

Homotopy Analysis Method for Solving Non-linear Various Problem of Partial Differential Equations

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Abstract
In this paper, solve several important equations such as korteweg-devries (kdv) problem, Boussinesq equation of non-homogeneous problem and non-homogeneous system Hirota-Satsuma problem of partial differential equation by Homotopy analysis method (HAM). Studied comparison exact solution with numerical results , this method have shown that is very effective and convenient and gives numerical solutions in the form of convergent series with easily computable components for solving non-linear various problem of partial differential equation .
Keywords: Homotopy analysis method , Approximate solution , non-linear problems of partial differential equation , analytical solutions .

1. Introduction

Analytical methods have made a comeback in research methodology after taking a backseat to the numerical techniques for the latter half of the preceding century. The advantage of analytical methods are manifolds, the main being that they give a much better insight than the numbers crunched by a computer using a purely numerical algorithm. Most new nonlinear equations do-not have a precise analytic solution; so numerical methods have largely been used to handle these equation[8]. Nonlinear differential equations are usually arising from mathematical modeling of many physical systems. Some of them are solved using numerical methods and some are solved using the analytic methods such as perturbation [1, 4]. The numerical methods such as Rung-Kutta method are based on discretization techniques, and they only permit us to calculate the approximate solutions for some values of time and space variables, which cause us to overlook some important phenomena, in addition to the intensive computer time required to solve the problem[3]. It is well known that nonlinear dynamical systems arise in various fields. A wealth of methods have been developed to find these exact physically significant solutions of a partial equation though it is rather difficult. Some of the most important methods are Backlund transformation [5]. In 1992, Liao [6, 9] proposed a new analytical technique; namely the HAM based on homotopy of topology. However, in Liao's PhD dissertation [6], he did not introduce the auxiliary parameter h , but simply followed the traditional concept of homotopy to construct the following one-parameter family of equations. The HAM [6] , is a powerful method to solve non-linear problems. Based on homotopy of topology, the validity of the HAM is independent of whether or not there exist small parameters in the considered equation.

2.Basic idea of HAM

consider the following differential equation

$$N[u(t)]=0 \qquad ....(1)$$

where N is a nonlinear operator, τ denotes independent variable, u(τ) is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [6] construct the so-called zero-order deformation equation

$$(1-q)L[\varphi(t;q)-u_0(t)]=qhH(t)N[\varphi(t;q)] \qquad ....(2)$$

where L is an auxiliary linear operator, N is a nonlinear operator related to the original nonlinear problem $N[\varphi(t;q)]$ and q is the embedding parameter. An improved two parameters family of equations was proposed to avoid divergence of solution by introducing an auxiliary parameter h [11,10] and the auxiliary function H(t) , Liao [12] constructs, using $q\in[0,1]$. where $\varphi(t;q)$ is the solution which depends on h , H(t), L, $u_0(t)$ and q ,when q=0 and q=1 , it holds:

$$\varphi(t;0)=u_0(t) \qquad ....(3)$$

$\varphi(t;1) = u(t)$

Thus, as q increases from 0 to 1, the solution $\varphi(t;q)$ varies from the initial guess $u_0(t)$ to the solution $u(t)$. Expanding $\varphi(t;q)$ by Taylor series with respect to q, we get

$$\varphi(t;q) = u_0(t) + \sum_{m=1}^{\infty} u_m(t) q^m \quad \text{...........(4)}$$

$$u_m(t) = \frac{1}{m!} \frac{\partial^m \varphi(t;q)}{\partial q^m}\bigg|_{q=0} \quad \text{...........(5)}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter . and the auxiliary function are so properly chosen, the series (5) converges at q = 1, then we have

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t) \quad \text{...........(6)}$$

which must be one of the solutions of the original nonlinear equation, as proven by Liao [2]. As h = −1 and H(t) = 1, equation (3) become

$$(1-q)L[\varphi(t;q) - u_0(t)] + qN[\varphi(t;q)] = 0 \quad \text{..........(7)}$$

The governing equation can be deduced from the zero-order deformation equation (3). Define the vector

$$\vec{u}_n(t) = \{u_1(t), u_2(t), \ldots\ldots, u_n(t)\} \quad \text{...........(8)}$$

Differentiating equation (3) m-times with respect to the embedding parameter q, then setting q= 0 and finally dividing them by m!, we obtain the mth-order deformation equation.

$$L[u_m(t) - x_m u_{m-1}(t)] = hH(t) R_m(\vec{u}_{m-1}) \quad \text{..........(9)}$$

Where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\varphi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} \quad \text{..........(10)}$$

$$x_m = \begin{cases} 0, m \le 1 \\ 1, m > 1 \end{cases} \quad \text{............(11)}$$

it should be emphasized that $u_m(t)$ for $m \geqslant 1$ is governed by the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple, Mathematica and Matlab. If equation (2) admits unique solution, then this method will produce the unique solution. If equation (2) does not posses a unique solution, the HAM will give a solution among many other possible solutions.

3. Numerical experiments

In this section we give some computational results of numerical experiments with methods for solving important problems such as korteweg-devries (kdv) problem , Boussinesq equation of non-homogeneous problem and non-homogeneous system Hirota-Satsuma of partial differential equation . By solving the above issues we get accurate and good results as shown in the tables and figures below:

Example(1):

Conseder the korteweg-devries (kdv) problem of partial differential equation:

$$u_t + u_{xxx} + 6uu_x + 5(1-x)e^t = 0 \quad \text{............(12)}$$

With initial condition

$$u(x,0)=u_0(x)=(1-x)$$ ....(13)

To solve the equation (12) by means of homotopy analysis method, according to the initial conditions (13).

............(14)

$$L[\varphi(x,t;q)=\frac{\partial\varphi(x,t;q)}{\partial t},L[c_1]=0$$

where c_1 is the constant coefficients and φ is the real function, where L is linear operator.

Now define the nonlinear operator as:

$$N[\varphi(x,t;q)]=\frac{\partial\varphi(x,t;q)}{\partial t}+\frac{\partial^3\varphi(x,t;q)}{\partial x^3}+6\varphi(x,t;q)\frac{\partial\varphi(x,t;q)}{\partial x}$$
$$-(1-x)e^t$$

............(15)

Using above definition, with assumption we construct the zero order deformation equation.

$$(1-q)L[\varphi(r,t;q)-u_0(r,t)]=qhH(r,t)N[\varphi(r,t;q)]$$ ....(16)

with assumption H(r , t)=1 .It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when p = 0 and p = 1, it holds

$$\varphi_1(x,t;0)=u_0(x,t)$$
$$\varphi_2(x,t;1)=u(x,t)$$ ....(17)

Thus, we obtain the m^th-order deformation equation.

$$L[u_m-x_m u_{m-1}]=hH(r,t)R_m(\vec{u}_{m-1},t)$$ ....(18)

where

…..........(19) $$x_m=\begin{cases}0, m\le 1\\ 1, otherwise\end{cases}$$

and

............(20)

$$R_m(u_{m-1},x,t)=\frac{\partial u_{m-1}}{\partial t}+\frac{\partial^3 u_{m-1}}{\partial x^3}+6u_{m-1}\frac{\partial u_{m-1}}{\partial x}-(1-x)e^t$$

Applying L^{-1} both sides of (18) and uesd (HAM) to Eq. (12) and (13), as follows:

$$u_m=x_m u_{m-1}(x,t)+hH(r,t)L^{-1}[R_m(\vec{u}_{m-1},t)]$$ ....(21)

Now,apply HAM of equation (12),(13) and since m≥1 , $x_m=1$,h=-1, H(r , t)=1

In equation (21) then give:

$$u_m(x,t) = u_{m-1}(x,0) - L^{-1}(R_m(u_{m-1},x,t)) \qquad ....(22)$$

And

$$L^{-1} = \int_0^t (.)dt$$

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t) \qquad ....(23)$$

then by using iteration formula of HAM give:

$$u_0(x,t) = (1-x)$$
$$u_1(x,t) = (1-x)(1+t)$$
$$u_2(x,t) = (1-x)[1+t+\frac{t^2}{2!}]$$
$$u_3(x,t) = (1-x)[1+t+\frac{t^2}{2!}+\frac{t^3}{3!}]$$
$$.................... \qquad ....(24)$$

Table 1. approximate values and exact solutions for equation (24)

t	x	u approximate solution	u exact solution	Error
0.01	0	1	1	0
	0.1	0.9090000	0.9090451	0.0000451
	0.2	0.8080400	0.8080401	0.0000001
	0.3	0.7070331	0.7070351	0.0000002
	0.4	0.6060300	0.6060303	0.0000003
	0.5	0.5050250	0.5050261	0.0000011
	0.6	0.4040200	0.4040221	0.0000021
	0.7	0.3030151	0.3030171	0.0000002
	0.8	0.2020100	0.2020130	0.000003
	0.9	0.1010051	0.1010091	0.000004
	1	1.0100502	1.0100502	0

Now, compare the numerical results with exact solution obtained by the HAM gives much better numerical results shows that by above table and below figures(1),(2).

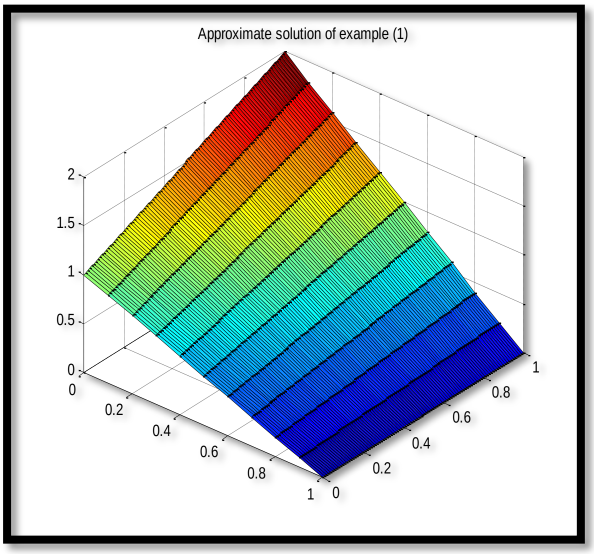


Figure 1. Exact solution of (kdv)

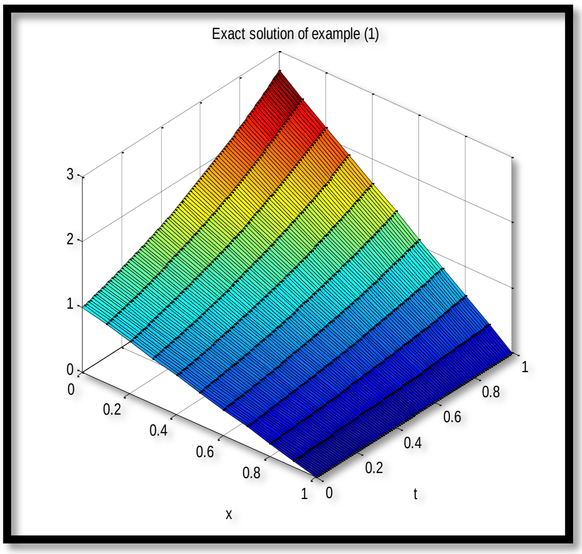


Figure 2. Approximate solution of (kdv)

The approximate solution of (1) is give:

$$u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+......+$$

and so on

$$u(x,t)=(1-x)e^t$$

which is an exact solution and is same as obtained by HAM , and shown the comparison numerical results exact solution in figure (1) and(2).

Example(2):

Consider the tow-dimensional weave equation:

$$u_{tt}-(u_{xx}+u_{yy})=0, \qquad .... (25)$$

together with the initial conditions

$$u(x,0)=\sin(\pi x)\cos(\pi y),$$
$$u_t(x,0)=\pi\sin(\pi x)\cos(\pi y), \qquad ....(26)$$

to solve the problem by using the HAM , substitute (25), (26) into following equation

$$u_m(x,y,t)=u_{m-1}(x,y,0)-L^{-1}[R(u_{m-1},x,y,t)] \qquad ....(27)$$

$$R(u_{m-1},x,y,t)=\frac{\partial^2 u_{m-1}}{\partial t^2}-\frac{1}{2}(\frac{\partial^2 u_{m-1}}{\partial x^2}+\frac{\partial^2 u_{m-1}}{\partial y^2}) \qquad ....(28)$$

and $L^{-1} = \int_0^t \int_0^t (.)dtdt$

Now, obtain the following recurrence relation:

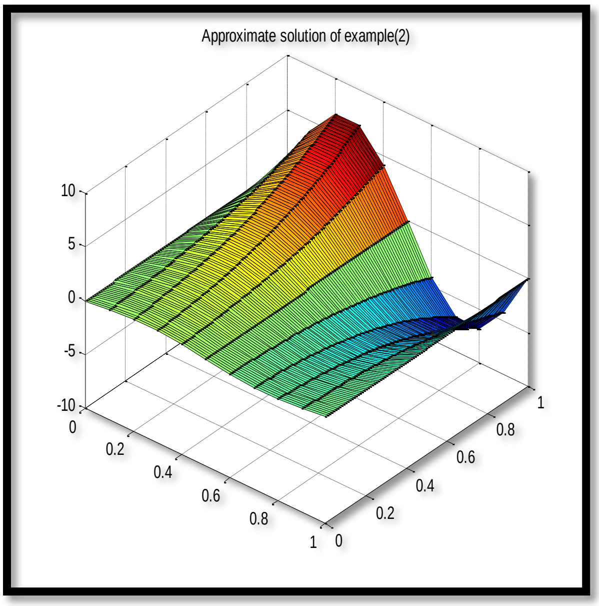
$$u_0(x,y,t) = \sin(\pi x)\cos(\pi y)(1+\pi t)$$
$$u_1(x,y,t) = \sin(\pi x)\cos(\pi y)(\frac{(\pi t)^2}{2!} + \frac{(\pi t)^3}{3!})$$
$$u_2(x,y,t) = \sin(\pi x)\cos(\pi y)(\frac{(\pi t)^4}{4!} + \frac{(\pi t)^5}{5!})$$
$$u_3(x,y,t) = \sin(\pi x)\cos(\pi y)(\frac{(\pi t)6}{6!} + \frac{(\pi t)7}{7!})$$

....(29)

Table 2. approximate values and exact solutions for equation (29)

t	y	x	u approximate solution	u exact solution	Error
0.01	0	0	0	0	0
	0.1	0.1	0.0054662	0.0056579	0.0001917
	0.2	0.2	0.0112042	0.0113153	0.0001111
	0.3	0.3	0.0167933	0.0169712	0.0001779
	0.4	0.4	0.0202253	0.0203638	0.0001385
	0.5	0.5	0.0251246	0.0254511	0.0003265
	0.6	0.6	0.0301350	0.0305359	0.0004009
	0.7	0.7	0.0346811	0.0356177	0.0009366
	0.8	0.8	0.0394513	0.0406960	0.0012447
	0.9	0.9	0.0446793	0.0457704	0.0010911
	1	1	0.0508403	0.0508403	0

In above table (2) computed absolute error for example (2) obtained by the HAM and It also illustrates the figure below that the error rate obtained accurate and excellent shown below:



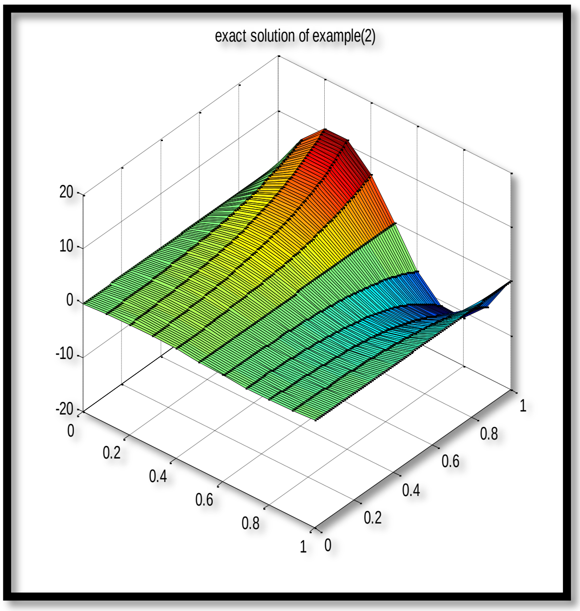


Figure 3. Exact solution of wave equation

Figure 4. Approximate solution of wave equation

Hence

$$u(x,y,t)=\sin(\pi x)\cos(\pi y)(1+\pi t+\frac{(\pi t)^2}{2!}+\frac{(\pi t)^3}{3!}+\frac{(\pi t)^4}{4!}+\frac{(\pi t)^5}{5!}+....)$$

................(30)

The result shows that the method provides an excellent approximation.

Example(3):

Consider the Boussinesq equation of non-homogeneous problem:

$$u_{tt}-u_{xx}-u_{xxxx}-3(u^2)_{xx}=-\pi^4\cos(\pi x)\cos(\pi t)+6\pi^2\cos^2(\pi t)[\cos^2(\pi x)-\sin^2(\pi x)]$$

................(31)

together with the initial conditions

$$u(x,0)=\cos(\pi x),$$
$$u_t(x,0)=0,$$

...............(32)

Now, to solve problem (31), (32) by HAM writing equation (33) in form yields:

$$u_m(x,t)=u_{m-1}(x,0)-L^{-1}(R_m(u_{m-1},x,t))$$

...............(33)

Where

$$R(u_{m-1},x,t)=\frac{\partial^2 u_{m-1}}{\partial t^2}-\frac{\partial^2 u_{m-1}}{\partial t^2}-\frac{\partial^4 u_{m-1}}{\partial x^4}-3\frac{\partial^2 u_{m-1}^2}{\partial x^2}+\pi^4\cos(\pi x)\cos(\pi t)$$
$$-6\pi^2\cos^2(\pi t)[\cos^2(\pi x)-\sin^2(\pi x)]$$

and

$$L^{-1}=\int_0^t\int_0^t(.)dtdt$$

this equation (31)can be easily solved by using this method to find the approximate solution beginning with $u(x,0)=\cos(\pi x),$

can obtain:

$$u_0(x,t)=\cos(\pi x)$$
$$u_1(x,t)=\cos(\pi x)(-\frac{(\pi t)^2}{2!}+\frac{(\pi t)^4}{4!})$$
$$u_2(x,t)=\cos(\pi x)(-\frac{(\pi t)^6}{6!}+\frac{(\pi t)^8}{8!})$$
$$u_3(x,t)=\cos(\pi x)(-\frac{(\pi t)^{10}}{10!}+-\frac{(\pi t)^{12}}{12!})$$

....(34)

Table 3. approximate values and exact solutions for equation (34)

t	x	u approximate solution	u exact solution	Error
0.01	0	0.9999991	0.9999991	0
	0.1	0.9995687	0.9999877	0.0004304
	0.2	0.9997531	0.9999511	0.0001980
	0.3	0.9995332	0.9998902	0.0003570
	0.4	0.9993532	0.9998051	0.0004519
	0.5	0.9992243	0.9996955	0.0004712
	0.6	0.9991258	0.9995616	0.0004358
	0.7	0.9990188	0.9994033	0.0003845
	0.8	0.9990243	0.9992207	0.0001964
	0.9	0.9990021	0.9990138	0.0000117
	1	0.9987251	0.9987251	0

Also in table (3), compare the numerical results with exact solution obtained by the HAM gives accurate numerical results shows that by above table and below.

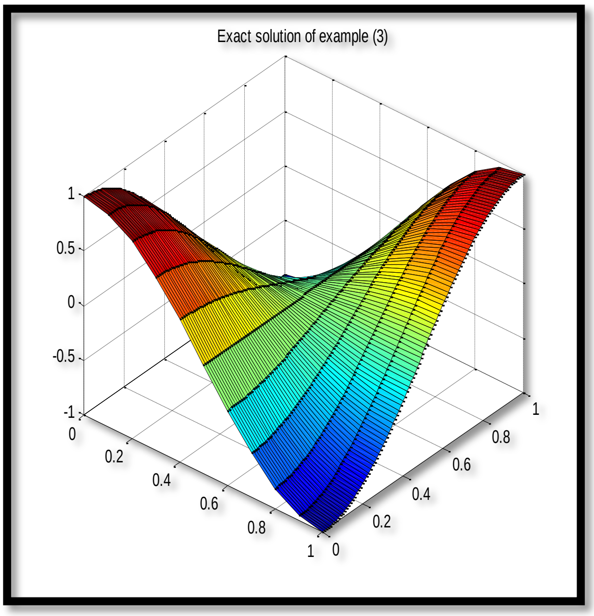


Figure 5.Exact solution of example (3)

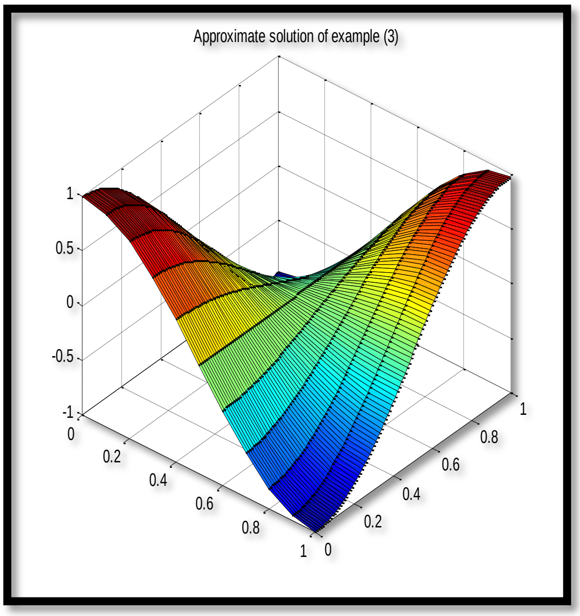


Figure 6.Approximate solution of example(3)

Which implies that:

$$u(x,t)=\cos(\pi x)(1-\frac{(\pi t)^2}{2!}+\frac{(\pi t)^4}{4!}-\frac{(\pi t)^6}{6!}+\frac{(\pi t)^8}{8!}-\frac{(\pi t)^{10}}{10!}+....)$$

which converges to the exact solution

Example(4):

Consider the non-homogeneous system Hirota-Satsuma of partial differential equation:

$$u_t-\frac{1}{2}u_{xxx}-3uu_x+6ww_x=\frac{1}{2}e^t\sinh(x)+3e^{2t}\sinh(x)\cosh(x)$$

....................(35)

$$w_t-w_{xxx}+3uw_x=e^t\cosh(x)+e^t\sinh(x)+3e^{2t}\sinh^2(x)$$

together with initial condition: ....(36)

$$u(x,0)=\sinh(x),$$
$$w(x,0)=\cosh(x),$$

....................(37)

Now, application of homotopy analysis method:

....................(38)

$$L_u[\varphi_1(x,t;q)=\frac{\partial\varphi_1(x,t;q)}{\partial t},L_w[\varphi_2(x,t;q)=\frac{\partial\varphi_2(x,t;q)}{\partial t},$$

with the property $L_u[c_1],L_w[c_2]$ where c_1 and c_2 are constant coefficients, ϕ and φ are real functions. Furthermore , define the nonlinear operators

$$N_u[\varphi_1(x,t;q),\varphi_2(x,t;q)=\frac{\partial\varphi_1}{\partial t}-\tfrac{1}{2}\frac{\partial^3\varphi_1}{\partial x^3}-3\varphi_1\frac{\partial\varphi_1}{\partial x}+6\varphi_2\frac{\partial\varphi_2}{\partial x}$$ ....(39)

.

$$N_w[\varphi_1(x,t;q),\varphi_2(x,t;q)=\frac{\partial\varphi_2}{\partial t}+\frac{\partial^3\varphi_2}{\partial x^3}+3\varphi_1\frac{\partial\varphi_2}{\partial x}$$

....................(40)

$q\in[0,1], \varphi_1(x,t;q) and\ \varphi_2(x,t;q)$ are real functions of x , t and q . Let h_u h_w denote the non-zero auxiliary parameters. Using the above definition, with assumption, $H_u(x, t)$, $H_w(x, t)$ construct the zero-order deformation equations as follows:

$$(1-q)L_u[\varphi_1(x,t;q)-u_0(x,t)]=qh_uH_u(t)N_u[\varphi_1(x,t;q),\varphi_2(x,t;q)]$$

...................(41)

$$(1-q)L_w[\varphi_2(x,t;q)-w_0(x,t)]=qh_wH_w(t)N_w[\varphi_1(x,t;q),\varphi_2(x,t;q)]$$

....................(42)

when q = 0 and q = 1 , it is clear that :

$$\varphi_1(x,t,0)=u_0(x,t),\varphi_2(x,t,0)=w_0(x,t),\varphi_1(x,t,1)=u(x,t),\varphi_2(x,t,1)=w(x,t)$$

....................(43)

Both of h_u and h_w are properly chosen so that the terms

$$u_n(x,t)=\frac{1}{n!}\frac{\partial^n\varphi_1(x,t;q)}{\partial q^n}|_{q=0}\ and\ \ w_n(x,t)=\frac{1}{n!}\frac{\partial^n\varphi_2(x,t;q)}{\partial q^n}|_{q=0}$$

.....................(44)

exist for $n \geqslant 1$ and the power series of q in the following forms

$$\varphi_1(x,t;q)=u_0(x,t)+\sum_{n=1}^{\infty}u_n(x,t)q^n,\varphi_2(x,t;q)=w_0(x,t)+\sum_{n=1}^{\infty}w_n(x,t)q^n$$

...................(45)

are convergent at q = 1 . So using (44), we obtain

$$\varphi_1(x,t;q)=u_0(x,t)+\sum_{n=1}^{\infty}u_n(x,t),\varphi_2(x,t;q)=w_0(x,t)+\sum_{n=1}^{\infty}w_n(x,t)$$

...................(46)

According to the fundamental theorem of HAM, we have the n^{th}-order deformation equation

$$L_u[u_n(x,t)-x_nu_{n-1}(x,t)]=h_uR_n^u(\vec{u}_{n-1,}\overrightarrow{w}_{n-1}),L_w[w_n(x,t)-x_nw_{n-1}(x,t)]=h_wR_n^w(\vec{u}_{n-1,}\overrightarrow{w}_{n-1})$$

.................(47)

$$R_n^u(\vec{u}_{n-1},\vec{w}_{n-1}) = h_u[\frac{\partial u_{n-1}}{\partial t} - \tfrac{1}{2}\frac{\partial^3 u_{n-1}}{\partial x^3} - 3u_{n-1}\frac{\partial u_{n-1}}{\partial x} + 6w\frac{\partial w_{n-1}}{\partial x} - \frac{1}{2}e^t\sinh(x)$$
$$-3e^{2t}\sinh(x)\cosh(x)$$

.................(48)

$$R_n^w(\vec{u}_{n-1},\vec{w}_{n-1}) = h_w[\frac{\partial w_{n-1}}{\partial t} - \frac{\partial^3 w_{n-1}}{\partial x^3} - 3u_{n-1}\frac{\partial w_{n-1}}{\partial x} - e^t\cosh(x) - e^t\sinh(x)$$
$$-3e^{2t}\sinh^2(x)$$

...................(49)

Now, the solution of the n th-order deformation equation (47) for n ≥ 1, becomes

$$u_n(x,t) = x_n u_{n-1}(x,t) + h_u L^{-1}[R_n^u(\vec{u}_{n-1},\vec{w}_{n-1})]$$

................(50)

$$w_n(x,t) = x_n w_{n-1}(x,t) + h_w L^{-1}[R_n^w(\vec{u}_{n-1},\vec{w}_{n-1})]$$

...............(51)

Where $h_u = h_w = h = -1$ and xn defined by (19). Now, write the differential equations need to calculate $u_1,u_2,u_3,\ldots,u_n$ and $w_1,w_2,w_3,\ldots,w_n$ as follows:

$$\begin{cases} u_0(x,t) = \sinh(x) \\ w_0(x,t) = \cosh(x) \end{cases}$$
$$\begin{cases} u_1(x,t) = t\sinh(x) \\ w_1(x,t) = t\cosh(x) \end{cases}$$
$$\begin{cases} u_2(x,t) = \dfrac{t^2}{2!}\sinh(x) \\ w_2(x,t) = \dfrac{t^2}{2!}\cosh(x) \end{cases}$$
$$\begin{cases} u_3(x,t) = \dfrac{t^3}{3!}\sinh(x) \\ w_3(x,t) = \dfrac{t^3}{3!}\cosh(x) \end{cases}$$
....

..................(52)

Table 4. approximate values and exact solutions of u for equation (52)

t	x	$u_{\text{approximate solution}}$	$u_{\text{exact solution}}$	Error
0.01	0	0	0	0
	0.1	0.0014642	0.0015865	0.0001223
	0.2	0.0030431	0.0031732	0.0001301
	0.3	0.0045225	0.0047597	0.0002372
	0.4	0.0061687	0.0063462	0.0001775
	0.5	0.0077253	0.0079328	0.0002075
	0.6	0.0093522	0.0095193	0.0001671
	0.7	0.0102151	0.0111059	0.0008908
	0.8	0.0113422	0.0126923	0.0013501
	0.9	0.0123625	0.0142787	0.0019162
	1	0.0158652	0.0158652	0

Table 5. approximate values and exact solutions of w for equation (52)

t	x	$u_{\text{approximate solution}}$	$u_{\text{exact solution}}$	Error
0.01	0	1	1	0
	0.1	1.0100325	1.0100489	0.0000164
	0.2	1.0100224	1.0100452	0.0000228
	0.3	1.0100201	1.0100389	0.0000188
	0.4	1.0100154	1.0100302	0.0000148
	0.5	1.0100117	1.0100190	0.0000073
	0.6	1.0100044	1.0100053	0.0000009
	0.7	1.0085122	1.0099891	0.0014769
	0.8	1.0073421	1.0099704	0.0026283
	0.9	1.0062962	1.0099492	0.0036530
	1	1.0099256	1.0099256	0

In above tables (4),(5) gives approximate values and exact solutions of non-homogeneous system Hirota-Satsuma of partial differential equation obtained by HAM gives much accurate numerical rustles shown that in below figures:

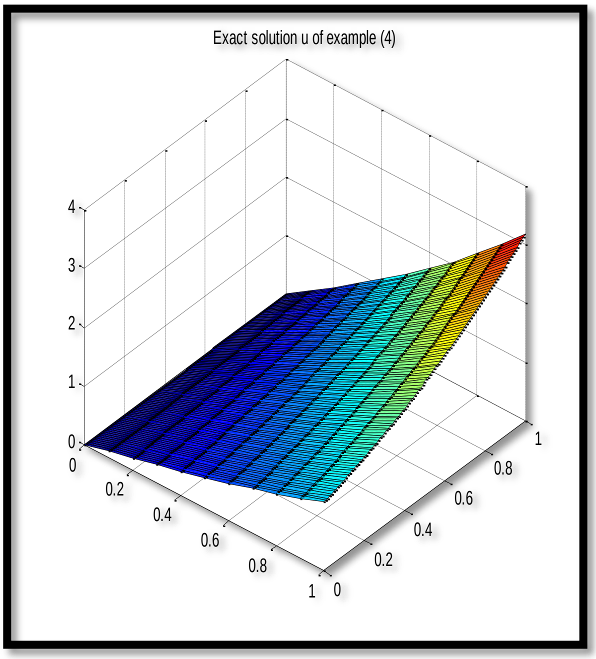


Figure 7.Exact solution for u of example (4)

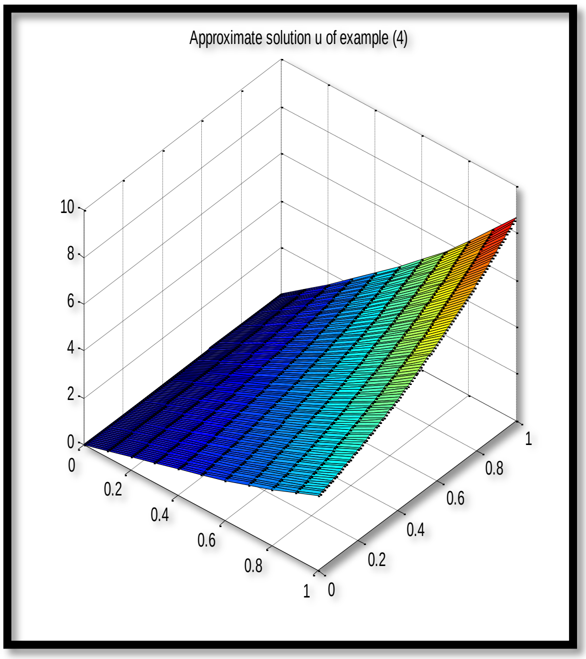


Figure 8.Approximate solution for u of example(4)

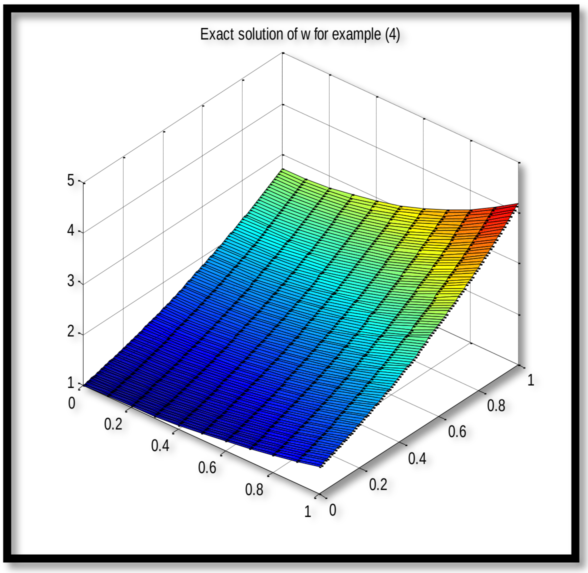


Figure 9.Exact solution for w of example (4)

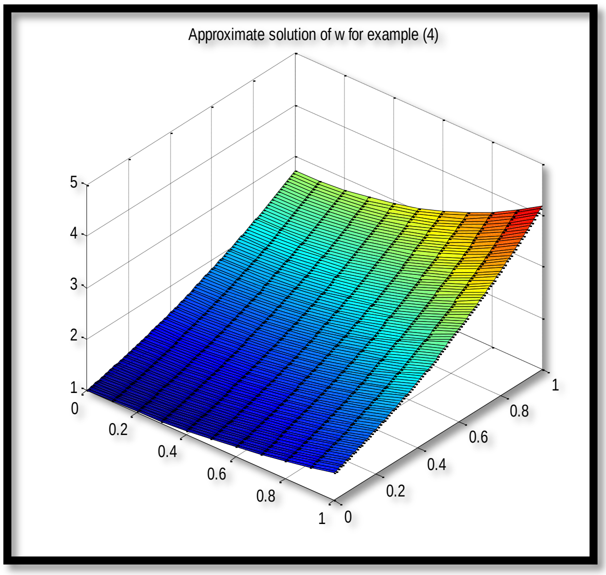


Figure10.Approximate solution for w of example(4)

Consequently, give:

$u(x, t)=e^{t}\sinh(x)$

$w(x, t)=e^{t}\cosh(x)$

which give best approximation result.

4.Discussion

In this paper, the HAM is employed to obtain the analytical and approximate solutions of PDE and it's successfully applied to solve many nonlinear problems of PDE's and many other equation such as (kdv)equation, non-homogeneous Boussinesq equation ,wave equation and non-homogeneous Hirota-Satsuma system. This method is very powerful and efficient technique in finding analytical solutions for wider class of problems. Moreover gives us a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. In conclusion, it provides accurate exact solution for various problems.

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