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Some Types of Ideals on KS-Semigroups

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Abstract:
In this paper we introduce a new types of ideals in KS- Semigroups in ordinary and fuzzy sense,we called it KS-H- ideal and fuzzy KS-H-ideal and study its properties

1.Introduction

The notation of BCK algebra introduced by Y.Imai and K.Ise'ki [3] in 1966 . In the same year , K.Ise'ki [2]introduced the notation of BCI algebra which is a generalization of BCK algebra. In 2006 ,Kyung Ho Kim [5] introduced a new class of algebraic structure called KS semigroup .In 2009 Jocelyns S. Paradero Vilea and Mila Cawi [10] characterized ideals of KS- Semigroups and prove some properties .In 2007 , D.R. Prince Wiliams and Husain Shamshad[9] fuzzify KS semigroup and called it fuzzy KS Semigroups and introduced the notations of fuzzy subKS- Semigroups,,fuzzy KS ideal ,fuzzy KS P ideal and investigated some of their related properties in this paper we define a KS –H ideal and a fuzzy KS H- ideal on KS –Semigroups , we prove some of properties on it .

keywords: Semigroup, BCK algebra, H-ideal, P-ideal,ideal, Ks –semigroup,

2.Preliminary

This section contains some basic concepts we needed it in this paper

Definition (2.1)[9]: An algebraic system $(X, *, 0)$ is called a ***BCK algebra*** if it satisfies the following conditions:

1. $((x * y) * (x * z)) * (z * y) = 0$,

2. $((x * (x * y)) * y = 0$,

3. $x * x = 0$,

4. $0 * x = 0$

5.If $x * y = 0$ *and* $y * x = 0$ *then* $x = y$, *for all* $x, y, z \in X$.

Remarks (2.2)[6] : Let X be a BCK algebra then:

a) A partial ordering " $\le$ " on X can be defined by $x \le y$ if and only if $x * y = 0$.

b) A BCK-algebra X has the following properties:

1. $x * 0 = x$.

2.If x*y=0 and y*z=0 imply x*z=0 .

3.If x*y=0 implies (x*z)*(y*z)=0 and (z*y)*(z*x)=0 .

4. If (x*y)*z=(x*z)*y.

Definition (2. 3)[9]

A ***KS-semigroup*** is a non-empty set X with two binary operation " $*$ " and " **.** " , and a constant **0** satisfies the following axioms:

1. $(X, *, 0)$ is a *BCK-algebra*

2. (X, .) is a *semigroup*,

3. $x.(y * z) = (x.y) * (x.z)$ *and* $(x * y).z = (x.z) * (y.z)$, for all $x, y, z \in X$.

Definition (2. 4) [9] A non empty subset S of X with binary operation * and **.** is called ***sub KS-semigroup*** of X if it satisfies the following condition :

1- $x*y \in S \quad \forall \; x, y \in S$.

2- $x.y \in S \quad \forall \quad x, y \in S$

Definition (2. 5) [7] A ***strong KS-semigroup*** is a KS-semigroup X satisfying : $x*y=x*x.y$ *for all* $x, y \in X$

Lemma(2.6) [7]:Let X be a strong KS-semigroup then :

1- $x.y*y = 0 \quad for\ all \quad x, y \in X$.

2- $x*y = 0 \leftrightarrow x*x.y = 0 \quad for\ all \quad x, y \in X$.

Definition (2. 7) [11] A non empty subset I of a BCK –algebra X is called a ***H- ideal*** of X if the following conditions hold :

1- $0 \in I$.

2- If $x*(y*z) \in I$ *and* $y \in I \Rightarrow x*z \in I$, *for all* $x, y, z \in X$

Definition(2. 8) [7] Let X and Y be KS-semigroups . a mapping $f : X \to Y$ is called a ***KS-Semigroup. homomorphism*** (briefly ***homomorphism***) if $f(x*y) = f(x)*f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in X$. Let $f : X \to Y$ KS-Semigroup homomorphism . then the set $\{x \in X / f(x) = 0\}$ is called ***the kernel of f*** , and denote by ***ker f*** . moreover the set $\{f(x) \in Y / x \in X\}$ is called ***the image of f*** and denote by ***Im f*** .

Definition (2. 9) [9] A non-empty subset A of a semigroup (X, .) is said to be ***left (resp. right) stable*** *if* $xa \in A$ *(resp.*$ax \in A$*) whenever* $x \in X$ and $a \in A$.

Both left and right stable is called ***two-sided stable*** or simply *stable*.

Definition. (2.10) [9]A non-empty subset A of a KS-semigroup X is said to be ***left (resp. right) ideal*** of X if :

1. A is left (resp.right) stable subset of (X,.) and

2. $x * y \in A$ *and* $y \in A$ *imply that* $x \in A$, *for all* $x, y \in X$.

If A is both left and right ideal then A is called two-sided ideal or simply an ideal .

Remarks (2.11)

■ *let* A be a KS-ideal then $0 \in A$ *for all* $x \in X$ since $A \neq \theta$ *then* $\exists\, a \in A$ *such that* $xa, ax \in A$, *put* $x = 0$ *we get* $0 \in A$

■ *let* $f : X \to Y$ KS-Semigroup homomorphism then $f(0) = 0$ and if $x \le y$, then $f(x) \le f(y)$,[7] .

■ ker f is a KS-ideal[7] .

Definition (2.12) [9] A non-empty subset A of a KS-Semigroup X is said to be ***left (resp.right) p-ideal*** of X if :

1. A is a left (resp. right) stable subset of (X, .) and,

2. $(x * y) * z \in A$ *and* $y * z \in A$ *imply that* $x * z \in A$, *for all* $x, y, z \in X$.

If A is both left and right p- ideal then A is called *two sided ideal* or *simply p-ideal*

Theorem (2.13) [7] Every p-ideal of a *KS-Semigroup* X is an ideal but convers is not true

Definition (2.14) [10] The element e is called a unity in a KS-semigroup X if $e.x = x.e = x \quad \forall x \in X$.

Definition (2.15) [1] Let X be a non-empty set a ***fuzzy subset*** of X is a function $\mu : X \to [0, 1]$.

Remarks (2.16)[1]

Let X be a non-empty set then :

1) each fuzzy subset λ and μ of X , if $\lambda \subseteq \mu$ mean that $\lambda(a) \le \mu(a) \quad for\ all \; a \in X$.

2) *if* $x \le y$ *implies that* $\mu(x) \ge \mu(y) \quad for\ all \quad x, y \in X$.

3) If μ, ν be two fuzzy set of X and $a \le b$ such that $a, b \in [0,1]$, then $\mu_b \subseteq \mu_a$.

Definition (2.17) [9]Let X be a non-empty set and let μ be the fuzzy subset of X for a fixed $0 \le t \le 1$,the set μ_t = $\{x \in X \ / \ \mu(x) \ge t\}$ is called an ***upper level set*** of μ

Definition (2. 18) [9] Let $f : X \to Y$ be a mapping of KS-Semigroup and μ be a fuzzy subset of Y . The map μ^f is ***the pre-image of*** μ under f if $\mu^f = \mu(f(x)) \ \forall x \in X$.

Definition (2.19) [5] Let X be a BCK –algebra a fuzzy subset μ of X is called a ***fuzzy subalgebra*** of X if it satisfies the fallowing condition : $\mu(x*y) \ge \min\{\mu(x), \mu(y)\} \quad \forall x, y \in X$.

Definition (2.20) [11] A fuzzy set μ of BCK –algebra X is called a ***fuzzy H-ideal*** if it satisfies :

1- $\mu(0) \ge \mu(x) \quad \forall x \in X$,

2- $\mu(x*z) \ge \min\{\mu(x*(y*z)), \mu(y)\} \quad \forall x, y, z \in X$.

Definition (2.21) [9] A fuzzy set μ defined on X is called a ***fuzzy subKS-semigroup*** of X if it satisfies the fallowing conditions :

1. $\mu(x_1 * x_2) \ge \min\{\mu(x_1), \mu(x_2)\}$,
2. $\mu(x_1 x_2) \ge \min\{\mu(x_1), \mu(x_2)\} \quad \forall x_1, x_2 \in X$

Definition (2.22) [9] A fuzzy subset μ of X is called a ***left fuzzy KS-ideal*** if :

KSI1. $\mu(0) \ge \mu(x)$

KSI2. $\mu(x) \ge min\{\mu(x * y), \mu(y)\}$

KSI3. $\mu(xa) \ge min\{\mu(x), \mu(a)\}$ *for all* $x, y, a \in X$.

A fuzzy subset μ is called a ***right fuzzy KS-ideal*** if it satisfies **KSI1, KSI2** and **KSI4:** $\mu(ax) \ge min\{\mu(x), \mu(a)\}$, for all $x, y, a \in X$.

A fuzzy subset μ of X is called a ***fuzzy KS-ideal*** if it is both left and right fuzzy KS-ideal of X.

Definition (2.23) [9] A fuzzy subset μ of X is called a ***left fuzzy p-ideal*** if :

KSP1. $\mu(0) \ge \mu(x)$

KSP2. $\mu(x * z) \ge min\{\mu((x * y) * z), \mu(y * z)\}$

KSP3. $\mu(xa) \ge min\{\mu(x), \mu(a)\}$ for all $x, y, z, a \in X$.

A fuzzy subset μ is called a ***right fuzzy p-ideal*** if it satisfies **KSP1, KSP2** and **KSP4 :** $\mu(ax) \ge min\{\mu(x), \mu(a)\}$ for all $x, y, a \in X$.

A fuzzy subset μ of X is called a ***fuzzy p-ideal*** if it is both left and right fuzzy p-ideal of X.

Theorem (2.24) [9] Every left (resp.right) fuzzy p-ideal of X is a left (resp.right) fuzzy KS-ideal of X .

Definition (2.25) [9]Let λ and μ be the fuzzy subsets in a set X The ***cartesian product*** $\lambda \times \mu : X \times X \longrightarrow [0, 1]$ is defined by $(\lambda \times \mu)(x, y) = min\{\lambda(x), \mu(y)\}$ for all $x, y \in X$.

Definition (2.26) [9]Let V be a fuzzy subset in X ***the strong fuzzy relation*** on X that is a fuzzy relation on ν is ρ_V given by $\rho_\nu(x, y) = \min\{\nu(x), \nu(y)\}$

3. *KS-H-Ideal*

Definition (3.1)

A non-empty subset I of a KS-semigroups X is said to be ***left KS-H-ideal*** of X if it satisfies :

1) *If* $x*(y*z) \in I$ *and* $y \in I$ *then* $x*z \in I$

2) $xa \in I$ (*resp*. $ax \in I$) *whenever* $x \in X$ *and* $a \in I$.

A non-empty subset I is said to be ***right KS-H-ideal*** of X if it satisfies (1) and (3) : $ax \in I$ *whenever* $x \in X$ *and* $a \in I$.

If I is both left and right KS-H- ideal then I is called *two-sided KS-H- ideal* or simply *KS-H- ideal* .

Example(3. 2)

Let X = {0, 1, 2, 3} be defined by the following tables:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	3	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Then by usual calculations we can prove that X *is a **KS-semigroup***. If A ={0,1} then A is a *KS-H- ideal* of a KS-semigroup X .

Proposition (3. 3)

Let X be a KS-semigroup and let A be left (resp. right) KS-H-ideal of X then A is a left (resp. right) KS-ideal of X .

Proof:

Let A be a left KS-H-ideal of X then A is a stable. Now, let $x, y \in X$ such that $x * y \in A$ *and* $y \in A$ *then* $x * y = x*(y*0) \in A$ *and* $y \in A$ *then* $x \in A$ and since A is a left KS-H-ideal then .Hence A is a left KS-H-ideal

.

Proposition (3.4)

Let I and J are left (resp. right) KS-H-ideal of KS-Semigroups X then $I \cap J$ is a left (resp. right) KS-H-ideal of X .

Proof: *it is clear*

Proposition (3.5)

Let I and J are left (resp. right) KS-H-ideal of KS-Semigroups X then $I \cup J$ is a left (resp. right) KS-H-ideal *if* $I \subseteq J$ *or* $J \subseteq I$.

Proof: it is clear

Proposition (3. 6)

Let I and J are left (resp. right) KS-H-ideal of KS-Semigroups X then $I \times J$ is a left (resp. right) KS-H-ideal of $X \times X$.

Proof:

Let I and J are left KS-H-ideal of KS-Semigroups X

For any $x_1, x_2, a_1, a_2 \in X$ and $(x_1, x_2) \in X \times X$, $(a_1, a_2) \in I \times J$ then

$(x_1, x_2).(a_1, a_2) = (x_1 a_1, x_2 a_2)$, since I, J *are left* $KS-H-ideal$ so $x_1 a_1 \in I$ *and* $x_2 a_2 \in J$ *then* $(xa_1, xa_2) \in I \times J$ therefore $(x_1, x_2).(a_1, a_2) \in I \times J$

let $x*(y*z) \in I \times J$ *and* $y \in I \times J$ *, where* $x = (x_1, x_2)$, $y = (y_1, y_2)$ *and* $z = (z_1, z_2) \in X \times X$

if $(x_1, x_2)*[(y_1, y_2)*(z_1, z_2)] \in I \times J$ *and* $(y_1, y_2) \in I \times J$ *then* $(x_1, x_2)*(y_1*z_1, y_2*z_2) \in I \times J$ *and* $(y_1, y_2) \in I \times J$ *then* $(x_1*(y_1*z_1), x_2*(y_2*z_2) \in I \times J$ *and* $(y_1, y_2) \in I \times J$ *then* $(x_1*(y_1*z_1)) \in I$, $(x_2*(y_2*z_2)) \in J$

, $y_1 \in I$ *and* $y_2 \in J$ *then* $x_1*z_1 \in I$ *and* $x_2*z_2 \in J$ [since I, J *are left* $KS-H-ideal$]so $(x_1*z_1, x_2*z_2) \in I \times J$ *so* $(x_1, x_2)*(z_1, z_2) \in I \times J$ *then* $x*z \in I \times J$

hence $I \times J$ is a left KS-H-ideal .

Proposition (3. 7)

Let $f : X \to Y$ be a KS-semigroup epimorphism if A is a left (resp. right) KS-H-ideal in X then $f(A)$ is a left (resp. right) KS-H-ideal in Y .

Proof:

Let A be a left KS-H-ideal of X . let $a^{-} = f(a) \in f(A)$ *and* $y \in Y$ *where* $a \in A$
Since f onto then there exists $x \in X$ *such that* $f(x) = y$
since $xa \in A$ $\forall\ x \in X$ *and* $a \in A$ so $f(xa) \in f(A)$ but $f(xa) = f(x)f(a) = ya^{-}$ [since f is epimorphism]

therefore $f(A)$ is stable . Now , Suppose that $f(x), f(y), f(z) \in f(A)$ *for some* $x, y, z \in A$ Such that $f(x)*[f(y)*f(z)] \in f(A)$ *and* $f(y) \in f(A)$, since f is a homomorphism then $f(x)*[f(y)*f(z)] = f(x*(y*z)) \in f(A)$ *and* since $f(y) \in f(A)$,
thus $x*(y*z) \in A$, $y \in A \to x*z \in A$ [since A is KS-H-ideal] therefore $f(x*z) \in f(A)$ but $f(x)*f(z) = f(x*z) \in f(A)$ so hence $f(A)$ is a left KS-H-ideal .

Proposition (3. 8)

Let $f : X \to Y$ be a KS-semigroup homomorphism then ker f is a KS-H-ideal of X .

Proof:

Let $f : X \to Y$ be a KS-semigroup homomorphism , since ker f is an ideal of X [3] it follows that ker f is a stable , now, let $x, y, z \in X$ such that $x*(y*z) \in \ker f$ *and* $y \in \ker f$,
so $f(x*(y*z)) = 0$ *and* $f(y) = 0$ so $f(x)*[f(y)*f(z)] = 0$ *and* $f(y) = 0$ so $f(x)*[0*f(z)] = 0$
so $f(x) = 0$ so $x \in$ ker f ,now , $f(x*z) = f(x)*f(z) = 0*f(z) = 0$
therefore $x*z \in$ ker f hence ker f is a KS-H-ideal .

Proposition (2.1.9)

Let I be a KS-ideal of KS-semigroup X such that $x*y = y*x$ for all $x \neq 0$ *and* $y \neq 0$ and $x*y = 0$ just when $x = 0$. Then I is a KS-H-ideal of X .

Proof:

First since I is a KS-ideal so $xa \in I$ $\forall x \in X$ *and* $a \in I$, Now
let $x, y, z \in X$ *and* $x*(y*z), y \in I$ *to prove* $x*z \in I$. There are several cases :

1) *If* $x, y, z \neq 0$ *and* $x \neq y \neq z$ *so*

$$\begin{aligned} x*(y*z) &= x*(z*y) && [\text{since } x*y = y*x\ \forall\ x, y \neq 0] \\ &= (z*y)*x && [\text{since } x*y = y*x\ \forall\ x, y \neq 0 \text{ and } x*y \neq 0] \\ &= (z*x)*y && [\text{since } (x*y)*z = (x*z)*y \text{ in } BCK] \\ &= (x*z)*y \in I && [\text{since } x*y = y*x\ \forall\ x, y \neq 0 \text{ and } x*y \neq 0\ ] \end{aligned}$$

and $y \in I$ *then* $x*z \in I$ [since I is a $KS-ideal$].

2) *If* $x = 0$ *and* $y, z \neq 0$ *so*
$0*(y*z) \in I$ *and* $y \in I$ *then*
$0 \in I$ *and* $y \in I$, I *is a* $KS-ideal$ *so*
$x*z = 0*z \in I$.

3) *If* $x = y = 0$ *and* $z \neq 0$ *then*
$0*(0*z) \in I$ *and* $z \in I$ *so* $x*z = 0*z \in I$.

4) *If* $x = 0, z = 0$ *then* $x*z \in I$ [*by the same way of* (3)].

5) $If\ \ x \neq 0\ ,\ \ and\ \ y = 0\ \ then$
$x*(0*z) \in I\ ,0 \in I$ $\qquad then\ \ x \in I, 0 \in I$.

$so\, If\ \ z = 0\ \ then\ \ x*z = x \in I\ \ \ and$
$If\ \ z \neq 0\ \ then\ \ (x*z)*x = 0\ \ \ and\ \ \ x \in I\ \ so\ \ x*z \in I\ \ \ [I\, is\, a\, KS-ideal]$.

6) $If\ \ x = 0, y = 0, z = 0\ \ then\ \ 0*(0*0) = 0 \in I$ so $x*z \in I$.

7) $If\ \ x \neq 0\ ,\ y \neq 0\ ,\ z = 0\ \ so$
$x*(y*0) \in I\ \ and\ \ y \in I\ \ then\ \ x*y \in I\ \ and\ \ y \in I\ \ [since\ I\ is\, a\ KS-ideal]$
$so\ \ x*z = x*0 \in I$

8) $If\ x = 0, y \neq 0\ , z = 0\ \ then\ \ 0*(y*0) \in I\ \ \ so\ \ \ x*z \in I$.

9) $If\ \ x = 0,, z \neq 0, y \neq 0\ \ then\ \ 0*(y*z) \in I\ \ \ so\ \ x*z \in I$.

10) $If\ z = 0, x \neq 0, y = 0\ \ \ x*(y*z) = x*0 \in I\ \ and\ \ z = 0 \in I \rightarrow x*z = x*0 \in I$, [since I is a KS-ideal] .

Then I is a KS-H-ideal .

4. *fuzzyKS-H-Ideal*

In this section , we define the notion of the fuzzy KS-H-ideal of KS-semigroup X and and prove some results and examples.

Definition (4..1)

A fuzzy subset μ of KS-semigroup X is called a ***left fuzzy KS-H-ideal*** if the following conditions hold :

$KSH1\ \ \ \mu(0) \geq \mu(x),$

$KSH2\ \ \ \mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\},$

$KSH3\ \ \ \mu(xa) \geq \min\{\mu(x), \mu(a)\}\ ,\ \ \ for\ all\ \ \ x, y, z, a \in X$.

A fuzzy subset μ is called a ***right fuzzy KS-H-ideal*** if it satisfies $KSH1$, $KSH2$ and

$KSH4\text{:}\ \ \mu(ax) \geq \min\{\mu(x), \mu(a)\},\ for\ all\ \ x, y, a\ \in X$.

A fuzzy subset μ is called a ***fuzzy KS-H-ideal*** it if is both left and right fuzzy KS -H-ideal of X .

Example (4.2)

Let X = {0, 1, 2, 3} be a KS-semigroup defined by the following tables:

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	3	2	0	0
3	3	3	3	0

.	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

Define a fuzzy subset $\mu : X \rightarrow [0,1]$ by $\mu(0) = 0.4$, $\mu(x) = 0.2\ \ \forall\, x \neq 0 \in X$. by usual calculations, we can prove that μ is a left fuzzy KS-H-ideal of X .

Remark(4.3)

Every fuzzy KS-H-ideal is a fuzzy KS- ideal .

Proof:

let μ be a fuzzy KS-H-ideal of X since $x*0 = x\ \ \forall\ x \in X$ [by Remark (2.2)]

$\mu(x)=\mu(x*0)\geq \min\{\mu(x*(y*0)),\mu(y)\}$
$=\min\{\mu(x*y),\mu(y)\}$

thus $\mu(x)\geq \min\{\mu(x*y),\mu(y)\}$

and since $\mu(0)\geq\mu(x)\ \forall x\in X$ and $\mu(xa)\geq \min\{\mu(x),\mu(a)\},\mu(ax)\geq\min\{\mu(a),\mu(x)\}\ \forall x,a\in X$

Hence μ is a fuzzy KS- ideal .

Proposition (4.4)

Let μ and λ are left (resp. right) fuzzy KS-H-ideal of KS-semigroup X then $\mu\cap\lambda$ is a left (resp. right) fuzzy KS-H-ideal .

Proof: Let μ and λ are left fuzzy KS-H-ideal of X then

$(\mu\cap\lambda)(0)=\min\{\mu(0),\lambda(0)\}\geq\min\{\mu(x),\lambda(x)\}=(\mu\cap\lambda)(x)\ \forall x\in X$ [$since\ \mu,\lambda\ are\ left\ fuzzy\ KS-H-ideal$]

$, now, (\mu\cap\lambda)(xa)=\min\{\mu(xa),\lambda(xa)\}\geq\min\{\min\{\mu(x),\mu(a)\},\min\{\lambda(x),\lambda(a)\}\}=\min\{\min\{\mu(x),\lambda(x),\min\{\mu(a),\lambda(a)\}$
$=\min\{(\mu\cap\lambda)(x),(\mu\cap\lambda)(a)\}\ \forall x,a\in X$

$so, (\mu\cap\lambda)(x*z)=\min\{\mu(x*z),\lambda(x*z)\}\geq\min\{\min\{\mu(x*(y*z),\mu(y)\},\min\{\lambda(x*(y*z),\lambda(y)\}\}$
$=\min\{\min\{\mu(x*(y*z),\lambda(x*(y*z)\},\min\{\mu(y),\lambda(y)\}\}$
$=\min\{\mu(x*(y*z)),\lambda(x*(y*z))\},\min\{\mu(y),\lambda(y)\}\}\ \forall x,y,z\in X$

hence $\mu\cap\lambda$ is a left fuzzy KS-H-ideal .

Proposition(4.5)

Let μ and ν be two fuzzy KS-H-ideal of KS-semigroup X if $\mu\subseteq\nu$ or $\nu\subseteq\mu$ then $\mu\cup\nu$ is a fuzzy KS-H-ideal .

Proof:

Let μ and ν are fuzzy KS-H-ideal of X , without loss of generality we may assume that let $\mu\subseteq\nu$ since μ and ν are fuzzy KS-H-ideal and $x,y,a\in X$ so $\mu(0)\geq\mu(x)$ and $\nu(0)\geq\nu(x)$ $,\forall\ x\in X$ therefore,

$(\mu\cup\nu)(0)=\max\{\mu(0),\nu(0)\}\geq\max\{\mu(x),\nu(x)\}=(\mu\cup\nu)(x)$,now ,since μ and ν are fuzzy KS-H-ideal so $\mu(xa)\geq\min\{\mu(x),\mu(a)\}$ and $\nu(xa)\geq\min\{\nu(x),\nu(a)\}$

max { $\mu(xa),\nu(xa)$ } $\geq\max\{\min\{\mu(x),\mu(a)\},\min\{\nu(x),\nu(a)\}\}$ since $\mu\subseteq\nu$ therefore

$(\mu\cup\nu)(xa)\geq\min\{\max\{\mu(x),\mu(a)\},\max\{\nu(x),\nu(a)\}\}=\min\{\max\{\mu(x),\nu(x)\},\max\{\mu(a),\nu(a)\}\}=\min\{(\mu\cup\nu)(x),(\mu\cup\nu)(a)\}$

and so, since $\mu(x*z)\geq\min\{\mu(x*(y*z)),\mu(y)\}$ and $\nu(x*z)\geq\min\{\nu(x*(y*z)),\nu(y)\}$ so

$\max\{\mu(x*z),\nu(x*z)\}\geq\max\{\min\{\mu(x*(y*z)),\mu(y)\},\min\{\nu(x*(y*z)),\nu(y)\}\}$ since $\mu\subseteq\nu$ therefore

$(\mu\cup\nu)(x*z)\geq\min\{\max\{\mu(x*(y*z)),\mu(y)\},\max\{\nu(x*(y*z)),\nu(y)\}\}$
$=\min\{\max\{\mu(x*(y*z)),\nu(x*(y*z))\},\max\{\mu(y),\nu(y)\}\}=\min\{(\mu\cup\nu)(x*(y*z)),(\mu\cup\nu)(y)\}$

hence $\mu\cup\nu$ is a fuzzy KS-H-ideal .

Proposition (4.6)

Let I and J are left (resp. right) fuzzy KS-H-ideal of KS-semigroup X then $I\times J$ is a left (resp. right) fuzzy KS-H-ideal of $X\times X$.

Proof:

Let I and J are left fuzzy KS-H-ideal of X then

$(I\times J)(0,0)=\min\{I(0),J(0)\}\geq\min\{I(x),J(y)\}=(I\times J)(x,y)\quad\forall(x,y)\in X\times X.$ [since I,J are left fuzzy KS-H-ideal of X], let $(x,x)\in X\times X$ and $(a_1,a_2)\in I\times J$ so ,

$(I\times J)(x,x)(a_1,a_2)=(I\times J)(xa_1,xa_2)=\min\{I(xa_1),J(xa_2)\}\geq\min\{\min\{I(x),I(a_1)\},\min\{J(x),J(a_2)\}\}$
$=\min\{\min\{I(x),J(x)\},\min\{I(a_1),J(a_2)\}\}\ =\min\{(I\times J)(x,x),(I\times J)(a_1,a_2)\}$

$now,\ let\ (x_1,x_2)\ ,(y_1,y_2)\ and\ (z_1,z_2)\in X\times X,$

$(I\times J)((x_1,x_2)*(z_1,z_2))=(I\times J)(x_1*z_1,x_2*z_2)=\min\{I(x_1*z_1),J(x_2*z_2)\}$
$\geq\min\{\min\{I(x_1*(y_1*z_1),I(y_1)\},\min\{J(x_2*(y_2*z_2)),J(y_2)\}\}$
$=\min\{\min\{I(x_1*(y_1*z_1),J(x_2*(y_2*z_2)\},\min\{I(y_1),J(y_2)\}\}$
$=\min\{(I\times J)((x_1,x_2)*(y_1*z_1,y_2*z_2)),(I\times J)(y_1,y_2)\}$
$=\min\{(I\times J)((x_1,x_2)*((y_1,y_2)*(z_1,z_2))),(I\times J)(y_1,y_2)\}$

hence $I \times J$ is a left fuzzy KS-H-ideal .

Proposition (4.7)

If A be a left (resp. right) KS-H-ideal of KS-semigroup X then $\forall\ 0 < t \le 1$ their exist a left (resp. right) fuzzy KS-H-ideal μ such that $A = \mu_t$.

Proof:

Let A be a left KS-H-ideal and μ be defined by

$$\mu(x) = \begin{cases} t & if \quad x \in A \\ 0 & if \quad x \notin A \end{cases} \qquad where \quad o < t \le 1$$

$let\ x \in A \quad then\ \mu(x) = t, then\ x \in \mu_t, so\ A \subseteq \mu_t, and\ if\ x \in \mu_t\ then\ \mu(x) \ge t, then\ x \in A\ so\ A = \mu_t$

Since A is a left KS-H-ideal $so\ 0 \in A then\ \mu(0) = t \ge \mu(x)\ \forall x \in X$, $now\ let\ x, a \in X$ there are several cases :

1. $If\ x, a \in X\ so$ so $xa \in A\ since\ A\ is\ a\ left\ H-ideal\quad \mu(xa) = t \ge \min\{\mu(x), \mu(a)\}$. so ,
2. $If\ x \notin A\ and\ a \notin A\quad then\quad \mu(xa) \ge \min\{\mu(x), \mu(a)\} = 0$
3. $If\ at\ most\ one\ of\ x, a\ belong\ to\ A, then\ at\ most\ one\ of\ \mu(x)\ and\ \mu(a)$ $is\ equal\ to\ 0$. $therefore\quad \mu(xa) \ge \min\{\mu(x), \mu(a)\} = 0$

$\mu(xa) \ge \min\{\mu(x), \mu(a)\}\quad \forall\ x, a \in X\ let\ x*(y*z), y \in X\ there\ are\ saveral\ cases:$

1. $If\ x*(y*z), y \in A then\ x*z \in A since\ A\ is\ a left\ KS-H-ideal\ so\ (x*z) = t \ge \min\{\mu(x*(y*z)), \mu(y)\}$
2. $If\ x*(y*z), y \notin A\ then\quad \mu(x*(y*z)) = \mu(y) = 0\quad so\quad \mu(x*z) \ge \min\{\mu(x*(y*z)), \mu(y)\} = 0$
3. $If\ at\ most\ one\ of\ x*(y*z), y\ belong\ to\ A, then\ at\ most one\ of\ \mu(x*(y*z))\ and\ \mu(y)\ is\ equal\ to\ 0, therefor$ $x*z \notin A\ \ \mu(x*z) \ge \min\{\mu(x*(y*z)), \mu(y)\} = 0\ so\ \mu(x*z) \ge \min\{\mu(x*(y*z)), \mu(y)\}\quad for\ all\ x, y, z \in X.$

hence μ is a left fuzzy KS-H-ideal .

Proposition (4.8)

Let μ be a left (resp. right) fuzzy KS-H-ideal in KS-semigroup X then a fuzzy set μ^+ defined by $\mu^+ = \mu(x) + 1 - \mu(0)$ is a left (resp. right) fuzzy KS-H-ideal such that $\mu \subseteq \mu^+$.

Proof:

Let μ be a left fuzzy KS-H-ideal and μ^+ is a fuzzy set then

$\mu^+(0) = \mu(0) + 1 - \mu(0) = 1 \ge \mu^+(x)\quad \forall\ x \in X$. now , let $x, a \in X$ so

$$\begin{aligned} \mu^+(xa) &= \mu(xa) + 1 - \mu(0) \\ &\ge \min\{\mu(x), \mu(a)\} + 1 - \mu(0) \quad [since\ \mu\ is\ a\ left\ fuzzy\ KS-H-ideal] \\ &= \min\{\mu(x) + 1 - \mu(0), \mu(a) + 1 - \mu(0)\} = \min\{\mu^+(x), \mu^+(a)\}. \end{aligned}$$

let $x, y, z \in X$ so

$$\begin{aligned} \mu^+(x*z) &= \mu(x*z) + 1 - \mu(0) \\ &\ge \min\{\mu(x*(y*z)), \mu(y)\} + 1 - \mu(0) \quad [since\ \mu\ is\ a\ left\ fuzzy\ KS-H-ideal] \\ &= \min\{\mu(x*(y*z)) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\} \\ &= \min\{\mu^+(x*(y*z)), \mu^+(y)\} \end{aligned}$$

hence μ^+ is a left fuzzy KS-H-ideal .

Proposition (4.9)

Let $f : X \to Y$ be a homomorphism if μ is a left (resp. right) fuzzy KS-H-ideal of Y then μ^f is a left (resp. right) fuzzy KS-H-ideal of X .

Proof :

Let μ be a left fuzzy KS-H-ideal of Y then $\mu^f(0) = \mu(f(0)) \geq \mu(f(x)) = \mu^f(x) \quad \forall\ x \in X$.

now, let $x, a \in X$ so

$$\mu^f(xa) = \mu(f(xa)) = \mu(f(x)f(a)) \qquad [f \text{ is a homomorphism}]$$
$$\geq \min\{\mu(f(x)), \mu(f(a))\} \qquad [\text{since } \mu \text{ is a left fuzzy KS-H-ideal}]$$
$$= \min\{\mu^f(x), \mu^f(a)\}$$

let $x,y,z \in X$, $\mu^f(x*z) = \mu(f(x*z)) = \mu(f(x)*f(z))$

$$\geq \min\{\mu(f(x)*[f(y)*f(z)]), \mu(f(y))\} \qquad [\text{since } \mu \text{ is a left H-ideal}]$$
$$= \min\{\mu(f(x*(y*z))), \mu(f(y))\}$$
$$= \min\{\mu^f(x*(y*z)), \mu^f(y)\}$$

hence μ^f is a left fuzzy KS-H-ideal of X .

Proposition (4.10)

Let $f : X \to Y$ be an epimorphism if μ^f is a left (resp. right) fuzzy KS-H-ideal of X then μ is a left (resp. right) fuzzy KS-H-ideal of Y.

Proof :

Let μ^f is a fuzzy KS-H-ideal of X and

let $y \in Y$ *then* $\exists\ x \in X$ *such that* $f(x) = y$

$$\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(0) \qquad [\text{since } \mu^f \text{ is a left fuzzy KS-H-ideal}]$$
$$= \mu(f(0)) = \mu(0) . \qquad [\text{by remark 2.13}]$$

now, let $x, a \in Y$ *then* $\exists\ t, m \in X$ *such that*

$f(t) = x$, $f(m) = a$ *then* $\mu(xa) = \mu(f(t)f(m)) = \mu(f(tm))$

$$= \mu^f(tm) \geq \min\{\mu^f(t), \mu^f(m)\} = \min\{\mu(f(t)), \mu(f(m))\} = \min\{\mu(x), \mu(a)\}$$

so, let $x, y, z \in Y$ *then* $\exists\ a, b, c \in X$ *such that* $f(a) = x$, $f(b) = y$, $f(c) = z$ *then*

$$\mu(x*z) = \mu(f(a)*f(c)) = \mu(f(a*c)) = \mu^f(a*c)$$
$$\geq \min\{\mu^f(a*(b*c)), \mu^f(b)\} = \min\{\mu(f(a*(b*c))), \mu(f(b))\} = \min\{\mu(f(a)*[f(b)*f(c)]), \mu(f(b))\}$$
$$= \min\{\mu(x*(y*z)), \mu(y)\}$$

hence μ is a fuzzy KS-H-ideal of Y.

Proposition (4.13)

Let I be a non-empty subset of a strong KS-semigroup X then I is a left (resp. right) KS-H-ideal of X if and only if χ_I is a left (resp. right) fuzzy KS-H-ideal where $\chi_I : X \to [0,1]$ define as follows :

$$\chi_I = \begin{cases} 1 & \text{if} \quad x \in I \\ 0 & \text{if} \quad x \notin I \end{cases}$$

Proof:

It is clear that χ_I is a fuzzy set .

suppose that I is a left KS-H-ideal of X and $x, y, a \in X$

since $0 \in X$ *so* $0.a = 0 \in I \quad \forall\, a \in I$ *then* $\chi_I(0) = 1 \geq \chi_I(x) \quad \forall\ x \in X.$

there are several cases : *let* $x, a \in X$

$1 - If\ x \in I, a \in I \quad so\ xa \in I \quad [since\ I\ is\ a\ left\ fuzzy\ KS-H-ideal]$

$\chi_I(x)=1\ , \chi_I(a)=1\ \ and\ \ \chi_I(xa)=1\ \ then\ \ \chi_I(xa) \geq \min\{\chi_I(x), \chi_I(a)\}$

$2 - If\ x \in I, a \notin I \quad so\ xa \notin I \quad thus\ \chi_I(x)=1\ , \chi_I(a)=0\ \ and\ \ \chi_I(xa)=0$

$then\ \ \chi_I(xa) \geq \min\{\chi_I(x), \chi_I(a)\}$

$3 - If\ x \notin I, a \in I \quad so\ xa \in I \quad thus\ \chi_I(x)=0\ , \chi_I(a)=1\ \ and\ \ \chi_I(xa)=1$

$then\ \ \chi_I(xa) \geq \min\{\chi_I(x), \chi_I(a)\}$

$4 - If\ x \notin I, a \notin I \quad so\ xa \notin I \quad thus\ \chi_I(x)=0\ , \chi_I(a)=0\ \ and\ \ \chi_I(xa)=0$

$then\ \ \chi_I(xa) \geq \min\{\chi_I(x), \chi_I(a)\}$

$so\ \ \chi_I(xa) \geq \min\{\chi_I(x), \chi_I(a)\}$

In similar way we can prove that $\chi_I(x*z) \geq \min\{\chi_I(x*(y*z)), \chi_I(y)\} \quad \forall\ x, y, z \in X$.

Hence χ_I is a left KS-H-ideal .

Conversely, assume that X is a strong KS-semigroup and let χ_I is a fuzzy KS-H-ideal of X

$let\ x \in X \quad and \quad a \in I\ since\ X\ is\ a\ strong\ KS-semigroup\ so \quad xa*a=0 \quad and\ since\ \ 0 \in I \quad so$

$xa*a \in I \quad and \quad a \in I \quad then \quad \chi_I(xa) \geq \min\{\chi_I((xa*a), \chi_I(a)\} = \min\{\chi_I(0), \chi_I(a)\} = \chi_I(a) = 1$

$so \quad xa \in I$

$now, let \quad x*(y*z) \in I \quad and \quad y \in I\ \ so$

$\chi_I(x*(y*z)) = \chi_I(y) = 1 \quad since \quad \chi_I\ is\ fuzzy\ KS-H-ideal\ we\ have$

$\chi_I(x*z) \geq \min\{\chi_I(x*(y*z)), \chi_I(y)\} = 1\ so\ x*z \in I$

therefore *I* is a left KS-H-ideal

Proposition (4.14)

If μ be a right fuzzy KS-H-ideal of KS-semigroup X with left identity *e* and satisfying the condition $(xy)z = (xz)y \quad \forall\ x, y, z \in X$ then μ is a left fuzzy H- ideal of X.

Proof:

Let μ be a right fuzzy KS-H-ideal of KS-semigroup X with left identity and let $x, a \in X$

$\mu(xa) = \mu((ex)a) = \mu((ea)x) = \mu(ax) \geq \min\{\mu(a), \mu(x)\} \qquad [by\ hypothesis]$

$\mu(xa) \geq \min\{\mu(x), \mu(a)\}$

$since \quad \mu(0) \geq \mu(x) \quad \forall x \in X \quad and$

$\mu(x*z) \geq \min\{\mu(x*(y*z)), \mu(y)\} \qquad [\mu\ is\ a\ right\ fuzzy\ KS-H\ ideal]$

therefore μ is a left fuzzy KS- H- ideal.

Corollary (4.15)

Every right fuzzy KS-H-ideal of KS-semigroup X with left identity *e* satisfying the condition is a fuzzy KS-H- ideal of X .

Proof:

Let μ be a right fuzzy KS-H- ideal with left identity then μ is a left fuzzy KS -H- ideal therefore μ is a fuzzy KS-H-ideal .

Proposition (4.16)

Let μ be a fuzzy set of strong KS-semigroup X if μ is a left fuzzy KS-H-ideal then μ_t left KS-H-ideal where $t \in [0, \mu(0)]$.

Proof:

Let μ be a left fuzzy KS-H-ideal of X ,and $t \in [0, \mu(0)]$. let $x \in \mu_t$

since $\mu(0) \geq t$ *then* $0 \in \mu_t$ then $\mu_t \neq \varphi$,
now, let $x \in X$ *and* $a \in \mu_t$ *so* $\mu(a) \geq t$ since X is a strong so $xa * a = 0$ and
since μ is a left fuzzy KS-H-ideal so
$\mu(xa) \geq \min\{\mu(xa*(a*0)), \mu(a)\} = \min\{\mu(xa*a), \mu(a)\} = \min\{\mu(0), \mu(a)\} = \mu(a) \geq t$
$\Rightarrow \quad xa \in \mu_t$
let $x*(y*z) \in \mu_t$ *and* $y \in \mu_t$ then $\mu(x*(y*z)) \geq t$ *and* $\mu(y) \geq t$, since μ is a left fuzzy KS-H-ideal so $\mu(x*z) \geq t$ *so* $x*z \in \mu_t$ .Therefore μ_t is a left KS-H-ideal .

Theorem (4.17)

Let X be a KS-semigroup and μ , λ be two fuzzy sets in X such that $\mu \times \nu$ is a fuzzy KS-H-ideal of X then :

1. either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \lambda(0)$ *for all* $x \in X$.
2. If $\mu(x) \leq \mu(0)$ *for all* $x \in X$ then either $\mu(x) \leq \lambda(0)$ *or* $\lambda(x) \leq \lambda(0)$.
3. If $\lambda(x) \leq \lambda(0)$ *for all* $x \in X$ then either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \mu(0)$.
4. either μ or λ is a fuzzy KS-H-ideal of X .

Proof:

since $\mu \times \nu$ is a fuzzy KS-H-ideal of X then it is fuzzy sub KS semigroup by [ ], so (1),(2) and (3) satisfied by [12]. Now, to prove 4, Since by (1) either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \lambda(0)$ *for all* $x \in X$ without loss of generality we may assume that $\lambda(x) \leq \lambda(0)$ it follows from (3) that either $\mu(x) \leq \mu(0)$ *or* $\lambda(x) \leq \mu(0)$ if $\lambda(x) \leq \mu(0)$ $\forall$ $x \in X$ then

$$\lambda(x.\ a) = \min\{\mu(0), \lambda(x.\ a)\} = (\mu \times \lambda)(0, x.\ a) = (\mu \times \lambda)(0.0, x.\ a) = (\mu \times \lambda)((0, x).(0, a)) \geq \min\{(\mu \times \lambda)(0, x), (\mu \times \lambda)(0, a)\}$$
$$= \min\{\min\{\mu(0), \lambda(x)\}, \min\{\mu(0), \lambda(a)\}\} = \min\{\lambda(x), \lambda(a)\ \} .$$

Now,

$$\lambda(x*z) = \min\{\mu(0), \lambda(x*z)\} = (\mu \times \lambda)(0, x*z) = (\mu \times \lambda)(0*0, x*z) = (\mu \times \lambda)((0, x)*(0, z))$$
$$\geq \min\{(\mu \times \lambda)[(0, x)*((0, y)*(0, z))], (\mu \times \lambda)(0, y)\} = \min\{(\mu \times \lambda)[(0, x)*(0, y*z)], (\mu \times \lambda)(0, y)\}$$
$$= \min(\mu \times \lambda)(0, x*(y*z)), (\mu \times \lambda)(0, y)\} = \min\{\min\{\mu(0), \lambda(x*(y*z))\}, \min\{\mu(0), \lambda(y)\}\}$$
$$= \min\{\lambda(x*(y*z)), \lambda(y)\} .$$

so λ is a fuzzy KS-H-ideal in X .
If $\lambda(x) \leq \mu(0)$ is not satisfied then $\lambda(y) > \mu(0)$ *for some* $y \in X$ and by our assumption ,
$\mu(x) \leq \mu(0)$ *for all* $x \in X$ we have $\lambda(0) \geq \lambda(y) > \mu(0) \geq \mu(x)$ *i.e* $\lambda(0) \geq \mu(x)$ $\forall x \in X$.
therefore $(\mu \times \lambda)(x,0) = \min\{\mu(x), \lambda(0)\} = \mu(x)$ and ,

$$\mu(x.a) = (\mu \times \lambda)(x.a,0)$$
$$= (\mu \times \lambda)(x.a,0.0) = (\mu \times \lambda)((x,0).(a,0)) \geq \min\{(\mu \times \lambda)(x,0), (\mu \times \lambda)(a,0)\} = \min\{\mu(x), \mu(a)\} .$$

so ,

$$\mu(x*z) = (\mu \times \lambda)(x*z,0) = (\mu \times \lambda)(x*z)(0*0) = (\mu \times \lambda)((x,0)*(z,0)) \geq \min\{(\mu \times \lambda)[(x,0)*((y,0)*(z,0))], (\mu \times \lambda)(y,0)\}$$
$$= \min\{(\mu \times \lambda)[(x,0)*(y*z,0)], (\mu \times \lambda)(y,0)\} = \min\{(\mu \times \lambda)((x*(y*z)),0), (\mu \times \lambda)(y,0)\}$$
$$= \min\{\min\{\mu(x*(y*z)), \lambda(0)\}, \min\{\mu(y), \lambda(0)\}\} = \min\{\mu(x*(y*z)), \mu(y)\} .$$

therefore μ is a fuzzy KS-H-ideal in X **.**

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