values of α and with a comparison with the exact solution when α = 2.

Table (2)The approximate solution of example (2) using different values of α with a comparison with the exact solution when α = 2

t	HAM α= 1.5	HAM α=1.75	HAM α=2	Exact
0	0	0	0	0
0.1	0.048	0.022	0.01	0.01
0.2	0.137	0.075	0.04	0.04
0.3	0.255	0.153	0.09	0.09
0.4	0.397	0.254	0.16	0.16
0.5	0.563	0.377	0.25	0.25
0.6	0.75	0.521	0.36	0.36
0.7	0.958	0.686	0.49	0.49
0.8	1.186	0.872	0.64	0.64
0.9	1.434	1.077	0.81	0.81
1	1.7	1.303	1	1

Example (3):-

Consider the FDDES

$$ {}^{c}D_t^{\alpha} y(t) = 1 - 2y^2\left(\frac{t}{2}\right) \quad , \quad 0 \le t \le 1 \ , \quad 0 < \alpha \le 1 \tag{38} $$

$$ y(0) = 0 $$

The exact solution was given in (Evans,2004), as $y(t) = \sin t$.

And in order to solve equation (38) using HAM

First we choose the initial approximation $y_0(t)$ to be

$$ y_0(t) = t \tag{39} $$

And according to equation (17) then

$$ N\left[t, y(t), y\left(\frac{t}{2}\right)\right] = {}^{c}D_t^{\alpha} y(t) - 1 + 2y^2\left(\frac{t}{2}\right) \tag{40} $$

Set $L = {}^{c}D_t^{\alpha}$, $h = -1$ $\quad and \quad$ $H = 1$ hence according to equation (25) we have

$$ {}^{c}D_t^{\alpha}[y_m(t) - X_m y_{m-1}(t)] = -R_{y_m}(\vec{y}_{m-1}, t) \tag{41} $$

Where

$$ R_{y_m}(\vec{y}_{m-1}, t) = {}^{c}D_t^{\alpha} y_{m-1}(t) + 2\sum_{i=0}^{m-1} y_i\left(\frac{t}{2}\right) y_{m-1-i}\left(\frac{t}{2}\right) - (1 - X_m) \tag{42} $$

Operating I_t^{α} to the both sides of equation (41) and using equation (39) therefore one can get the functions $y_1, y_2, \ldots$ one after one in order by solving the linear high – order deformation (41).

Following table(3) represent the approximate solution of example (3) using HAM up to three terms for different values of α and with a comparison with the exact solution when α = 1

Table (3)The approximate solution of example (3) using different values of α with a comparison with the exact solution when α = 1

t	HAM α= 0.5	HAM α=0.75	HAM α=1	Exact
0	0	0	0	0
0.1	0.099	0.1	0.1	0.1
0.2	0.195	0.197	0.199	0.199
0.3	0.286	0.292	0.296	0.296
0.4	0.371	0.382	0.389	0.389
0.5	0.45	0.467	0.479	0.479
0.6	0.523	0.547	0.565	0.565
0.7	0.589	0.619	0.644	0.644
0.8	0.649	0.685	0.717	0.717
0.9	0.703	0.744	0.783	0.783
1	0.751	0.795	0.842	0.841

6. Conclusions

In this paper we have been used the HAM for solving variable order delay differential equations of fractional order. Three examples were solved in the view of the HAM with good approximation and agreement with the exact solution. The results presented in this paper shows that this method gave us rapidly and acceptate solution

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