Some Contractive Mappings On *S*-Metric Spaces

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Abstract

The present study prove some fixed point results for two self-mappings in a complete S-metric space under some contractive conditions.

Keywords: S-metric spaces,fixed point,nondecreasing map

1 *Introduction.*

Studies on generalized metric spaces have received serious attention in recent years. One reason for this interest is their potential applicability. Specifically [5, 6] introduced an improved version of the generalized metric space structure, which they called G-metric space and established the Banach contraction principle. For more details on G-metric space, one can refer to the papers [7, 8]. Recently Sedghi et al.[9] have introduced the concept of S-metric space and some properties. Also,in [3, 4] some new properties of S-metric spaces were represented. In this paper we attain some fixed point results for self-mappings in a complete S-metric space under some contractive conditions in terms of a nondecreasing map ϕ .

2 *Basic Concepts*

In this part we recast the concept of S-metric space introduced by [9] for our goals.

Definition 2.1 *Let* X *be a nonempty set. We call S-metric on* X *is a function* $S : X^3 \to [0,\infty)$ *which satisfies the following conditions for each* $x, y, z, a \in X$

(i) $S(x, y, z) \geq 0,$

(ii) $S(x, y, z) = 0$ if and only if $x = y = z$,

(iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The set X in which S-metric is defined is called *S*-metric space.
The examples of such S-metric spaces are:

(a) Let X be any normed space, then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is a *S*-metric on X .

(b) Let (X,d) be a metric space, then $S(x, y, z) = d(x, z) + d(y, z)$ is a *S*-metric on X . This S-metric is called the *usual* S-metric on X .

(c) Another S-metric on (X,d) is $S(x, y, z) = d(x, y) + d(x, z) + d(y, z)$ which is symmetric with respect to the arguments.

The following lemmas have important role in our work (See[9]).

Lemma 2.1 *In a S-metric space, we have* $S(x,x,y)=S(y,y,x)$.

Lemma 2.2 *Let* (X,S) *be a* S*-metric space. If there exist sequences* $\{x_n\}$ *and* $\{y_n\}$ *such that* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$, *then* $\lim_{n\to\infty} S(x_n,x_n,y_n)=S(x,x,y)$.

There exists a natural topology on a S-metric spaces, for more details we refer to [3].

Lemma 2.3 *(See[3]). Any* S*-metric space is a Hausdorff space.*

Definition 2.2 *Let* f *and* g *be self-mappings of a set* X. *If* $w=fx=gx$ *for some* x *in* X, *then* x *is called a coincidence point of* f *and* g, *and* w *is called a point of coincidence of* f *and* g.

Theorem 2.1 *[1] Let* f *and* g *be weakly compatible self-mappings of a set* X. *If* f *and* g *have a unique point of coincidence* $w=fx=gx$, *then* w *is the unique common fixed point of* f *and* g.

3 Main Result

Suppose by [2] a nondecreasing function $\phi:[0,+\infty)\to[0,+\infty)$ has the following properties (when the power of functions to be understand with respect to the composition operation):

$(M1)$ $\lim_{n\to+\infty}\phi^n(t)=0$, for all $t\in(0,+\infty)$,

$(M2)$ $\phi(t)<t$ for all $t\in(0,+\infty)$,

$(M3)$ $\phi(0)=0$.

Examples of such functions will appear in what follows. The set of all function ϕ is denoted by Φ.

The method of proof of the following theorem is similar to the proof of the respective fact from [10].

Theorem 3.1 *Let* X *be a complete* S*-metric space and a self-map T on X satisfy the following contraction condition:*

$$S(T(x),T(x),T(y))\le\phi(S(x,x,y)) \quad (1)$$

for a $\phi\in\Phi$ and for all $x,y\in X$. Then T has a unique fixed point $u\in X$ and T is continuous at u.

Proof. Choose $x_0\in X$ and suppose that $x_n=T(x_{n-1})$ for $n\in\mathbb{N}$. Assuming $x_n\ne x_{n-1}$ we will show that $\{x_n\}$ is a Cauchy sequence. For $n\in\mathbb{N}$ we get

$$S(x_n,x_n,x_{n+1})=S(T(x_{n-1}),T(x_{n-1}),T(x_n))$$

$$\leq \phi(S(x_{n-1}, x_{n-1}, x_n)) \tag{2}$$

$$\ldots$$

$$\leq \phi^n(S(x_0, x_0, x_1))$$

Let $\varepsilon > 0$ be given. By (M1) and (M2) we have , $\lim_{n\to+\infty}\phi^n(S(x_0, x_0, x_1)) = 0$ and $\phi(\varepsilon) < \varepsilon$, then there exists n_0 such that

$$\phi^n(S(x_0, x_0, x_1)) < \frac{1}{2}(\varepsilon - \phi(\varepsilon)) \qquad \forall n \geq n_0.$$

Therefore by (2)

$$S(x_n, x_n, x_{n+1}) < \frac{1}{2}(\varepsilon - \phi(\varepsilon)) \qquad \forall n \geq n_0. \tag{3}$$

Applying the induction on m we can assert that

$$S(x_n, x_n, x_m) < \varepsilon \quad \text{for all} \qquad m \geq n \geq n_0. \tag{4}$$

Since $\varepsilon - \phi(\varepsilon) < \varepsilon$, and by (3), holds for $m = k$. By (iii) and Lemma 2.1 for $m = k+1$, we have

$$S(x_n, x_n, x_{k+1}) \leq 2S(x_n, x_n, x_{n+1}) + S(x_{k+1}, x_{k+1}, x_{n+1})$$

$$= 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{k+1})$$

$$\leq \varepsilon - \phi(\varepsilon) + \phi(S(x_n, x_n, x_k))$$

$$\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon.$$

Therefore $\{x_n\}$ is a Cauchy sequence.

Since X is complete then $\{x_n\}$ convergent to some $u \in X$. By (iii) and Lemma 2.1, for $n \in \mathrm{N}$ we have

$$S(u, u, T(u)) \leq 2S(u, u, x_{n+1}) + S(T(u), T(u), x_{n+1})$$

$$= 2S(u, u, x_{n+1}) + S(x_{n+1}, x_{n+1}, T(u))$$

$$= 2S(u, u, x_{n+1}) + S(T(x_n), T(x_n), T(u))$$

$$\leq 2S(u, u, x_{n+1}) + \phi(S(x_n, x_n, u))$$

$$< 2S(u, u, x_{n+1}) + S(x_n, x_n, u)$$

By letting $n \to \infty$ we have $S(u,u,T(u)) = 0$, hence by (ii) we have $T(u) = u$. Therefore u is a fixed point of T. To prove the uniqueness suppose that v is another fixed point of T. By (1) and $(M2)$ we have

$$S(u,u,v) = S(T(u),T(u),T(v))$$

$$\leq \phi(S(u,u,v))$$

$$< S(u,u,v).$$

Then $u = v$. To prove the continuity of T at u, let $\{y_n\}$ be a sequence that convergent to u. For $n \in \mathrm{N}$ we get

$$S(u,u,T(y_n)) = S(T(u),T(u),T(y_n))$$

$$\leq \phi(S(u,u,y_n))$$

$$< S(u,u,y_n).$$

Letting $n \to \infty$, we have $\lim_{n\to\infty} S(u,u,T(y_n)) = 0$. Therefore $T(y_n)$ converges to $u = T(u)$.

Corollary 3.1 *Let* T *be a self map on complete* S *-metric space* (X,S) *satisfying on following contraction condition for a* $\phi \in \Phi$ *and all* $x, y \in X$ *and for some m:*

$$S(T^m(x),T^m(x),T^m(y)) \leq \phi(S(x,x,y)),$$

then T has a unique fixed point.

Proof. By Theorem 3.2 we deduce that T^m has a fixed point (say, u). Since

$$T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u),$$

therefore $T(u)$ is also a fixed point for T^m. By uniqueness of u, we have $T(u) = u$.

Corollary 3.2 *Let* T *be a self map on a complete* S *-metric space* (X,S). *Suppose there is* $k \in [0,1)$ *such that* T *satisfies the following two contraction conditions for all* $x, y \in X$ *:*

$$S(T(x),T(x),T(y)) \leq kS(x,x,y)), \tag{5}$$

$$S(T(x),T(x),T(y)) \leq \frac{S(x,x,y)}{1+S(x,x,y)}, \tag{6}$$

then T has a unique fixed point(say, u) and T is continuous at u.

Proof. For (5) define $\phi:[0,\infty)\to[0,\infty)$ by $\phi(t)=kt$ and for (6) define $\phi(t)=\frac{t}{1+t}$. It's clear that ϕ is nondecreasing function with $\lim_{n\to}\phi^n(t)=0$ for all $t>0$. Since (1) is holds, the result follows from Theorem 3.2.

In this paper we prove following theorem:

Theorem 3.2 *Let* X *be a* S*-metric space. Suppose the maps* $f,g:X\to X$ *satisfy:*

$$S(fx,fx,fy)\leq \phi(max\{S(gx,gx,gy),G(gx,gx,fx),G(gy,gy,fy)\}) \tag{7}$$

for all $x,y\in X$. If $f(X)\subseteq g(X)$ and $g(X)$ is a closed subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible , then f and g have a unique common fixed point.

Proof. Suppose f and g satisfy inequality (7). Let x_0 be an arbitrary point in X. Since $f(X)\subseteq g(X)$, choose $x_1\in X$ such that $f(x_0)=g(x_1)$. Continuing this process, we produce a sequence $\{x_n\}$ in X such that $f(x_n)=g(x_{n+1})$ for all $n\in \mathrm{N}$. For $n\in \mathrm{N}\cup 0$, we have

$$S(gx_n,gx_n,gx_{n+1})=S(fx_{n-1},fx_{n-1},fx_n)$$

$$\leq \phi(max\{S(gx_{n-1},gx_{n-1},gx_n),S(gx_{n-1},gx_{n-1},fx_{n-1}),$$

$$,S(gx_n,gx_n,fx_n)\}.$$

Since

$$S(gx_n,gx_n,fx_n)=S(gx_n,gx_n,gx_{n+1})$$

and

$$\phi(S(gx_n,gx_n,fx_n))<S(gx_n,gx_n,gx_{n+1})$$

we have

$$max\{S(gx_{n-1},gx_{n-1},gx_n),S(gx_{n-1},gx_{n-1},fx_{n-1}),S(gx_n,gx_n,fx_n)\}$$

$$=S(gx_{n-1},gx_{n-1},gx_n).$$

Thus for $n \in \mathbb{N}$, we have

$$S(gx_n, gx_n, gx_{n+1}) \le \phi(S(gx_{n-1}, gx_{n-1}, gx_n))$$

$$\le \phi^2(S(gx_{n-2}, gx_{n-2}, gx_{n-1}))$$

$$\ldots$$

$$\le \phi^n S((gx_0, gx_0, gx_1)).$$

Given $\varepsilon > 0$. Since $\lim_{n\to\infty} \phi^n(S(gx_0, gx_0, gx_1)) = 0$ and $\frac{1}{3}(\varepsilon - \phi(\varepsilon)) > 0$, there is an integer k_0 such that

$$\phi^n(gx_0, gx_1, gx_1) < \frac{1}{3}(\varepsilon - \phi(\varepsilon)) \qquad \text{for all} \quad n \ge k_0.$$

Hence

$$S(gx_n, gx_n, gx_{n+1}) < \frac{1}{3}(\varepsilon - \phi(\varepsilon)) \qquad \text{for all} \quad n \ge k_0. \tag{8}$$

For $k, n \in \mathbb{N}$ with $k > n$, we claim:

$$S(gx_n, gx_n, gx_k) < \varepsilon \qquad \text{for all} \quad k \ge n \ge k_0 : \tag{9}$$

By induction on k we prove inequality (9) . Inequality (9) holds for $k = n+1$ by using inequality (8) and the fact that $\frac{1}{3}(\varepsilon - \phi(\varepsilon)) < \varepsilon$. Assume inequality (9) holds for $k = m$, that is,

$$G(gx_n, gx_n, gx_m) < \varepsilon \qquad \text{for all} \quad m \ge n \ge k_0. \tag{10}$$

For $k = m+1$, we have

$$S(gx_n, gx_n, gx_{m+1}) \le 2S(gx_n, gx_n, gx_{n+1}) + S(gx_{n+1}, gx_{n+1}, gx_{m+1})$$

From inequality (7), we have

$$S(gx_{n+1}, gx_{n+1}, gx_{m+1}) = S(fx_n, fx_n, fx_m)$$

$$\le \phi(\max\{S(gx_n, gx_n, gx_m), S(gx_n, gx_n, fx_n), S(gx_m, gx_m, fx_m)\}).$$

If

$$\max\{S(gx_n, gx_n, gx_m), S(gx_n, gx_n, fx_n), S(gx_m, gx_m, fx_m)\}) = S(gx_n, gx_n, gx_m)$$

then

$$S(gx_n, gx_n, gx_{m+1}) \le 2S(gx_n, gx_n, gx_{n+1}) + \phi(S(gx_n, gx_n, gx_m))$$

By inequalities (8) and (10) , we get

$$G(gx_n, gx_n, gx_{m+1}) < \frac{2}{3}(\varepsilon - \phi(\varepsilon)) + \phi(\varepsilon) < \varepsilon$$

If

$$\max\{S(gx_n, gx_n, gx_m), S(gx_n, gx_n, fx_n), S(gx_m, gx_m, fx_m)\}\} = S(gx_n, gx_n, fx_n).$$

Then

$$S(gx_n, gx_n, gx_{m+1}) \le 2S(gx_n, gx_n, gx_{n+1}) + \phi(S(gx_n, gx_n, fx_n)) < 3S(gx_n, gx_n, gx_{n+1})$$

By inequality (8), we get

$$S(gx_n, gx_n, gx_{m+1}) < \varepsilon - \phi(\varepsilon) < \varepsilon.$$

If

$$\max\{S(gx_n, gx_n, gx_m), S(gx_n, gx_n, fx_n), S(gx_m, gx_m, fx_m)\}\} = S(gx_m, gx_m, fx_m),$$

then

$$S(gx_n, gx_n, gx_{m+1}) \le 2S(gx_n, gx_n, gx_{n+1}) + \phi(S(gx_m, gx_m, fx_m))$$

Since $\phi(S(gx_m, gx_m, fx_m)) < S(gx_m, gx_m, fx_m)$ and $m > n \ge k_0$, then by (8) we have

$$S(gx_n, gx_n, gx_{m+1}) < \varepsilon - \phi(\varepsilon) < \varepsilon.$$

By induction on k, we conclude that inequality (7) holds for all $k \ge n \ge k_0$. So $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since $g(X)$ is complete, there is a point q in $g(X)$ such that $\{gx_n\}$ is convergent to some q. Choose $p \in X$ such that $gp = q$. We claim $fp = gp$. If not, then for $n \in \mathrm{N} \cup \{0\}$ we have

$$S(gx_n, gx_n, fp) = S(fx_{n-1}, fx_{n-1}, fp)$$

$$\phi(max\{S(gx_{n-1}, gx_{n-1}, gp), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gp, gp, fp)\}).$$

If

$$max\{S(gx_{n-1}, gx_{n-1}, gp), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gp, gp, fp)\} = S(gx_{n-1}, gx_{n-1}, gp),$$

then

$$S(gx_n, gx_n, fp) \le \phi(S(gx_{n-1}, gx_{n-1}, gp)) < S(gx_{n-1}, gx_{n-1}, gp).$$

Letting $n \to \infty$, we get that $gp = fp$. If

$$max\{S(gx_{n-1}, gx_{n-1}, gp), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gp, gp, fp)\} = S(gx_{n-1}, gx_{n-1}, fx_{n-1}),$$

then

$$S(gx_n, gx_n, fp) \le \phi(S(gx_{n-1}, gx_{n-1}, fx_{n-1})) = \phi(S(gx_{n-1}, gx_{n-1}, gx_n))$$

Since $\{gx_n\}$ is a Cauchy sequence and $\phi(S(gx_{n-1}, gx_{n-1}, gx_n)) < S(gx_{n-1}, gx_{n-1}, gx_n)$, by letting

$n \to \infty$, we get $gp = fp$. If

$$max\{S(gx_{n-1}, gx_{n-1}, gp), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gp, gp, fp)\} = S(gp, gp, fp),$$

then $S(gx_n, gx_n, fp) \le \phi(S(gp, gp, fp))$. Letting $n \to \infty$ we get

$$S(gp, gp, fp) \le \phi(S(gp, gp, fp))$$

Since $\phi(S(gp, gp, fp)) < S(gp, gp, fp)$, we have $S(gp, gp, fp) < S(gp, gp, fp)$ which is a contradiction.Therefore $gp = fp$. For uniqueness p, suppose that there exists another q in X such that $fq = gq$. If $gp \neq gq$, then we have

$$S(gq, gq, gp) = S(fq, fq, fp)$$

$$\phi(\max\{S(gq, gq, gp), S(gq, fq, fq), S(gp, gp, fp)\}).$$

Since $G(gq, gq, fq) = 0$, $S(gp, gp, fp) = 0$, and $\phi(S(gq, gq, gp)) < S(gq, gq, gp)$,we have $S(gq, gq, gp) < S(gq, gp, gp)$ which is a contradiction. So $gp = gq$. From Theorem 2.1, f and g have a unique common fixed point.

Theorem 3.2 generalizes Theorems 2.3 and 2.4 in [1] for S-metric spaces.

Corollary 3.3 *Let X be a S-metric space. Suppose the maps $f, g : X \to X$ satisfy on following inequality:*

$$S(fx, fx, fy) \le aS(gx, gx, gy) + bS(gx, gx, fx) + cS(gy, gy, fy)$$

for all $x, y \in X$, where $a + b + c < 1$. If $f(X) \subseteq g(X)$ and $g(X)$ is a closed subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. For $x, y \in X$, suppose

$$H(x, x, y) = \max\{S(gx, gx, gy), S(gx, gx, fx), S(gy, gy, fy)\}.$$

Then

$$aS(gx, gx, gy) + bS(gx, gx, fx) + cS(gy, gy, fy) \le (a + b + c)H(x, x, y).$$

So if,

$$S(fx,fx,fy)\leq aS(gx,gx,gy)+bS(gx,gx,fx)+cS(gy,gy,fy)$$

then $S(fx,fx,fy)\leq(a+b+c)H(x,x,y)$. Define $\phi:[0,+\infty)\to[0,+\infty)$ by $\phi(t)=(a+b+c)t$.

Then ϕ is a nondecreasing function. Also, if $a+b+c<1$ then $\lim_{n\to+\infty}\phi^n(t)=0$ for all $t>0$. Hence by Theorem 3.2, we get the result.

Corollary 3.4 *Let X be a S-metric space. Suppose the maps $f,g:X\to X$ satisfy on following inequality:*

$$S(fx,fx,fy)\leq k\max\{S(gx,gx,fx),S(gy,gy,fy)\} \tag{11}$$

for all $x,y\in X$, where $0\leq k<1$. If $f(X)\subseteq g(X)$ and $g(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible , then f and g have a unique common fixed point.

Proof. For all $x,y\in X$, we let

$$H(x,x,y)=\max\{S(gx,gx,fx),S(gy,gy,fy)\}.$$

if inequality (11) is hold,

then $S(fx,fx,fy)\leq kH(x,x,y)$. Define $\phi:[0,+\infty)\to[0,+\infty)$ by $\phi(t)=kt$. Then its clear that ϕ is nondecreasing and $\lim_{n\to+\infty}\phi^n(t)=0$ for all $t>0$. The result follows from Theorem 3.2.

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