

Numerical solution for the time-Fractional Diffusion-wave Equations by using Sinc-Legendre Collocation Method

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Abstract

In this paper the numerical solution of fractional diffusion wave equation is proposed. The fractional derivative will be in the Caputo sense. The proposed method will be based on shifted Legendre collocation scheme and sinc function approximation for time and space respectively. The problem is reduced to the problem into a system of algebraic equations after implementing this method. For demonstrating the validity and applicability of the proposed numerical scheme some examples are presented.

Keywords*:* Fractional diffusion equation, Sinc functions, shifted Legendre polynomials, Collocation method.

1.Introduction

Many phenomena in engineering physics, chemistry, and other sciences can be described very successfully by models using mathematical tools from fractional calculus, i.e. the theory of derivatives and integrals of fractional non-integer order [Oldham & Spanier(1974); Gorenflo & Mainardi(1997)]. Fractional differential equations are generalized from classical integer-order ones, which are obtained by replacing integer-order derivatives by fractional ones. Their advantages comparing with integer-order differential equations are the capability of simulating natural physical process and dynamic system more accurately [Chen(2007)].

The analytic results on existence and uniqueness of solutions to fractional differential equations have been investigated by many authors, see, say [Kilbas & Trujillo (2006)]. Most fractional differential equations do not have closed form solutions, so approximation and numerical techniques such as Laplace transform method [Podlubny(1999)], operational method [Luchko & Gorenflo(1999)], finite difference methods [Su & Xu(2010)], differential transform method [Odibat & Momani(2008)], wavelet method [Chen & Jin(2011)], Adomian's decomposition method [Ray(2009)], variational iteration method [Inc(2008)], homotopy analysis method [Dehghan & Saadatmandi(2010)], homotopy perturbation method [Momani & Odibat(2007)], tau method [Saadatmandi & Dehghan(2011)] and other methods [Podlubny & Chen(2009); Saadatmandi & Dehghan(2010)], must be used.

Fractional partial differential equations can be classified into two principal kinds: space-fractional differential equation and time-fractional one.

In the present paper, we shall consider the time-fractional diffusion- wave equation with variable coefficients:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = a(x,t)\frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad a < x < b, \quad 0 < t < \tau, \qquad \ldots(1)$$

with initial conditions

$$u(x,0) = \Psi(x), \qquad \frac{\partial u(x,0)}{\partial t} = \varphi(x) \quad a < x < b \qquad \ldots(2)$$

and boundary conditions

$$u(a,t) = u(b,t) = 0, \quad 0 < t < \tau \qquad \ldots(3)$$

Where $x \in [a,b]$ and $t \in (0,\tau]$ are space and time variables, respectively, the time fractional derivative is defined in the Caputo sense. $a(x,t)$ be a continuous function and $f(x,t)$, denotes the field variable where $a \leq x \leq b$ and $0 < \alpha \leq 2$. For $1 < \alpha < 2$, the fractional equ. (1) is known as the fractional diffusion-wave equation which fills the gaps between the diffusion equation and wave equation[Sun & Wu(2006)].

In this paper we develop a sinc-Legendre collocation method to solve numerically problem (1) - (3). Since a fractional derivative is a nonlocal operator, it is natural to consider a global scheme such as the collocation method for its numerical solution. The required approximate solution is expanded as a series with the elements of shifted Legendre polynomials in time and sinc functions in space with unknown coefficients. By utilizing the collocation technique and some properties of the shifted Legendre polynomials and sinc functions, the problem is reduced to the solution to a system of linear algebraic equations. And a matrix representation of the system is obtained to calculate the solution.

2.Fractional Derivative and Integration

In this section, we shall review the basic definitions and properties of fractional integral and derivatives, which are used further in this paper[Kilbas & Trujillo (2006)].

Definition(1):- The Riemann-Liouville fractional integral operator of order $v > 0$, is defined as

$$I^v f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt, \quad v > 0, x > 0. \qquad \ldots(4)$$

$$I^0 f(x) = f(x)$$

Definition(2):- The Riemann-Liouville fractional derivative operator of order $v > 0$,is defined as

$$_0D_x^v f(x) = \frac{1}{\Gamma(n-v)} \frac{d^n}{dx^n} \int_0^x (x-t)^{n-v-1} f(t)dt, \quad v > 0, x > 0. \qquad \ldots(5)$$

Where n is an integer and $n-1 < v \leq n$.

Definition(3):- The Caputo fractional derivative operator of order $v > 0$, is defined as

$$^cD_x^v f(x) = \frac{1}{\Gamma(n-v)} \int_0^x (x-t)^{n-v-1} \frac{d^n}{dx^n} f(t)dt, \quad v > 0, x > 0 \qquad \ldots(6)$$

Where n is an integer and $n-1 < v \leq n$.

Caputo fractional derivative has an useful property:

$$I^v\ ^cD_x^v f(x) = f(x) - \sum_{k=0}^{n-1} f^{(k)}(0^+) \frac{x^k}{k!} \qquad \ldots(7)$$

Where n is an integer and $n-1 < v \leq n$.

Also, for the Caputo fractional derivative we have

$$^cD_x^v x^\beta = \begin{cases} 0 & \text{for } \beta \in N_0 \text{ and } \beta < \lceil v \rceil \\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-v)} x^{\beta-v}, & \text{for } \beta \in N_0 \text{ and } \beta \geq \lceil v \rceil \text{ or } \beta \notin N \text{ and } \beta > \lfloor v \rfloor. \end{cases} \qquad \ldots(8)$$

We use the ceiling function $\lceil v \rceil$ to denote the smallest integer greater than or equal to v, and the floor function $\lfloor v \rfloor$ to denote the largest integer less than or equal to v. Also $N = \{1,2,\ldots\}$ and $N_0 = \{0,1,2,\ldots\}$.

Recall that for $v = 0$, the Caputo differential operator concides with the usual differential operator of an integer order. Similar to the integer-order differentiation, the Caputo fractional differentiation is a linear operator; i.e.

$$^cD_x^v\big(\lambda f(x) + \mu g(x)\big) = \lambda\ ^cD_x^v f(x) + \mu\ ^cD_x^v g(x) \qquad \ldots (9)$$

Where λ and μ are constants.

3.Sinc functions

Sinc function prosperities are discussed thoroughly in [Stenger(1993)]. In this section an overview of the basic formulation of the Sinc function required for subsequent application is presented.

The sinc function is defined on the whole real by

$$\mathrm{Sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0 \\ 1 & , x = 0 \end{cases} \quad \ldots(10)$$

For each integer k and the mesh size $h > 0$, the sinc basis functions are defined on R by

$$S_k(h,x) \equiv \mathrm{Sinc}\left(\frac{x-kh}{h}\right) = \begin{cases} \frac{\sin\left(\frac{\pi}{h}(x-kh)\right)}{\frac{\pi}{h}(x-kh)}, & x \neq kh, \\ 1 & , x = kh. \end{cases} \quad \ldots(11)$$

If a function f(x) is defined on $\mathcal{R}$, then for h > 0 the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\mathrm{Sinc}\left(\frac{x-kh}{h}\right), \quad \ldots(12)$$

is called the Whittaker cardinal expansion of f whenever this series converges. The properties of the Whittaker cardinal expansion have been extensively studied in [Lund & Bowers(1992)]. These properties are derived in the infinite strip DS of the complex ω-plane, where for d > 0,

$$D_S = \left\{\omega = t + is: \ |S| < d \leq \frac{\pi}{2}\right\}, \quad \ldots(13)$$

To construct the approximation over the finite interval $[a,b]$, which is used in this paper, we consider the one-to-one conformal map

$$\omega = \phi(z) = \mathrm{In}\left(\frac{Z-a}{b-Z}\right),$$

which maps the eye-shaped region

$$D_E = \left\{z = x + iy: \ \left|\arg\left(\frac{Z-a}{b-Z}\right)\right| < d \leq \frac{\pi}{2}\right\}, \quad \ldots(14)$$

onto the infinite strip D_S. We also define the range of $\Psi = \phi^{-1}$ on the real line as

$$\Gamma = \{\Psi(t) \in D_E: -\infty < t < +\infty\} = (0, +\infty),$$

Thus we may define the inverse images of the real line and of the evenly spaced nodes $\{kh\}_{k=-\infty}^{k=+\infty}$ as

$$x_K = \Phi^{-1}(kh) = \frac{a+be^{kh}}{1+e^{kh}}, \qquad k = 0, \mp 1, \mp 2, \ldots \quad \ldots(15)$$

Hence the numerical process developed in the domain containing the whole real line can be carried over to infinite interval by the inverse map. The basis functions on (a,b) are taken to be the composite translated sinc functions,

$$S_k(x) \equiv S(k,h)\circ\phi(x) = \mathrm{Sinc}\left(\frac{\phi(x)-kh}{h}\right), \quad \ldots(16)$$

Where $S(k,h)\circ\phi(x)$ is defined by $S(k,h)\big(\phi(x)\big)$.

Definition(4):- Let $B(D_E)$ be the class of functions f which are analytic in D_E, satisfy

$$\int_{\Phi^{-1}(t+L)} |f(z)dZ| \to 0, \qquad t \to \pm\infty,$$

Where $L = \left\{iv: |v| < d \leq \frac{\pi}{2}\right\}$, and on the boundary of D_E, (denoted ∂D_E), satisfy

$$N(F) = \int_{\partial D_E} |f(z)dZ| < \infty.$$

Interpolation for function in $B(D_E)$ is defined in the following theorem which is proved in [Stenger(1993)].

Theorem(1) (Interpolation, see [Stenger(1993); Lund & Bowers(1992)]. If $f\phi' \in B(D_E)$ then for all $x \in \Gamma$

$$|f(x) - \sum_{k=-\infty}^{\infty} f(x_K)S_K(x)| \leq \frac{N(f\phi')}{2\pi d \sinh\left(\pi d/h\right)} \leq 2\frac{N(f\phi')}{\pi d} e^{-\pi d/h}.$$

Moreover, if $|f(x)| \leq Ce^{-\beta|\Phi(x)|}$, $x \in \Gamma$, for some positive constants C and β, and if the selection

$$h = \sqrt{\frac{\pi d}{\beta N}} \leq \frac{2\pi d}{\ln 2}, \quad \text{then} \quad \left|f(x) - \sum_{k=-N}^{N} f(x_K)S_K(x)\right| \leq C_2\sqrt{N}\exp\left(-\sqrt{\pi d\beta N}\right), \ x \in \Gamma,$$

Where C_2 depends only on f, d and β.
The above expressions show Sinc interpolation on $B(D_E)$ converges exponentially. We also require derivatives of composite Sinc functions evaluated at the nodes. The expressions required for the present discussion are[Stenger(1993)].

$$\delta_{k,j}^{(0)} = [S_K(x)]|_{x=x_j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \qquad \ldots (17)$$

$$\delta_{k,j}^{(1)} = \frac{d}{d\Phi}[S_K(x)]|_{x=x_j} = \begin{cases} 0, & k = j, \\ \frac{(-1)^{j-k}}{j-k}, & k \neq j, \end{cases} \qquad \ldots (18)$$

$$\delta_{k,j}^{(2)} = \frac{d^2}{d\Phi^2}[S_K(x)]|_{x=x_j} = \begin{cases} \frac{-\pi^2}{3}, & k = j, \\ \frac{-2(-1)^{j-k}}{(j-k)^2}, & k \neq j, \end{cases} \qquad \ldots (19)$$

4.The Shifted Legendre Polynomials

The well-known Legendre polynomials are defined on the interval $[-1,1]$ and can be determined with the aid of the following recurrence formulae [Saadatmandi & Dehghan(2011)] as

$$L_{i+1}(z) = \frac{2i+1}{i+1} zL_i(t) - \frac{i}{i+1} L_{i-1}(z), \qquad i = 1,2, \ldots, .$$

where $L_0(z) = 1$ and $L_1(z) = z$. We also define the so-called shifted Legendre polynomials on the interval $[0, L]$ by using the change of variable $z = \frac{2x}{L} - 1$. So Shifted Legendre polynomials $L_i(\frac{2x}{L} - 1)$ are denoted by $L_i^h(x)$. Shifted Legendre polynomials of x can be determined with the aid of the formula:

$$L_{i+1}^h(x) = \frac{(2i+1)(2x-h)}{(i+1)h} L_i^h(x) - \frac{i}{i+1} L_{i-1}^h(x), \quad i = 1,2, \ldots . \qquad \ldots (20)$$

where $L_0^h(x) = 1$ and $L_1^h(x) = \frac{(2x-1)}{h}$. The analytic form of the n-degree shifted Legendre polynomials given by

$$L_i^\tau(t) = \sum_{k=0}^{n} (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k!)^2\tau^k} t^k, \qquad i = 1,2, \ldots . \qquad \ldots (21)$$

Note that $L_i^\tau(0) = (-1)^i$ and $L_i^\tau(L) = 1$ for all integer i.

5. Sinc-Legendre method

In order to solve problem (1)–(3) we consider $u_{m,n}$ as an approximate solution of equ. (1) namely $u(x,t)$.

$$u_{m,n}(x,t) = \sum_{i=-m}^{m} \sum_{j=0}^{n} c_{ij} S_i(x) L_j^{\tau}(t). \qquad \ldots(22)$$

where $\lim_{x\to a} S_k(x) = \lim_{x\to b} S_k(x) = 0$.

It is noticed that the above conditions guarantee boundary conditions to be satisfied.

Caputo's fractional derivative of order $v > 0$ for the shifted Legendre polynomials $L_i^{\tau}(t)$

[Saadatmandi & Azizi(2012)] is given by

$${}^{c}D_0^{v} L_i^{\tau}(t) = \sum_{k=\lceil v\rceil}^{i} b_{i,k} t^{k-v}, \quad i = \lceil v\rceil, \lceil v\rceil + 1, \ldots, \qquad \ldots(23)$$

$${}^{c}D_0^{v} L_i^{\tau}(t) = 0, \quad i = 0,1,\ldots,\lceil v\rceil - 1, \quad v > 0 \qquad \ldots(24)$$

Where

$$b_{i,k} = (-1)^{i+k} \frac{(i+k)!}{(i-k)!(k)!\tau^k \Gamma(k-v+1)} \qquad \ldots(25)$$

<u>Lemma1</u>[Saadatmandi & Azizi(2012)] Let $1 < v < 2$ and x_k be spatial collocation points given in (15). Then the following relations hold:

$$\frac{\partial^v u_{m,n}(x_k,t)}{\partial t^v} = \sum_{j=1}^{n} \sum_{r=1}^{j} c_{kj} b_{jr} t^{r-v}, \qquad \ldots(26)$$

$$\frac{\partial u_{m,n}(x_k,t)}{\partial x} = \sum_{i=-m}^{m} \sum_{j=0}^{n} c_{ij} q_{ik}^{(1)} L_j^{\tau}(t), \qquad \ldots(27)$$

$$\frac{\partial^2 u_{m,n}(x_k,t)}{\partial x^2} = \sum_{i=-m}^{m} \sum_{r=0}^{n} c_{kj} q_{ik}^{(2)} L_j^{\tau}(t), \qquad \ldots(28)$$

Where $q_{ik}^{(1)} = \phi'(x_k)\delta_{i,k}^{(1)}$, and $q_{ik}^{(2)} = \phi''(x_k)\delta_{i,k}^{(1)} + (\phi'(x_k))^2 \delta_{i,k}^{(2)}$,

And $b_{jr} = (-1)^{j+r} \frac{(j+r)!}{(j-r)!(r)!\Gamma(r-v+1)\tau^r}$,

for the proof of equations (26)-(28) see [Saadatmandi & Azizi(2012)].

Now, we substitute $u_{m,n}$ in equ. (1) and conclude

$$\frac{\partial^v u_{m,n}(x,t)}{\partial t^v} = a(x,t)\frac{\partial^2 u_{m,n}(x,t)}{\partial x^2} + f(x,t), \quad a < x < b, \quad 0 < t < \tau, \qquad \ldots(29)$$

We set $x = x_k$ in equ. (29) and now substituting equations (26)-(28) into equ. (29) yields

$$\sum_{i=2}^{n} \sum_{r=2}^{j} c_{kj} b_{jr} t^{r-\alpha} - a(x_k,t) \sum_{i=-m}^{m} \sum_{j=0}^{n} c_{kj} q_{ik}^{(2)} L_j^{\tau}(t) + f(x_k,t_\ell), \qquad \ldots(30)$$
$$k = -m,\ldots,m, \quad \ell = 1,\ldots,n.$$

For suitable collocation points we use the shifted Legendre roots t_ℓ, $\ell = 1,\ldots,n+1$ of $L_{\ell+1}(t)$. we applay the collocation method to equ. (30), we have

$$\sum_{i=2}^{n} \sum_{r=2}^{j} c_{kj} b_{jr} t_\ell^{r-v} - a(x_k,t_\ell) \sum_{i=-m}^{m} \sum_{j=0}^{n} c_{kj} q_{ik}^{(2)} L_j^{\tau}(t_\ell) + f(x_k,t_\ell), \qquad \ldots(31)$$

From equ. (2) and equ. (22) we have

$$\begin{cases} \sum_{i=-m}^{m} \sum_{j=0}^{n} c_{ij} S_i(x) L_j^{\tau}(0) = \Psi(x) \\ \sum_{i=-m}^{m} \sum_{j=0}^{n} c_{ij} S_i(x) L_j^{\tau}(\tau) = \varphi(x) \end{cases} \quad \ldots(32)$$

if we set $x = x_j$ and using equ. (17) we obtain

$$\begin{cases} \sum_{i=-m}^{m} \sum_{j=0}^{n} c_{ij} (-1)^j = \Psi(x_k), & k = -m, \ldots, m. \\ \sum_{i=-m}^{m} \sum_{j=0}^{n} c_{ij} (-1)^{j-1} \frac{j(j+1)}{\tau} = \varphi(x_k) & k = -m, \ldots, m. \end{cases} \quad \ldots(33)$$

Equations (30) and (33) provide $(n+1)(2m+1)$ linear equations. Using these equations we obtain the unknown coefficient c_{ij}, $i = 1,2, \ldots, 2m+1,\ j = 1,2, \ldots, n+1$.
And therefore the approximate solution $u_{m,n}$ can be obtained from equ. (22).

6.Numerical examples

In this section we shall consider some tested examples in order to justify the accuracy and efficiency of the proposed method.

Example 1. Consider the following time-fractional diffusion equation

$$\frac{\partial^v u_{m,n}(x,t)}{\partial x^v} = \frac{\partial^2 u(x,t)}{\partial x^2} + \sin(\pi x), \quad 0 < x < 1,\ 0 < t \leq 1,$$

Where the initial conditions

$$u(x,0) = u_t(x,0) = 0,$$

And the boundary conditions

$$u(0,t) = u(1,t) = 0.$$

The exact solution to this problem is [Mao & Shi(2014)].

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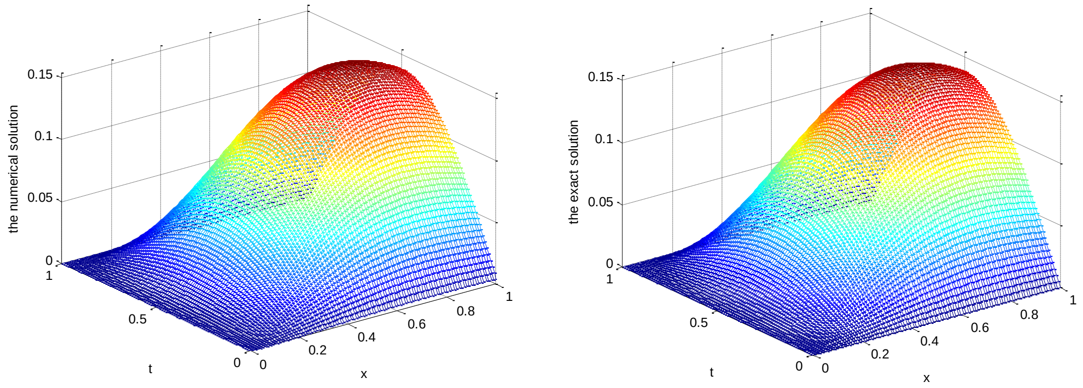
$$u(x,t) = \frac{1}{\pi^2}[1 - E_v(-\pi^2 t^v)] \sin(\pi x).$$

where $E_v(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(vk+1)}$ is the one-parameter Mittag-Leffler function.
following Figure.1 represent a comparision between the exact and the numerical solution given by the proposed method for $m = 15$ and $n = 8$, Furthermore Fig.2 show the absolute error function $|u(x,t) - u_{m,n}(x,t)|$ obtained by the present method. we solve the above problem with $v = 1.7$ by using the method described in section 5, we choose $\beta = 1$ and $d = \frac{\pi}{2}$ and this leads to $h = \frac{\pi}{\sqrt{2m}}$.



.Figure 1: Comparison of the numerical and exact solution in the domain [0,1] × [0,1] for Example 1

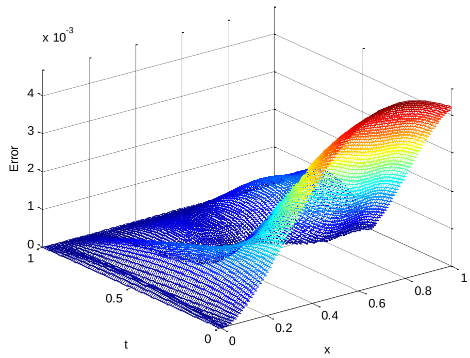


Figure 2: Plot of the absolute error for Example 1

Example 2. Consider the following fractional diffusion equation with the Caputo fractional derivative

$$\frac{\partial^{v} u_{m,n}(x,t)}{\partial x^{v}} - \frac{\partial^{2} u(x,t)}{\partial x^{2}} = \frac{2}{\Gamma(3-v)} t^{2-\alpha} \sin(2\pi x) + 4\pi^{2} t^{2} \sin(2\pi x), \qquad 0 < x < 1, \ \ 0 < t \le 1,$$

The initial condition

$$u(x,0) = 0, \quad 0 < x < 1,$$

And the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad 0 < t \le 1,$$

It is remarkable that the initial conditions of this example can be considered as a special case of equ.(2) and the exact solution for this problem is given by [Ren & Sun(2013)] as: $u(x,t) = t^{2} \sin(2\pi x)$.

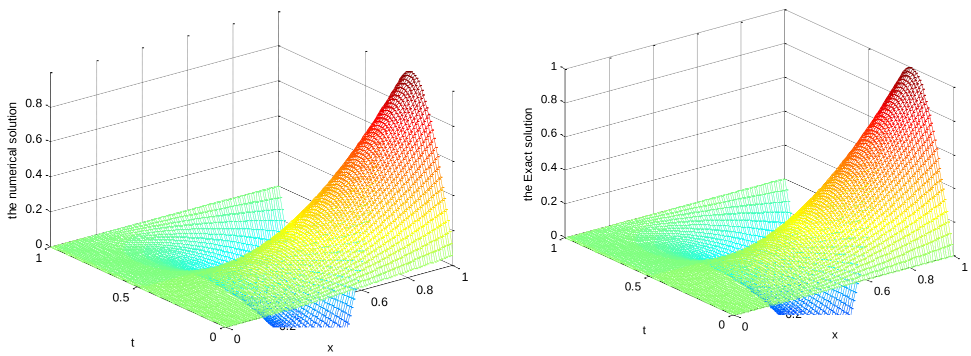


Figure 3: Comparison of the numerical and exact solution for Example 2

To solve the above problem with $v = 0.7$ by using the method described in section 5, we choose $\beta = 1$ and $d = \frac{\pi}{2}$ and this leads to $h = \frac{\pi}{\sqrt{2m}}$, Fig.3 gives the Numerical and exact solution on the whole computational domain with $m = 15, n = 8$. A good agreement of the numerical solution with the exact one is achieved. Furthermore Fig.4 show the absolute error function $|u(x,t) - u_{m,n}(x,t)|$ obtained by the present method.

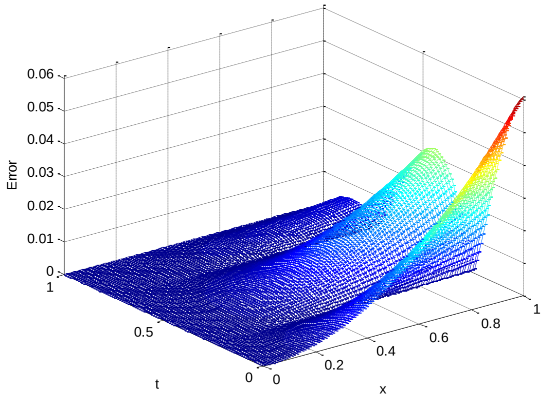


Figure 4: Plot of the absolute error for Example 2

7. Conclusion

In this paper, we develop and analyze the efficient numerical algorithm for the fractional diffusion wave-equation Based on the collocation technique, the sinc functions and shifted Legendre polynomials are used to reduce the problem to the solution of a system of linear algebraic equations. From the computational point of view, the solution obtained by this method is in excellent agreement with the exact one.

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