The Influence of c-normality Subgroups on the Structure of Finite Groups

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Abstract. A subgroup H of a group G is c-normal in G if there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$, where $H_G = Core(H) = \cap_{g \in G} H^g$ is the maximal normal subgroup of G which is contained in H. We study the structure of a finite group G under the assumption that certain subgroups of prime power order of nilpotent residual of G are c-normal in G.

Key Words: c-Normal subgroup; Nilpotent residual; Solvable group; Supersolvable group.

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1. INTRODUCTION

In this paper only finite groups are considered. A subgroup of a group G which permutes with each subgroup of G is called a quasinormal subgroup of G. We say, following Kegel [7], that a subgroup of G is S-quasinormal in G if it permutes with each Sylow subgroup of G. Agrawal [1], defined the generalized center $genz(G)$ of a group G is the subgroup generated by all elements g of G such that $< g >$ is S-quasinormal in G. The generalized

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hypercenter $genz_\infty(G)$, is the largest term of the chain $1 = genz_0(G) \leq genz_1(G) = genz(G) \leq genz_2(G) \leq \ldots$, where $genz_{i+1}(G)/genz_i(G) = genz(G/genz_i(G))$ for all $i > 0$.

A 2-group is called quaternion-free if it has no section isomorphic to the quaternion group of order 8. If P is a p-group, we denote $\Omega(P) = \Omega_1(P)$ if $p > 2$ and $\Omega(P) = \langle \Omega_1(P),\ \Omega_2(P) \rangle$ if $p = 2$, where $\Omega_i(P) = \langle x \in P \mid |x| = p^i \rangle$. In addition, we define $D(G) = \cap\{H \mid H \quad G$ and G/H is nilpotent$\}$ and call it the nilpotent residual of G.

Following Wang [8], we say that a subgroup H of a group G is c-normal in G if there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$, where $H_G = Core(H) = \cap_{g \in G} H^g$ is the maximal normal subgroup of G which is contained in H. Furthermore, he proved that a group G is supersolvable if the subgroups of G of prime order or order 4 are c-normal in G. Asaad and Ezzat Mohamed [4], proved that if P is a Sylow p-subgroup of a group G, where p is the smallest prime dividing the order of G and the subgroups of P of order p or order 4 are c-normal in G, then G is p-nilpotent.

The present paper represents an attempt to extend and improve the above mentioned results.

2. PRELIMINARIES

In this section, we collect some results that are needed in the sequel.

Lemma 2.1. If $H \leq K \leq G$ and H is c-normal in G, then H is c-normal in K.

Proof. See [8; Lemma 2.1].

Lemma 2.2. **(i)** Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G . If $\Omega(P) \leq genz_\infty(G)$, then G is p-nilpotent.
(ii) If $\Omega(P) \leq genz_\infty(G)$, for each Sylow subgroup P of a group G, then G is supersolvable.
(iii) Let P be a normal p-subgroup of G, for some prime p. If G/P is supersolvable and $\Omega(P) \leq genz_\infty(G)$, then G is supersolvable.
(iv) If $\Omega(P) \leq genz_\infty(G)$ for each Sylow subgroup P of G with $(|P|, 2) = 1$, then G is solvable.

Proof. See [3].

Lemma 2.3. Let P be a Sylow 2-subgroup of a group G. If P is quaternion-free and $\Omega_1(P) \leq Z(G)$, then G is 2-nilpotent.

Proof. See Asaad [2; Corollary 1].

Lemma 2.4. If G is a minimal non-supersolvable group, then
(i) G has exactly one normal Sylow q-subgroup Q of exponent q (if $q > 2$) or 4 (if $q = 2$).
(ii) $Q/\Phi(Q)$ is a minimal normal subgroup of $G/\Phi(Q)$.

Proof. See Paper [6; VI, aufgabe 16).

Lemma 2.5. Let P be a normal p-subgroup of G and Q be a q-subgroup of G such that $q \neq p$. If Q is c-normal in G, then QP/P is c-normal in G/Q.

Proof. See [9; Lemma 2.4].

3. MAIN RESULTS

We begin to prove the following result:

Theorem 3.1. Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G . Suppose either
(i) P is quaternion-free and each minimal subgroup of $D(G) \cap P$ is c-normal in G; or
(ii) each maximal subgroup of $D(G) \cap P$ is c-normal in G.
Then G is p-nilpotent.

Proof. Suppose the result is false and let G be a counterexample of minimal order.

First, suppose that P is quaternion-free and each minimal subgroup of $D(G) \cap P$ is c-normal in G. Since G is a counterexample of minimal order, it follows that G is not p-nilpotent and so G contains a minimal non-p-nilpotent subgroup, K, say. By [6; IV, Satz 5.2], $|K| = p^n q^m$ for a prime $q \neq p$, K has a normal Sylow p-subgroup K_p of exponent p at $p > 2$ or at most 4 at $p = 2$ and a non-normal cyclic Sylow q-subgroup K_q. Without loss of generality, we can assume that $K_p \leq P$. Clearly, $D(K) = K_p$. Then $D(K) \cap K_p = K_p$. By hypothesis, each minimal subgroup of K_p is c-normal in G. Then by Lemma 2.1, each minimal subgroup of K_p is c-normal in K. Suppose that the exponent of K_p is p. Then $\Omega_1(K_p) = K_p$. If each minimal subgroup of K_p is a normal subgroup of K, then $\Omega_1(K_p) = K_p \leq genz_\infty(K)$. Hence K is p-nilpotent, by Lemma 2.2(i); a contradiction. Thus, there exists a subgroup H of K_p is not normal of K. By hypothesis and Lemma 2.1, H is c-normal in K. Then there exists a normal subgroup N of K such that $K = HN$ with $H \cap N \leq H_K$. Since H is not normal subgroup of K, we have that $N < K$. By the minimality of K, N is p-nilpotent and so also does K; a contradiction. Now, we may assume that the exponent of K_p is 4 and hence $p = 2$. So by [6; III, Satz 5.2], $K_2 = Z(K_2) = \Phi(K_2)$, K_2 is elementary abelian and K_2/K_2 is a chief factor of K. Then $\Omega_1(K_2) = K_2 \leq Z(K)$. By applying Lemma 2.3, we conclude that K is 2-nilpotent; contradiction.

Now suppose that each maximal subgroup of $D(G) \cap P$ is c-normal in G. Then we treat with the following two cases:

Case 1. $O_p\,(D(G)) \neq 1$.

Since $O_p\,(D(G))$ char $D(G)$ and $D(G)$ is a normal subgroup of G, we have that $O_p\,(D(G))$ is a normal subgroup of G. Consider the factor group $G/O_p\,(D(G))$. Put $D(G/O_p\,(D(G)) = L/O_p\,(D(G))$. Since

$$(G/O_p\,(D(G)))/(L/O_p\,(D(G))) \cong G/L$$

is nilpotent, we have that $D(G) \leq L$. Also, since

$$(G/O_p\,(D(G)))/D(G)/O_p\,(D(G))) \cong G/D(G)$$

is nilpotent, we have that $L/O_p\,(D(G)) \leq D(G)/O_p\,(D(G))$, i. e, $L \leq D(G)$. Then $D(G/O_p\,(D(G))) = L/O_p\,(D(G)) = D(G)/O_p\,(D(G))$. By hypothesis and Lemma 2.5, each maximal subgroup of

$$D(G/O_p\,(D(G)) \cap PO_p\,(D(G))/O_p\,(D(G)) =$$
$$D(G) \cap PO_p\,(D(G))/O_p\,(D(G)) = (D(G) \cap P)O_p\,(D(G))/O_p\,(D(G))$$

is c-normal in $G/O_p\,(D(G))$. Then $G/O_p\,(D(G))$ is p-nilpotent by the minimality of G, and so also does G ; a contradiction.

Case 2. $O_p\,(D(G)) = 1$.

Let M be a maximal subgroup of $D(G) \cap P$. By hypothesis, M is c-normal in G. Then by Lemma 2.1, M is c-normal in $D(G)$. Thus there exists a normal subgroup K of $D(G)$ such that $D(G) = MK$ with $M \cap K \leq M_{D(G)}$.

Since $M_{D(G)}$ char $D(G)$ and $D(G)$ is a normal subgroup of G, we have that $M_{D(G)}$ is a normal subgroup of G. Clearly, $D(G) \cap P \cap K$ is a Sylow p-subgroup of K. If $M_{D(G)} = 1$, then $D(G) \cap P \cap K$ is cyclic of order p. By Bernside's Theorem, K is p-nilpotent. If K is not p-group, then $O_p(D(G)) \neq 1$; a contraduction. Thus we may assume that K is p-group. Hence $D(G)$ is a p-group. Clearly each maximal subgroup M of $D(G)$ is normal of $D(G)$ and since $M_{D(G)} = 1$, we have that $D(G)$ is cyclic of order p. Then G is supersolvable and so it is p-nilpotent; a contradiction. Thus $M_{D(G)} \neq 1$, for each maximal subgroup M of $D(G) \cap P$. Let N be a minimal normal subgroup of G contained in M. Clearly, our hypothesis carries over to G/N. Then G/N is p-nilpotent by the minimality of G. Since the class of all p-nilpotent group is a saturated formation, we have that N is a unique minimal normal subgroup of G contained in M, for each maximal subgroup M of $D(G) \cap P$, such that G/N is p-nilpotent. Hence $N \leq \Phi(D(G) \cap P)$. By [5; Theorem 9.2(d)], $N \leq \Phi(D(G))$ and since $D(G)$ is a normal subgroup of G, we have that $N \leq \Phi(G)$. Then G is p-nilpotent as the class of all p-nilpotent group is a saturated formation; a final contradiction.

The following example shows that the hypothesis "P is quaternion-free" is necessary in Theorem 3.1.

Example. Set $G = SL(2,3)$. Then the Sylow 2-subgroup P of G is the quaternion group of order 8 and $D(G) = P$. Clearly, each minimal subgroup of $D(G) \cap P$ is normal (c-normal) subgroup of G but G is not p-nilpotent.

As an immediate consequence of Theorem 3.1, we have:

Theorem 3.2. If each minimal subgroup of $D(G) \cap P$ is c-normal in G for each Sylow subgroup P of G, then G is supersolvable or G has a section isomorphic to the quaternion group of order 8.

Proof. If G has a section isomorphic to the quaternion group of order 8, then the result holds. Thus we may assume that G has no section isomorphic to the quaternion group of order 8. If G is supersolvable, then the result is hold. Thus we may assume that G is not supersolvable. Let M be a maximal subgroup of G. Since $MD(G)/D(G) \cong M/M \cap D(G)$ is nilpotent, we have that $D(M) \leq D(G)$. By hypothesis and Lemma 2.1, our hypothesis carries over to M. Then M is supersolvable, by the induction on the order of G. Thus each proper subgroup of G is supersolvable. Hence G is a minimal non-supersolvable group. By Lemma 2.4, G has exactly one Sylow q-subgroup Q

of G is normal of G of exponent q (if $q > 2$) or 4 (if $q = 2$) and $Q/\Phi(Q)$ is a minimal normal subgroup of $G/\Phi(Q)$. By Theorem 3.1, G is r-nilpotent, where r is the smallest prime dividing the order of G. Then $G = RK$, where R is a Sylow r-subgroup of G and K is a normal Hall r'-subgroup of G. Then K is supersolvable, by the minimality of G. Hence K possesses an ordered Sylow tower and so also does G. Hence G has a normal Sylow s-subgroup S, where s is the largest prime dividing the order of G. Since G has exactly one Sylow subgroup Q of G is normal of G, we have that $S = Q$ and $s = q$. So the exponent of Q is q. By Schour Zassenhous's theorem, $G/Q \cong L$, where L is a q -Hall subgroup of G. Since G is a minimal non-supersolvable group, we have that $G/Q \cong L$ is supersolvable. Clearly, $G/D(G)\cap Q$ is supersolvable. If each minimal subgroup of $D(G)\cap Q$ is normal subgroup of G, then $D(G) \cap Q \leq genz_\infty(G)$. Hence by Lemma 2.2(ii); G is supersolvable; a contradiction. Thus, we may assume that there exists a subgroup H of $D(G) \cap Q$ of order q is not normal of G. By hypothesis, H is c-normal in G. Then there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$. Since H is not normal subgroup of G, we have that $N < G$. Clearly, $Q \cap N$ is a maximal subgroup of Q and so $(Q \cap N)/\Phi(Q)$ is a normal subgroup of G/Q. Since $Q/\Phi(Q)$ is a minimal normal subgroup of $G/\Phi(Q)$, we have that $Q \cap N = Q$ or $\Phi(Q)$. If $Q \cap N = Q$, then $N = G$; a contradiction. Thus $Q \cap N = \Phi(Q)$. Since G/Q and G/N are supersolvable, we have that $G/\Phi(Q) = G/Q\cap N$ is supersolvable and since Q is a normal subgroup of G, we have by [5; Theorem 9.2(e)], that $Q \cap N = \Phi(Q) \leq \Phi(G)$. Then $(G/\Phi(Q))/(\Phi(G)/\Phi(Q)) \cong G/\Phi(G)$ is supersolvable. Hence G is supersolvable as the class of all supersolvable groups is saturated formation; a final contradiction.

Now we prove:

Theorem 3.3. If each minimal subgroup of $D(G) \cap P$ is c-normal in G for each Sylow subgroup P of G with $(|P|, 2) = 1$, then G is solvable.

Proof. Suppose that the result is false and let G be a counterexample of minimal order. Let G^S be a solvable residual of G, that is, G^S is the smallest normal subgroup of G such that G/G^S is solvable. Since $G/D(G)$ is nilpotent, we have that $G^S \leq D(G)$. Clearly $G^S \cap P$ is a Sylow subgroup of G^S for each Sylow subgroup P of G. If each minimal subgroup of $G^S \cap P$ is normal subgroup of G^S, then $\Omega_1(G^S \cap P) \leq genz_\infty(G^S)$ for each Sylow subgroup $G^S \cap P$ of G^S with $(\left|G^S \cap P\right|, 2) = 1$. Hence G^S is solvable by

Lemma 2.2(iv) and so G is solvable; a contradiction. Thus we may assume that there exists a subgroup H of prime order ($\neq 2$) of G^S is not normal of G^S. By hypothesis H is c-normal in G. Then there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$. Since H is not normal subgroup of G and $|H| = prime$, we have that $H_G = 1$ and $N < G$. Then $G/N \cong H$ is solvable. Hence $G^S \leq N$ and so $N = G$; a final contradiction.

Asaad [2] proved the following result:

Theorem [2; Theorem 6]. Let P be a Sylow p-subgroup of a group G. Assume that $\Omega_1(D(G) \cap P \cap P^x) \leq Z(P)$ for all $x \in G - N_G(P)$. If P is quaternion-free and $N_G(P)$ is p-nilpotent, then G is p-nilpotent.

As an immediate consequence of Theorem 3.1 and Asaad's result we have:

Corollary 3.4. Let p be the smallest prime dividing the order of G and let P be a Sylow p-subgroup of G. If P is quaternion-free, $\Omega_1(D(G) \cap P \cap P^x) \leq Z(P)$ for all $x \in G - N_G(P)$ and each minimal subgroup of $D(N_G(P)) \cap P$ is c-normal in $N_G(P)$, then G is p-nilpotent.

Proof. Since each minimal subgroup of $D(N_G(P)) \cap P$ is c-normal in $N_G(P)$ and since P is quaternion-free, we have that $N_G(P)$ is p-nilpotent by Theorem 3.1. Then G is p-nilpotent by Asaad's result.

Now we can prove:

Theorem 3.5. If, for each Sylow subgroup P of G, each minimal subgroup of $D(N_G(P)) \cap P$ is c-normal in $N_G(P)$ and $\Omega_1(D(G) \cap P \cap P^x) \leq Z(P)$ for all $x \in G - N_G(P)$. Then G is supersolvable or G has a section isomorphic to the quaternion group of order 8.

Proof. If G has a section isomorphic to the quaternion group of order 8, then the result holds. Thus we may assume that G has no section isomorphic to the quaternion grou of order 8. Corollary 3.4, implies that G is r-nilpotent, where r is the smallest prime dividing the order of G. Then $G = RK$, where R is a Sylow r-subgroup of G and K is a normal Hall r'-subgroup of G. Clearly our hypothesis carries over K. Then K is supersolvable by the induction on the order of G. Hence K possesses an ordered Sylow tower and so Q is a normal Sylow q-subgroup of K, where q is the largest prime dividing the order of K. Clearly, $q > 2$. Since Q char K and K is a normal subgroup of G, we have that Q is a normal subgroup of G. Now we consider the factor group G/Q. By hypothesis, $\Omega_1(D(G/Q) \cap PQ/Q \cap P^xQ/Q) \leq \Omega_1(D(G) \cap P \cap P^x)Q/Q \leq Z(P)Q/Q \leq Z(PQ/Q)$ for all $x \in G - N_G(P)$ and since each minimal subgroup of $D(N_G(P)) \cap P$ is c-normal in $N_G(P)$, we have by Lemma 2.5, that each minimal subgroup of $D(N_{G/Q}(PQ/Q)) \cap PQ/Q = (D(N_G(P)) \cap P)Q/Q$ is c-normal in $N_G(PQ/Q)$, for each Sylow subgroup P of G, with $(|P|, |Q|) = 1$. Then G/Q is supersolvable by the induction on the order of G. Clearly, $G/D(G) \cap Q$ is supersolvable. Then $G^{\mathsf{U}} \leq D(G) \cap Q$, where G^{U} is the supersolvable residual subgroup of G, that is, G^{U} is the smallest normal subgroup of G such that G/G^{U} is supersolvable. If each minimal subgroup of G^{U} is normal of G, then $\Omega_1(G^{\mathsf{U}}) \leq genz_\infty(G)$. Hence G is supersolvable, by Lemma 2.2(iii). Thus, we may assume that there exists a subgroup H of G^{U} of order q is not normal subgroup of G. Since $G^{\mathsf{U}} \leq D(G) \cap Q$ and $G = N_G(Q)$, we have by hypothesis, that H is c-normal in $G = N_G(Q)$. Then there exists a normal subgroup N of G such that $G = HN$ with $H \cap N \leq H_G$. Since H is not normal subgroup of G and $|H| = q$, we have that $H_G = 1$ and $N < G$. Then $G/N \cong H$ is supersolvable. Hence $G^{\mathsf{U}} \leq N$ and so $N = G$; a contradiction.

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