Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.14, 2014

The Continuous Classical Optimal Control of a Coupled of Nonlinear Elliptic Equations

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Abstract

This paper is concern with the existence and the uniqueness solution of the state vector of a couple of nonlinear elliptic partial differential equations for a given continuous classical control vector. Also the existence theorem of a continuous classical optimal control vector governing by the considered couple of nonlinear partial differential equation of elliptic type with equality and inequality constraints is developed and proved. The existence and the uniqueness solution of the couple of adjoint equations associated with the considered couple equations of the state is studded. The derivation of the Frcéhet derivative of the Hamiltonian is obtained. The necessary conditions theorem so as the sufficient conditions theorem of optimality of the constrained problem are developed and proved.

Keywords: Classical optimal control, system of nonlinear elliptic, necessary and sufficient conditions.

1. Introduction

The optimal control problems play an important role in the many filed in life problems , for examples in robotics [Rubio et al 2011], in an electric power [Aderinto& Bamigbola 2012], in civil engineering [Amini & & Afshar 2008], in Aeronautics and Astronautics [Budigono& Wibowo2007], in medicine [El hiaet al 2012], in economic [Boucekkine& Fabbri 2013], in heat conduction [Borzabadi et al 2004], in biology [Agusto &Bamigbola2007] and many others field..

With Do to this importance and during the last decades many researchers interested to study the optimal control problems for systems governed either by nonlinear ordinary differential equations as in [Orpel2009] and many others, or governed either by linear partial differential equations as in [Lions1972] or by nonlinear partial differential equations either of a hyperbolic type as in [Farag 2014] and [Agusto&Bamigbola 2007], or by a parabolic type as in [Chryssoverghi & Al-Hawasy2004; El- Borai et al 2013] , or by an elliptic type as in [Bors & Walczak2005; Chryssoverghi et al 2006] or optimal control problems governed by semilinear elliptic equations as in [Casas& Kunisch2014] and an optimal control problem for a linear second order elliptic system as in [Bahaa& El-Shatery2013]. While the optimal control problem which is considered in this work is governed by a couple of nonlinear partial differential equations of elliptic type. The control is represented by a control vector and the state is represented by a vector state.

This paper is concern at first with the existence and the uniqueness of the state vector solution of a couple nonlinear elliptic partial differential equations for a given continuous classical control vector. Second the existence theorem of a continuous classical optimal control vector governing by the considered couple of nonlinear partial differential equation of elliptic type with equality and inequality constraints is developed and proved. The existence and uniqueness solution of the couple of adjoint vector equations associated with the considered couple equations of the state is studded. The derivation of the Fréchet derivative of the Hamiltonian is derived. Finally the theorem of necessary conditions so as the theorem of sufficient conditions of optimality of the constrained problem of are developed and proved.

2. Description of the problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\Gamma = \partial\Omega$. Consider the following nonlinear elliptic state equations with Dirichlet boundary value problem

$$-\Delta y_1 + y_1 - y_2 + f_1(x, y_1, u_1) = f_2(x, u_1), \text{ in } \Omega \quad (1)$$

$$-\Delta y_2 + y_2 + y_1 + h_1(x, y_2, u_2) = h_2(x, u_2), \text{in } \Omega \quad (2)$$

$$y_1 = 0 \text{ , on } \Gamma \quad (3)$$

$y_2 = 0$, *on* Γ (4)

where $(y_1, y_2) = (y_1(x), y_2(x)) \in (H_0^1(\Omega))^2$ is the state vector, the functions $f_1(x, y_1, u_1)$ and $h_1(x, y_2, u_2)$ are defined on $\Omega \times \mathbb{R} \times U_1$ and $\Omega \times \mathbb{R} \times U_2$ respectively and the functions $f_2(x, u_1)$ and $h_2(x, u_2)$ are defined on $\Omega \times U_1$ and $\Omega \times U_2$ respectively with $U_1 \subset \mathbb{R}$ and $U_2 \subset \mathbb{R}$.

The control constraint (The control set) is $(u_1, u_2) \in W_1 \times W_2 = \vec{W}, \vec{W} \subset (L^2(\Omega))^2$

where $\vec{W} = \vec{W}_{\vec{U}}$ is the set of controls with $\vec{U} \subset \mathbb{R}^2$ is defined by

$$\vec{W} = \{\vec{w} \in (L^2(\Omega))^2 \,|\, \vec{w} = (w_1, w_2) \in U_1 \times U_2 = \vec{U}, a.e. \, in \, \Omega\}$$

The cost functional is

$$G_0(\vec{u}) = \int_\Omega g_{01}(x, y_1, u_1)\,dx + \int_\Omega g_{02}(x, y_2, u_2)\,dx \qquad (5)$$

The constraints on the state and the control are

$$G_1(\vec{u}) = \int_\Omega g_{11}(x, y_1, u_1)\,dx + \int_\Omega g_{12}(x, y_2, u_2)\,dx = 0 \qquad (6)$$

$$G_2(\vec{u}) = \int_\Omega g_{21}(x, y_1, u_1)\,dx + \int_\Omega g_{22}(x, y_2, u_2)\,dx \le 0 \qquad (7)$$

The set of admissible controls is $\vec{W}_A = \{(u_1, u_2) \in W_1 \times W_2 \,|\, G_1(\vec{u}) = 0, G_2(\vec{u}) \le 0\}$ (8)

The continuous optimal control problem is to minimize the cost functional (5) subject to the constraints (6) and (7), i.e. to find $\vec{u}$ such that $G_0(\vec{u}) = \underset{\vec{w} \in \vec{W}_A}{Min}\, G_0(\vec{w})$ and $\vec{u} \in \vec{W}_A$.

Let $\vec{V} = V \times V = H_0^1(\Omega) \times H_0^1(\Omega)$. We denote by (v, v) and $\|v\|_0$ the inner product and the norm in $L^2(\Omega)$, by $(v, v)_1$ and $\|v\|_1$ the inner product and the norm in $H_0^1(\Omega)$, by $(\vec{v}, \vec{v})$ and $\|\vec{v}\|_0$ the inner product and the norm in $L^2(\Omega) \times L^2(\Omega)$ by $(\vec{v}, \vec{v})_1 = (v_1, v_1)_1 + (v_2, v_2)_1$ and $\|\vec{v}\|_1 = \|v_1\|_1 + \|v_2\|_1$ the inner product and the norm in $\vec{V}$ and $\vec{V}^*$ is the dual of $\vec{V}$.

3. The solution of the state equations

In order to find the classical solution of problem (1-4), first we find the weak forms of problem (1-4). Multiplying both sides of equations (1) and (2) by $v_1 \in V$ and $v_2 \in V$ respectively, integrating both sides of each one of the obtained equation with respect to x and then using the general Green's theorem for the 1st term in each obtained weak form the following weak forms are obtained

$$(\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (f_1(x, y_1, u_1), v_1) = (f_2(x, u_1), v_1), \ \forall v_1 \in V \qquad (9)$$

$$(\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_1, v_2) + (h_1(x, y_2, u_2), v_2) = (h_2(x, u_2), v_2), \ \forall v_2 \in V \qquad (10)$$

Adding (9) and (10), we get that

$$a(\vec{y}, \vec{v}) + (f_1(x, y_1, u_1), v_1) + (h_1(x, y_2, u_2), v_2) = (f_2(x, u_1), v_1) + (h_2(x, u_2), v_2) \forall (v_1, v_2) \in \vec{V} \qquad (11)$$

where $a(\vec{y}, \vec{v}) = (\nabla y_1, \nabla v_1) + (y_1, v_1) - (y_2, v_1) + (\nabla y_2, \nabla v_2) + (y_2, v_2) + (y_1, v_2)$ (12)

Assumptions (A): **a)** $a(\vec{y}, \vec{y})$ is coercive, $\vec{y} \in \vec{V}$.

b) $|a(\vec{y}, \vec{v})| \le \ell_1 \|\vec{y}\|_1 \|\vec{v}\|_1$, where $\ell_1 > 0$.

c) The functions $f_1(x, y_1, u_1)$ and $h_1(x, y_2, u_2)$ are of Caratheodory type on $\Omega \times \mathbb{R} \times U_1$ and $\Omega \times \mathbb{R} \times U_2$ respectively and satisfy the following sub linear conditions

$|f_1(x, y_1, u_1)| \le \phi_1(x) + c_1|y_1| + \bar{c}_1|u_1|$, where $\phi_1 \in L^2(\Omega)$, $c_1, \bar{c}_1 \ge 0$ and $|h_1(x, y_2, u_2)| \le \phi_2(x) + c_2|y_2| + \bar{c}_2|u_2|$, where $\phi_2 \in L^2(\Omega)$, $c_2, \bar{c}_2 \ge 0$.

d) f_1 is monotone w.r.t. y_1 for each $x \in \Omega$ and $u_1 \in U_1$, h_1 is monotone w.r.t. y_2 for each $x \in \Omega$, and $u_2 \in U_2$ and satisfy $f_1(x, 0, u_1) = 0$, $u_1 \in U_1$, $\forall x \in \Omega$ and $h_1(x, 0, u_2) = 0$, $u_2 \in U_2$, $\forall x \in \Omega$.

e) The functions $f_2(x, u_1)$ and $h_2(x, u_2)$ are of Caratheodory type on $\Omega \times U_1$ and $\Omega \times U_2$ respectively, and satisfy the following conditions

$|f_2(x, u_1)| \le \phi_3(x) + c_3|u_1|$, $\forall (x, u_1) \in \Omega \times U_1$, $\phi_3 \in L^2(\Omega)$, $c_3 \ge 0$ and

$|h_2(x, u_2)| \le \phi_4(x) + c_4|u_2|$, $\forall (x, u_2) \in \Omega \times U_1$, $\phi_4 \in L^2(\Omega)$, $c_4 \ge 0$.

Theorem 3.1: In addition to the assumptions (A), if one of the functions f_1 or h_1 in assumption (d) is strictly monotone. Then for each fixed control vector $\vec{u} \in \vec{W}$, the weak form (11) has a unique solution state vector $\vec{y} \in \vec{V}$.

Proof: Let $A: \vec{V} \longrightarrow \vec{V}^*$ then the weak form (11) can rewrite as

$$\langle A(\vec{y}),\vec{v}\rangle = (f_2(u_1),(h_2(u_2)),\ \forall \vec{v}\in \vec{V} \quad (13)$$

where $\langle A(\vec{y}),\vec{v}\rangle = a(\vec{y},\vec{v}) + (f_1(y_1,u_1),v_1) + (h_1(y_2,u_2),v_2)$ (14)

i) From the assumptions A(a & d) , A is coercive

ii) From the assumptions A (b and c) and using proposition 2.1 in [Chyssoverghi & Bacopoulos 1993] the mapping $\vec{y}\mapsto \langle A(\vec{y}),\vec{v}\rangle$ is continuous w.r.t. $\vec{y}$.

iii) From the assumptions A(a & d) and (i) A is strictly monotone with respect to $\vec{y}$.

By using Minty–Browder theorem in [Brezis 2011] we get that there exists is a unique weak solution $\vec{y}\in\vec{V}$ of the weak form (13).

Assumption (B): g_{l1} and g_{l2} are of Caratheodory type on $\Omega\times\mathbb{R}\times U_1$ and $\Omega\times\mathbb{R}\times U_2$ respectively and are satisfy the following sub quadratic conditions with respect to (y_1,u_1) and (y_2,u_2), i.e. $\forall l=0,1,2$

$$|g_{l1}(x,y_1,u_1)| \le \eta_{l1}(x) + c_{l1}y_1^2 + \bar{c}_{l1}u_1^2,\text{with } (y_1,u_1)\in\mathbb{R}\times U_1,\ \eta_{l1}\in L^1(\Omega),\text{ and } c_{l1},\bar{c}_{l1},\ge 0$$

and $$|g_{l2}(x,y_2,u_2)| \le \eta_{l2}(x) + c_{l2}y_2^2 + \bar{c}_{l2}u_2^2,\text{ with } (y_2,u_2)\in\mathbb{R}\times U_2,\ \eta_{l2}\in L^1(\Omega),\text{ and } c_{l2},\bar{c}_{l2}\ge 0$$

Lemma 3.1: In Addition to the assumptions (A), if the functions f_1 and h_1 are Lipschitz with respect to u_1 and the functions f_2 and h_2 are Lipschitz with respect to u_2. Then the operator $\vec{u}\mapsto\vec{y}_{\vec{u}}$, from $\vec{W}_A$ to $(L^2(\Omega))^2$ is Lipschitz continuous.

Proof: Let $u_1',u_2'\in\vec{W}_A$ be two controls of the weak forms (9) and (10) respectively, y_1' and y_2' be the corresponding state solutions of these controls. Setting $\delta y_1 = y_1'-y_1$, $\delta y_2 = y_2'-y_2$, $\delta u_1 = u_1'-u_1$ and $\delta u_2 = u_2'-u_2$, substituting these term in (9) and (10) with setting $v_1=\delta y_1$ and $v_2=\delta y_1$, then adding the obtained equations, we get

$$\begin{aligned}
&(\nabla\delta y_1,\nabla\delta y_1) + (\delta y_1,\delta y_1) + (\nabla\delta y_2,\nabla\delta y_2) + (\delta y_2,\delta y_2)\\
&+(f_1(y_1+\delta y_1,u_1+\delta u_1) - f_1(y_1,u_1+\delta u_1),\delta y_1)\\
&+(h_1(y_2+\delta y_2,u_2+\delta u_2) - h_1(y_2,u_2+\delta u_2),\delta y_2)\\
&\quad = -(f_1(y_1,u_1+\delta u_1) - f_1(y_1,u_1),\ \delta y_1) + (f_2(u_1+\delta u_1) - f_2(\delta u_1),\delta y_1)\\
&\quad\quad -(h_1(y_2,u_2+\delta u_2) - h_1(y_2,u_2),\delta y_2) + (h_2(u_2+\delta u_2) - h_2(\delta u_2),\delta y_2) \qquad (15)
\end{aligned}$$

Using assumption (A-d), (15) becomes

$$\begin{aligned}
\|\delta\vec{y}\|_1^2 \le &\left|\int_\Omega (f_1(x,y_1,u_1+\delta u_1) - f_1(x,y_1,u_1))\,\delta y_1\,dx\right|\\
&+\left|\int_\Omega (f_2(x,u_1+\delta u_1) - f_2(x,\delta u_1))\delta y_1\,dx\right|\\
&+\left|\int_\Omega (h_1(x,y_2,u_2+\delta u_2) - h_1(x,y_2,u_2))\delta y_2\,dx\right|\\
&+\left|\int_\Omega (h_2(x,u_2+\delta u_2) - h_2(x,\delta u_2))\delta y_2\,dx\right| \qquad (16)
\end{aligned}$$

Using the Lipschitz assumptions on f_1 and f_2 w.r.t. $u_1$and on h_1 and h_2 w.r.t. u_2 on the terms of the R.H.S. of (16) and then using the Cauchy-Schwartz inequality of the obtained inequality, the last one with $L_3 = L_1 + L_2$, $\bar{L}_3 = \bar{L}_1 + \bar{L}_2$ and $L = max(L_3,\bar{L}_3)$ becomes

$$\|\delta\vec{y}\|_1^2 \le L_3\ \|\delta u_1\|_0\|\delta y_1\|_0 + \bar{L}_3\ \|\delta u_2\|_0\|\delta y_2\|_0 \Rightarrow \|\delta\vec{y}\|_1^2 \le L\ \|\delta\vec{u}\|_0 \qquad (17)$$

*Lemma 3.2***:** With assumption (B), the functional $G_l(\vec{u})$, (for $=0,1,2$) defined on $(L^2(\Omega))^2$ is continuous.

Proof: From assumptions (B) and using Proposition 2.1 in [Chyssoverghi & Bacopoulos 1993] each of the functionals $\int_\Omega g_{l1}(x,Y_1)\,dx$ and $\int_\Omega g_{l2}(x,Y_2)\,dx$ (for $l=0,1,2$ and $Y_k=(y_k,u_k)$, for $k=1,2$) is continuous on $(L^2(\Omega))^2$. Hence the functional $G_l(\vec{u})$, (for $l=0,1,2$) is continuous on $(L^2(\Omega))^2$.

Lemma 3.3: Let $g: Q\times\mathbb{R}^2\longrightarrow\mathbb{R}$ is of Caratheodory type on $Q\times(\mathbb{R}\times\mathbb{R})$ and satisfies

$$|g(x,y,u)| \le \eta(x) + cy^2 + c'u^2,\text{ where } \in L^1(Q,\mathbb{R}),\ c\ge 0\ and\ c'\ge 0.$$

Then $\int_Q g(x,y,u)\,dx$ is continuous on $L^2(Q,\mathbb{R}^2)$, with $u\in U$, $U\subset\mathbb{R}$ is compact [Chryssoverghi 2010].

4. Existence of an optimal control:

In this section the existence of a classical optimal control of the considered problem under some conditions is studied. Also some other assumptions are added in or dear to study the adjoint equations of the sate equations (1-4) and to derive the Fréchet derivatives of the cost function (5) and of the functionals in (6 -7) as follows:

Theorem 4.1: In addition to assumptions (A) and (B), we assume that the set of controls $\vec{W}$ is of the form $\vec{W} = \vec{W}_{\vec{U}}$ with $\vec{U}$ convex and compact, $\vec{W}_A \neq \emptyset$, where f_1 and h_1 are independent of u_1 and u_2 respectively, and f_2 and h_2 are linear with respect to u_1 and u_2 respectively, i.e.

$$f_1(x, y_1, u_1) = f_1(x, y_1), \text{ with } |f_1(x, y_1)| \le \phi_1(x) + c_1|y_1|, \text{ where } \phi_1 \in L^2(\Omega), \text{ and } c_1 \ge 0$$

$$h_1(x, y_2, u_2) = h_1(x, y_2), \text{with } |h_1(x, y_2)| \le \phi_2(x) + c_2|y_2|, \text{ where } \phi_2 \in L^2(\Omega), \text{ and } c_2 \ge 0$$

$$f_2(x, u_1) = f_2(x)u_1 \text{ with, } |f_2(x)| \le \kappa_1 \text{ and } h_2(x, u_2) = h_2(x)u_2, \text{ with } |h_2(x)| \le \kappa_2$$

g_{l1} and g_{l2} is independent of u_1 and u_2, g_{l1} and g_{l2} $(l = 0,2)$ are convex with respect to u_1 and u_2 for fixed (x, y_1) and (x, y_2) respectively. If $\vec{W}$ is bounded. Then there exists a classical optimal control.

Proof: First, We need to prove that $W_1 \times W_2$ is weakly compact.

i) Since $U_i \subset \mathbb{R}$ is convex and bounded (for each $i = 1,2$), then W_i is convex and bounded then $W_1 \times W_2$ is also convex and bounded.

ii) Since U_i is closed $\forall i = 1,2$, then by using Egorov's theorem in [Warga 1972], W_i is closed and then $W_1 \times W_2$ is closed.

From (i) and (ii) we get $\vec{W} = W_1 \times W_2$ weakly compact.

Since $\vec{W}_A \neq \emptyset$, then there exists a point $\vec{u} \in \vec{W}_A$, such that $G_1(\vec{u}) = 0$, $G_2(\vec{u}) \le 0$, and then there exists a minimum sequence $\{\vec{u}_n\} = \{(u_{1n}, u_{2n})\} \in \vec{W}_A$, such that $\lim_{n\to\infty} G_0(\vec{u}_n) = \inf_{\vec{u}\in\vec{W}_A} G_0(\vec{u})$.

Then by using Alaoglu theorem [Adams 1975] there exists a subsequence of $\{\vec{u}_n\}$ say again $\{\vec{u}_n\}$ which converges weakly to some point $\vec{\bar{u}}$ in $\vec{W}$.

Now, by using equation (11), assumption (A-e) and the Cauchy Schwarz inequality ,we get

$$\|\vec{y}_n\|_1^2 \le \left((A(\vec{y}_n), \vec{y}_n)\right) = (f_2(x, u_{1n}), y_{1n}) + (h_2(x, u_{2n}), y_{2n})$$

$$\le \ell_3\|y_{1n}\|_0 + c_3\tilde{c}_1\|y_{1n}\|_0 + \ell_4\|y_{2n}\|_0 + c_4\tilde{c}_2\|y_{2n}\|_0 \le (r_3 + r_4)\|\vec{y}_n\|_1 = \bar{C}\|\vec{y}_n\|_1$$

where $r_3 = \max(\ell_3, c_3\tilde{c}_1)$, $r_4 = \max(\ell_4, c_4\tilde{c}_2)$, $\bar{C} = \max(r_3, r_4) > 0$, then $\|\vec{y}_n\|_1 \le \bar{C}$, $\forall n$.

Then by Alaoglu theorem there exists a subsequence of $\{\vec{y}_n\}$ say again $\{\vec{y}_n\}$ such that $\vec{y}_n \longrightarrow \vec{\bar{y}}$ weakly in $\vec{V}$ which means $\vec{y}_n \longrightarrow \vec{\bar{y}}$ weakly in $(L^2(\Omega))^2$ and by using the compactness theorem in [Temam, 1977] we get that $\vec{y}_n \longrightarrow \vec{\bar{y}}$ strongly in $(L^2(\Omega))^2$. Now and since for each n, $\vec{y}_n = (y_{1n}, y_{2n})$ satisfies the weak form (11) , then (with $f_1 = f_1(x, y_{1n}), h_1 = h_1(x, y_{2n})$, $f_2 = f_2(x)$, $h_2 = h_2(x)$)

$$(\nabla y_{1n}, \nabla v_1) + (y_{1n}, v_1) - (y_{2n}, v_1) + (\nabla y_{2n}, \nabla v_2) + (y_{2n}, v_2) + (y_{1n}, v_2)$$

$$+(f_1(y_{1n}), v_1) + (h_1(y_{2n}), v_2) = (f_2 u_{1n}, v_1) + (h_2 u_{2n}, v_2) \qquad (18)$$

Let $\vec{y}_n = (y_{1n}, y_{2n}) \in (C(\bar{\Omega}))^2$ and $(v_1, v_2) \in (C(\bar{\Omega}))^2$, We want to prove (18) converges to

$$(\nabla\bar{y}_1, \nabla v_1) + (\bar{y}_1, v_1) - (\bar{y}_2, v_1) + (\nabla\bar{y}_2, \nabla v_2) + (\bar{y}_2, v_2) + (\bar{y}_1, v_2)$$

$$+(f_1(\bar{y}_1), v_1) + (h_1(\bar{y}_2), v_2) = (f_2\bar{u}_1, v_1) + (h_2\bar{u}_2, v_2) \qquad (19)$$

i) Since for each $i = 1,2$, $y_{in} \longrightarrow \bar{y}_i \Longrightarrow$

$$y_{in} \longrightarrow \bar{y}_i \text{weakly in } L^2(\Omega) \text{ and } \nabla y_{in} \longrightarrow \nabla\bar{y}_i \text{weakly in } L^2(\Omega) \qquad (20)$$

ii) From the assumption on $f_1(x, y_{1n})$, $h_1(x, y_{2n})$) and using the result of lemma3.2, one gets that $\int_\Omega f_1(x, y_{1n})v_1\,dx$ and $\int_\Omega (h_1(x, y_{2n})v_2\,dx$ are continuous w.r.t. y_{1n} and y_{2n} respectively, since $y_{1n} \longrightarrow \bar{y}_1$ and $y_{2n} \longrightarrow \bar{y}_2$ strongly in $L^2(\Omega)$, then the L.H.S. of (18) $\longrightarrow$ the LH.S. of (19), i.e.

$$(f_1(y_{1n}), v_1) + (h_1(y_{2n}), v_2) \longrightarrow (f_1(\bar{y}_1), v_1 + (h_1(\bar{y}_2), v_2), \quad \forall (v_1, v_2) \in (C(\bar{\Omega}))^2$$

Also since $u_{1n} \longrightarrow \bar{u}_1$ & $u_{2n} \longrightarrow \bar{u}_2$ weakly in $L^2(\Omega)$, then the R.H.S. of (18) $\longrightarrow$ the R.H.S. of (19), i.e.

$$(f_2 u_{1n} - f_2\bar{u}_1, v_1) + (h_2 u_{2n} - h_2\bar{u}_2, v_2) = (f_2(u_{1n} - \bar{u}_1), v_1) + (h_2(u_{2n} - \bar{u}_2), v_2) \longrightarrow 0$$

But $(C(\bar{\Omega}))^2$ is dense in $\vec{V}$ then the above convergence hold for each $(v_1, v_2) \in \vec{V}$, which gives $\vec{y}_n \longrightarrow \vec{\bar{y}} = \vec{y}_{\vec{\bar{u}}}$ is a solution of the state equations.

From lemma (3.2) we get $G_l(\vec{u})$ is continuous on $(L^2(\Omega))^2$, for each l=0,1,2,

From the assumptions on g_{l1} & g_{l2} and $y_{1n} \longrightarrow \bar{y}_1$, $y_{2n} \longrightarrow \bar{y}_2$ strongly in $L^2(\Omega)$, then

$$G_1(\vec{\bar{u}}) = \lim_{n\to\infty} G_1(\vec{u}_n) = 0 \text{ , hence } G_1(\vec{\bar{u}}) = 0.$$

Now, we prove $G_l(\vec{\bar{u}})$, (for each l=0, 2) is W.L.S.C. with respect to $\vec{\bar{y}}$ and $\vec{\bar{u}}$.

From the assumptions (B), $(u_{1n}, u_{2n}) \in \vec{U}$ a.e. in Ω, $\vec{U}$ is compact and then from lemma (3.3), we get
$\int_\Omega g_{l1}(x, y_{1n}, u_{1n})dx \longrightarrow \int_\Omega g_{l1}(x, \bar{y}_1, u_{1n})dx$ and $\int_\Omega g_{l2}(x, y_{2n}, u_{2n})dx \longrightarrow \int_\Omega g_{l2}(x, \bar{y}_2, u_{2n})dx$

Since $g_{l1}(x, \bar{y}_1, \bar{u}_1)$ and $g_{l2}(x, \bar{y}_2, \bar{u}_2)$, (for each l=0, 2) are convex with respect to $\bar{u}_1$ and $\bar{u}_2$ respectively, then is convex $G_l(\vec{\bar{u}})$, (for each l=0, 2) with respect to $\vec{\bar{u}}$, i.e.

$$\int_\Omega g_{l1}(x, \bar{y}_1, \bar{u}_1)dx \leq \underline{\lim}_{n\to\infty} \int_\Omega g_{l1}(x, \bar{y}_1, u_{1n})dx$$

$$= \underline{\lim}_{n\to\infty} \int_\Omega \left(g_{l1}(x, \bar{y}_1, u_{1n}) - g_{l1}(x, y_{1n}, u_{1n})\right)dx + \underline{\lim}_{n\to\infty} \int_\Omega g_{l1}(x, y_{1n}, u_{1n})dx$$

$$= \underline{\lim}_{n\to\infty} \int_\Omega g_{l1}(x, y_{1n}, u_{1n})dx \quad \text{, (for } l = 0,2)$$

By the same way we get $\int_\Omega g_{l2}(x, \bar{y}_2, \bar{u}_2)dx \leq \underline{\lim}_{n\to\infty} \int_\Omega g_{l2}(x, y_{2n}, u_{2n})dx$, (for $l = 0,2$) , i.e.

$$G_l(\vec{\bar{u}}) \leq \underline{\lim}_{n\to\infty} G_l(\vec{u}_n) \text{ , (for each } l\text{=0, 2)} \Longrightarrow \text{ then } G_2(\vec{\bar{u}}) \leq \underline{\lim}_{n\to\infty} G_2(\vec{u}_n) = 0 \Longrightarrow G_2(\vec{\bar{u}}) \leq 0$$

on the other hand we have that

$$G_0(\vec{\bar{u}}) \leq \underline{\lim}_{n\to\infty} G_0(\vec{u}_n) = \lim_{n\to\infty} G_0(\vec{u}_n) = \inf_{\vec{w}\in\vec{W}_A} G_0(\vec{u}) \text{ . i.e. } \vec{\bar{u}} \text{ is an optimal control.}$$

Assumptions C: Assume f_{1y_1}, f_{1u_1}, f_{2u_1}, h_{1y_1}, h_{1u_1}, h_{2u_1}, g_{l1y_1}, g_{l1u_1}, g_{l2y_2} and g_{l2u_2} are of the Caratheodory type and satisfy: $|f_{1y_1}| \leq c_1$, $|f_{1u_1}| \leq c_2$, $|f_{2u_1}| \leq c_3$, $|h_{1y_1}| \leq \acute{c}_1$, $|h_{1u_1}| \leq \acute{c}_2$, $|h_{2u_1}| \leq \acute{c}_3$, $|g_{ljy_j}| \leq \eta_{lj} + c_{lj}|y_j| + c_{lj}|u_j|$, and $|g_{lju_j}| \leq \acute{\eta}_{lj} + \acute{c}_{lj}|y_j| + \acute{c}_{lj}|u_j|$ for $j = 1,2$ and $l = 0,1,2$.

Theorem 4.2: With assumptions (A) , (B) and (C), the Hamiltonian is defined by:

$$H(x, y_1, y_2, u_1, u_2) = z_1(f_2(x, u_1) - f_1(x, y_1, u_1)) + g_{01}(x, u_1, y_1)$$
$$+ z_2(h_2(x, u_2) - h_1(x, y_2, u_2)) + g_{02}(x, u_2, y_2)$$

the adjoint vector $(z_1, z_2) = (z_{1u_1}, z_{2u_2})$ equations of the state equations (1- 4) are given by

$$-\Delta z_1 + z_1 + z_2 + z_1 f_{1y_1}(x, y_1, u_1) = g_{01y_1}(x, y_1, u_1), \text{ in } \Omega \tag{21}$$
$$-\Delta z_2 + z_2 - z_1 + z_2 h_{1y_2}(x, y_2, u_2) = g_{02y_2}(x, y_2, u_2), \text{ in } \Omega \tag{22}$$
$$z_1 = 0, \text{ on } \Gamma \tag{23}$$
$$z_2 = 0, \text{ on } \Gamma \tag{24}$$

Then the Fréchet derivatives of G_0 are given by

$$\acute{G}_0(\vec{u}).\overrightarrow{\delta u} = \int_\Omega H_{\vec{u}}^T.\overrightarrow{\delta u}\, dx \quad \text{, where } H_{\vec{u}} = \begin{pmatrix} H_{u_1}(x,y_1,u_1,y_2,u_2) \\ H_{u_2}(x,y_1,u_1,y_2,u_2) \end{pmatrix} = \begin{pmatrix} z_1(f_{2u_1} - f_{1u_1}) + g_{1u_1} \\ z_2(h_{2u_2} - h_{1u_2}) + g_{2u_2} \end{pmatrix}$$

and the operator $\vec{u} \longmapsto \vec{z}_{\vec{u}}$ is continuous.

Proof: Rewriting the adjoint equations (21-23)-(22-24) by their weak forms, adding these two weak forms, we get

$$(\nabla z_1, \nabla v_1) + (z_1, v_1) + (z_2, v_1) + (\nabla z_2, \nabla v_2) + (z_2, v_2) - (z_1, v_2) + (f_{1y_1} z_1, v_1)$$
$$+ (h_{1y_2} z_2, v_2) = (g_{01y_1}(y_1, u_1), v_1) + (g_{02y_2}(y_2, u_2), v_2), \forall (v_1, v_2) \in \vec{V} \tag{25}$$

It is clear that the weak form of the adjoint equation (25) has a unique solution $\vec{z} = \vec{z}_{\vec{u}}$ for a given control $\vec{u} \in \vec{W}$.

Now, by substituting $v_1 = \delta y_1$ and $v_2 = \delta y_2$ in (25) we obtain

$$(\nabla z_1, \nabla \delta y_1) + (z_1, \delta y_1) + (z_2, \delta y_1) + (\nabla z_2, \nabla \delta y_2) + (z_2, \delta y_2) - (z_1, \delta y_2) + (f_{1y_1} z_1, \delta y_1)$$
$$+ (h_{1y_2} z_2, \delta y_2) = (g_{01y_1}(y_1, u_1), \delta y_1) + (g_{02y_2}(y_2, u_2), \delta y_2) \tag{26}$$

Substituting the solution y_1 once in (9) and then again the solution $y_1 + \delta y_1$, subtracting the obtained equations one from the other, finally substituting $v_1 = z_1$, we have

$$(\nabla\delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1)$$
$$+(f_1(y_1 + \delta y_1, u_1 + \delta u_1) - f_1(x, y_1, u_1), z_1) = ((f_2(u_1 + \delta u_1) - f_2(u_1), z_1)) \quad (27)$$

Also substituting the solutions y_2 once in (10) and then again the solution $y_2 + \delta y_2$, subtracting the obtained equations one from the other, then substituting $v_2 = z_2$ we have

$$(\nabla\delta y_2, \nabla z_2) + (\delta y_2, z_2) + (\delta y_1, z_2) + (h_1(y_2 + \delta y_2, u_2 + \delta u_2) - h_1(y_2, u_2)z_2)$$
$$= ((h_2(u_2 + \delta u_2) - h_2(u_2), z_2)) \quad (28)$$

Adding (27) and (28), we obtain

$$(\nabla\delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) + (\nabla\delta y_2, \nabla z_2) + (\delta y_2, z_2) + (\delta y_1, z_2)$$
$$+(f_1(y_1 + \delta y_1, u_1 + \delta u_1) - f_1(y_1, u_1), z_1) + (h_1(y_2 + \delta y_2, u_2 + \delta u_2) - h_1(y_2, u_2)z_2)$$
$$= ((f_2(u_1 + \delta u_1) - f_2(u_1), z_1)) + ((h_2(u_2 + \delta u_2) - h_2(u_2), z_2)) \quad (29)$$

From the assumptions on f_1 ,h_1 , f_2 and h_2 , up on using the proposition2.1 in [Chyssoverghi & Bacopoulos 1993] and the Mean value theorem the Fréchet derivatives of f_1 , h_1 , f_2 and h_2 are exist , then from Lemma 3.1 and the Minkowiski inequality, we get that

$$(\nabla\delta y_1, \nabla z_1) + (\delta y_1, z_1) - (\delta y_2, z_1) + (\nabla\delta y_2, \nabla z_2) + (\delta y_2, z_2) + (\delta y_1, z_2) + (f_{1y_1}\delta y_1 - f_{1u_1}\delta u_1, z_1)$$
$$+(h_{1y_2}\delta y_2 - h_{1u_2}\delta u_2, z_2) = (f_{2u_1}\delta u_1, z_1) + (h_{2u_2}\delta u_2, z_2) + \varepsilon(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\|_0 \quad (30)$$

where $\varepsilon(\overrightarrow{\delta u}) \longrightarrow 0$, and $\|\overrightarrow{\delta u}\|_0 \longrightarrow 0$, as $\overrightarrow{\delta u} \longrightarrow 0$.

Now, from the assumptions on g_{01} and g_{02}, the definition of the Fréchet derivative and then using the result of Lemma (3.1), we have

$$G_0(\vec{u} + \delta\vec{u}) - G_0(\vec{u}) = \int_\Omega (g_{01y_1}\delta y_1 + g_{01u_1}\delta u_1)dx + \int_\Omega (g_{02y_2}\delta y_2 + \delta u_2)dx + \tilde{\varepsilon}(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\|_0 \quad (31)$$

where $\tilde{\varepsilon}(\overrightarrow{\delta u}) \longrightarrow 0$, and $\|\overrightarrow{\delta u}\|_0 \longrightarrow 0$ as $\overrightarrow{\delta u} \longrightarrow 0$.

By subtracting (26) from (30), substituting the obtained equation in (31) , the last becomes

$$G_0(\vec{u} + \delta\vec{u}) - G_0(\vec{u}) = \int_\Omega (z_1(f_{2u_1} - f_{1u_1}) + g_{01u_1})\delta u_1 dx$$
$$+ \int_\Omega (z_2(h_{2u_2} - h_{1u_2}) + g_{02u_2})\delta u_2 dx + \tilde{\varepsilon}(\overrightarrow{\delta u})\|\overrightarrow{\delta u}\|_0 \quad (32)$$

But from the definition of the Fréchet derivative we have that

$$\vec{G}_0(\vec{u}).\overrightarrow{\delta u} = \int_\Omega H_{\vec{u}}{}^T.\overrightarrow{\delta u}\, dx\, dx$$

It is Clear that the operator $\vec{u} \longmapsto \vec{z}_{\vec{u}}$ is continuous in $(L^2(\Omega))^2$.

Remark 4.1: Of course the Fréchet derivatives of the functions $G_1(\vec{u})$ and $G_2(\vec{u})$ can be derived by the same this way.

5. Necessary and sufficient conditions for optimality:

In this section the necessary conditions for optimality under prescribed assumptions is considered so as the sufficient condition for optimality as follows:

Theorem 5.1: Necessary conditions for optimality:

a) with assumptions (A),(B) , (C) and with $\vec{W}$ is convex, if the control $\vec{u} \in \vec{W}_A$ is optimal, then there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0$, $\lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$ such that the following Kuhn-Tucker-Lagrange (K.T.L.) conditions are satisfied:

$$\int_\Omega H_{\vec{u}}.\overrightarrow{\delta u}\, dx \geq 0 \ , \ \forall \vec{w} \in \vec{W} \ , \ \overrightarrow{\delta u} = \vec{w} - \vec{u} \quad (33a)$$

where $g_j = \sum_{l=0}^{2} \lambda_l g_{lj}$ and $z_j = \sum_{l=0}^{2} \lambda_l z_{lj}$ $(j = 1,2)$ in the definition of H, and also

$$\lambda_2 G_2(\vec{u}) = 0 \quad (33b)$$

b) (Minimum principle in weak form): If $\vec{W} = \vec{W}_{\vec{U}}$ then inequality (33a) is equivalent to the minimum principle in point wise: $H_{\vec{u}}{}^T.\vec{u} = \min_{\vec{v} \in \vec{U}} H_{\vec{u}}{}^T.\vec{v}$, a.e. on Ω

Proof: a)From Theorem(4.2)we get that the functional $G_l(\vec{u})$ has a continuous Fréchet derivative at each $\vec{u} \in \vec{W}$, since the control $\vec{u} \in \vec{W}_A$ is optimal, then using the K.T.L. theorem there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$

with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$ such that

$$\left(\lambda_0 \vec{G}_{0\vec{u}}(\vec{u}) + \lambda_1 \vec{G}_{1\vec{u}}(\vec{u}) + \lambda_2 \vec{G}_{2\vec{u}}(\vec{u})\right).(\vec{w} - \vec{u}) \geq 0 \quad , \forall \vec{w} \in \vec{W}$$

and $\lambda_2 G_2(\vec{u}) = 0$

Substituting the Fréchet derivatives of $G_l(\vec{u})$ in the above inequality we get:

$$\int_\Omega (z_1 f_{1u_1} + g_{1u_1}, z_2 h_{1u_2} + g_{2u_2}).\overrightarrow{\delta u}\, dx \geq 0 \Rrightarrow \int_\Omega H_{\vec{u}}^T.\overrightarrow{\delta u}\, dx \geq 0 \ , \ \forall \vec{w} \in \vec{W} \ , \ \overrightarrow{\delta u} = \vec{w} - \vec{u}$$

where $z_j = \sum_{l=0}^{2} \lambda_l z_{lj}$ and $g_{ju_j} = \sum_{l=0}^{2} \lambda_l g_{lju_j}$

b) Let $\{\vec{u}_k\}$ be a dense sequence in $\vec{W}$, and let $D \subset \Omega$ be a measurable set such that

$$\vec{w}(x) = \begin{cases} \vec{w}_k(x), & if\ x \in D \\ \vec{u}(x), & if\ x \notin D \end{cases}$$

Hence (33a) becomes

$\int_D H_{\vec{u}}^T.(\vec{w}_k - \vec{u})\, dx \geq 0$, for each measurable set $D \Rrightarrow H_{\vec{u}}^T.(\vec{w}_k - \vec{u}) \geq 0$, a.e. on Ω

the above inequality holds in a set $Q = \cap_k Q_k$ where $Q_k = \Omega - \Omega_k$, with $\mu(\Omega_k) = 0$. But $Q$is independent on k with $\mu\left(\Omega/_Q\right) = 0$ and since $\{\vec{u}_k\}$ is dense sequence in $\vec{W}$ then the above inequality becomes

$H_{\vec{u}}^T.(\vec{w} - \vec{u}) \geq 0$, a.e. on $\Omega \Rrightarrow H_{\vec{u}}^T.\vec{u} = \min_{\vec{v} \in \vec{U}} H_{\vec{u}}^T.\vec{v}$, a.e. on Ω

The converse is clear.

Theorem 5.2: (Sufficient conditions for optimality)

In addition to assumptions (A), (B) and (C), assume $\vec{W}$ is convex, f_1, g_{11} are affine w.r.t. (y_1, u_1), h_1, g_{12} are affine w.r.t. (y_2, u_2), f_2, h_2 are affine w.r.t. u_1 and u_2 respectively for each x ,and g_{l1}, g_{l2} $l =$ 0,2 are convex w.r.t. (y_1, u_1) and (y_2, u_2) respectively for each x. Then the necessary conditions in Theorem (5.1) with $\lambda_0 > 0$ are also sufficient.

Proof: Assume $\vec{u} \in \vec{W}_A$ and $\vec{u}$ is satisfied the K.T.L. and the Transversality conditions, i.e.

$$\int_\Omega H_{\vec{u}}(x, y_1, u_1, y_2, u_2)\overrightarrow{\delta u}\, dx \geq 0, \ \forall \vec{w} \in \vec{W} \quad \text{and} \quad \lambda_2 G_2(\vec{u}) = 0$$

Let $G(\vec{u}) = \sum_{l=0}^{2} \lambda_l G_l(\vec{u})$, then $\vec{G}'(\vec{u})\overrightarrow{\delta u} = \sum_{l=0}^{2} \lambda_l \vec{G}_l'(\vec{u})\overrightarrow{\delta u} \Longrightarrow$

$$\vec{G}'(\vec{u})\overrightarrow{\delta u} = \sum_{l=0}^{2} \lambda_l \int_\Omega [(z_{1l}(f_{2u_1} - f_{1u_1}) + g_{1lu_1})\delta u_1 + (z_{2l}(h_{2u_2} - h_{1u_2}) + g_{2lu_2})\delta u_2]\, dx$$
$$= \int_\Omega H_{\vec{u}}(x, \vec{y}, \vec{u})\overrightarrow{\delta u}\, dx \geq 0$$

Since $f_1(x, y_1, u_1) = f_{11}(x)y_1 + f_{12}(x)u_1 + f_{13}(x)$, $f_2(x, u_1) = f_{21}(x)u_1 + f_{22}(x)$,

$h_1(x, y_2, u_2) = h_{11}(x)y_2 + h_{12}(x)u_2 + h_{13}(x)$, and $h_2(x, u_2) = h_{21}(x)u_2 + h_{22}(x)$

Let $\vec{u} = (u_1, u_2)$ and $\bar{\vec{u}} = (\bar{u}_1, \bar{u}_2)$ are given controls then $y_1 = y_{1u_1}$, $y_2 = y_{2u_2}$, $\bar{y}_1 = \bar{y}_{1\bar{u}_1}$ and $\bar{y}_2 = \bar{y}_{2\bar{u}_2}$ are their corresponding solutions , substituting the pair $(\vec{u}, \vec{y})$ in (1-4) and multiplying all the obtained equations by $\alpha \in [0,1]$ once and then substituting the pair $(\bar{\vec{u}}, \bar{\vec{y}})$ in (1-4) and multiplying all the obtained equations by $(1 - \alpha)$, finally adding each pair from the corresponding equations together one gets:

$$-\Delta(\alpha y_1 + (1 - \alpha)\bar{y}_1) + (\alpha y_1 + (1 - \alpha)\bar{y}_1) - (\alpha y_2 + (1 - \alpha)\bar{y}_2) + f_{11}(x)(\alpha y_1 + (1 - \alpha)\bar{y}_1)$$
$$+ f_{12}(x)(\alpha u_1 + (1 - \alpha)\bar{u}_1) + f_{13}(x) = f_{21}(x)(\alpha u_1 + (1 - \alpha)\bar{u}_1) + f_{22}(x) \qquad (34a)$$
$$\alpha y_1 + (1 - \alpha)\bar{y}_1 = 0 \qquad (34b)$$

and

$$-\Delta(\alpha y_2 + (1 - \alpha)\bar{y}_2) + (\alpha y_2 + (1 - \alpha)\bar{y}_2) + (\alpha y_1 + (1 - \alpha)\bar{y}_1) + h_{11}(x)(\alpha y_2 + (1 - \alpha)\bar{y}_2)$$
$$+ h_{12}(x)(\alpha u_2 + (1 - \alpha)\bar{u}_2) + h_{13}(x) = h_{21}(x)(\alpha u_2 + (1 - \alpha)\bar{u}_2) + h_{22}(x) \qquad (35a)$$
$$\alpha y_2 + (1 - \alpha)\bar{y}_2 = 0 \qquad (35b)$$

Now, if we have the control vector $\bar{\bar{\vec{u}}} = (\bar{\bar{u}}_1, \bar{\bar{u}}_2)$ with

$\bar{\bar{u}}_1 = \alpha u_1 + (1 - \alpha)\bar{u}_1$ and $\bar{\bar{u}}_2 = \alpha u_2 + (1 - \alpha)\bar{u}_2$

Then from (34a&b) and (35a&b), we get that $\bar{\bar{\vec{y}}} = \bar{\bar{\vec{y}}}_{\bar{\bar{\vec{u}}}} = (\bar{\bar{y}}_1, \bar{\bar{y}}_2)$ with

$\bar{\bar{y}}_1 = y_{1\bar{\bar{u}}_1} = y_{1(\alpha u_1+(1-\alpha)\bar{u}_1)} = \alpha y_1 + (1-\alpha)\bar{y}_1$ and $\bar{\bar{y}}_2 = y_{2(\alpha u_2+(1-\alpha)\bar{u}_2)} = \alpha y_2 + (1-\alpha)\bar{y}_2$

are there corresponding solution, i.e. are satisfied (1-4) respectively. So we get the operators $u_1 \mapsto y_{1u_1}$ and $u_2 \mapsto y_{2u_2}$ are convex- linear w.r.t. (y_1, u_1) and (y_2, u_2) respectively, for each $x \in \Omega$.

Now, since $g_{11}(x, y_1, u_1)$ and $g_{12}(x, y_2, u_2)$ are affine w.r.t. (y_1, u_1) and (y_2, u_2) respectively, for each $x \in \Omega$, and from the convex –linear property of the above two operators we get that $G_1(\vec{u})$ is convex-linear w.r.t. $(\vec{y}, \vec{u})$, $\forall x \in \Omega$.

The convexity of $G_l(\vec{u})$ (for , l =0,2) w.r.t. $(\vec{y}, \vec{u})$, $\forall x \in \Omega$ is obtained form the assumptions of g_{l1} and g_{l2}, l =0,2 are convex w.r.t. (y_1, u_1) and (y_2, u_2) respectively, $\forall x \in \Omega$.

Hence $G(\vec{u})$ is convex w.r.t. $(\vec{y}, \vec{u})$, in the convex set $\vec{W} = \vec{W}_{\vec{U}}$ and has a continuous Fréchet derivative satisfies $\vec{G}'(\vec{u})\delta\vec{u} \geq 0 \Rightarrow G(\vec{u})$ has a minimum at $\vec{u} \Rightarrow G(\vec{u}) \leq G(\vec{w})$, $\forall \vec{w} \in \vec{W} \Rightarrow$

$$\lambda_0 G_0(\vec{u}) + \lambda_1 G_1(\vec{u}) + \lambda_2 G_2(\vec{u}) \leq \lambda_0 G_0(\vec{w}) + \lambda_1 G_1(\vec{w}) + \lambda_2 G_2(\vec{w}) \quad (36)$$

Now, let $\vec{w}$ be an admissible control and since $\vec{u}$ is also admissible and satisfies the Transversality condition then (36) becomes $G_0(\vec{u}) \leq G_0(\vec{w})$, $\forall \vec{w} \in \vec{W}$ i.e. $\vec{u}$ is an optimal control for the problem.

6.Conclusions

The Minty–Browder theorem can be used successfully to prove the existence and the uniqueness solution of the continuous state vector of a couple nonlinear elliptic partial differential equations for fixed continuous classical control vector. The existence theorem of a continuous classical optimal control vector governing by the considered couple of nonlinear partial differential equation of elliptic type with equality and inequality constraints is developed and proved. The existence and the uniqueness solution of the couple of adjoint equations associated with the considered couple equations of the state is studded. The Frcéhet derivation of the Hamiltonian is derived. The necessary conditions theorem so as the sufficient conditions theorem of optimality of the constrained problem are developed and proved.

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