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# On s*g-a-Proper Functions

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**Abstract**

In this paper we introduce a new class of functions in topological spaces, namely, s*g-α-proper functions. Also, we study the basic properties and characterizations of these functions. One of the most important of equivalent definitions to the s*g-α-proper functions gives by using s*g-α-limit points of nets. Moreover we define and study s*g-α-perfect functions and s*g-α-compact functions in topological spaces and we study the relation between s*g-α-proper functions and each of proper functions, s*g-α-perfect functions, closed functions, s*g-α-closed functions and s*g-α-compact functions and we give an example when the converse may not be true.

**Key words:** s*g-α-proper functions, s*g-α-perfect functions, s*g-α-closed functions, s*g-α-compact functions, s*g-α-limlt points, compactly s*g-α-closed sets and s*g-α-K-spaces.

## Introduction

Levine, N. [6] introduced the concept of semi open sets. Also, Khan, M. and et.al. [5] introduced and investigated s*g-open sets by using the concept of semi-closed sets. Mahmood, S. and Tareq, J. [7] we introduced and study s*g-α-open sets and we can prove that the family of all s*g-α-open subsets of a topological space $(X,\tau)$ from a topology on X which is finer than $\tau$. The purpose of this paper is to introduce a new class of functions, namely, s*g-α-proper functions. We give the definition by depending on the definition of s*g-α-closed functions. Also, we give useful characterizations of s*g-α-proper functions. The second equivalent definition to s*g-α-proper functions by using s*g-α-limit points of nets is more interesting than the first equivalent definition. Moreover we study the relation between s*g-α-proper functions and certain types of functions such as proper functions, s*g-α-perfect functions, closed functions, s*g-α-closed functions and s*g-α-compact functions and we give an example when the converse may not be true. Recall that a subset A of a topological space $(X,\tau)$ is called a semi-open set if there exists an open subset U of X such that $U \subseteq A \subseteq cl(U)$ [6]. The complement of a semi-open set is said to be semi-closed [6]. An s*g-open set is also called ĝ-open [9], s*-open [2] and w-open [8].

## 1. Preliminaries

**1.1 Definition [5]:** A subset A of a topological space $(X,\tau)$ is called s*g-open if $F \subseteq A^{o}$ whenever $F \subseteq A$ and F is semi-closed in X. The complement of an s*g-open set is defined to be s*g-closed.

**1.2 Definition [5]:** Let $(X,\tau)$ be a topological space and $A \subseteq X$. Then:

**i)** The s*g-closure of A, denoted by $\overline{A}^{s^*g}$ is the intersection of all s*g-closed subsets of X which contains A.
**ii)** The s*g-interior of A, denoted by $A^{s^*go}$ is the union of all s*g-open subsets of X which are contained in A.

**1.3 Definition[7]:** A subset A of a topological space $(X,\tau)$ is called an s*g-α-open set if $A \subseteq {\overline{A^{o}}^{s^*g}}^{o}$. The complement of an s*g-α-open set is defined to be s*g-α-closed. The family of all s*g-α-open subsets of X is denoted by $\tau^{s^*g-\alpha}$.

**1.4 Definition [7]:** A subset A of a topological space $(X,\tau)$ is called an s*g-α-neighborhood of a point x in X if there exists an s*g-α-open set U in X such that $x \in U \subseteq A$. The family of all s*g-α-neighborhoods of a point $x \in X$ is denoted by $N_{s^*g-\alpha}(x)$.

**1.5 Proposition [7]:** Let $(X,\tau)$ be a topological space and B be a subset of X. Then B is s*g-α-closed in X if and only if $\overline{\overline{B}^{s^*go}} \subseteq B$.

**1.6 Definition [7]:** Let $(X,\tau)$ be a topological space and $A \subseteq X$. Then the s*g-α-closure of A, denoted by $\overline{A}^{s^*g-\alpha}$ is the intersection of all s*g-α-closed subsets of X which contains A.

**1.7 Theorem [7]:** Let $(X,\tau)$ be a topological space and $A,B \subseteq X$. Then:

**i)** $A \subseteq \overline{A}^{s*g-\alpha} \subseteq \overline{A}$

**ii)** $\overline{A}^{s*g-\alpha}$ is an s*g-α-closed set in $X$.

**iii)** If $A \subseteq B$, then $\overline{A}^{s*g-\alpha} \subseteq \overline{B}^{s*g-\alpha}$.

**iv)** A is s*g-α-closed iff $\overline{A}^{s*g-\alpha} = A$.

**v)** $\overline{\overline{A}^{s*g-\alpha}}^{s*g-\alpha} = \overline{A}^{s*g-\alpha}$.

**vi)** $x \in \overline{A}^{s*g-\alpha}$ iff for every s*g-α-open set U containing x, $U \cap A \neq \phi$.

**1.8 Proposition:** Let $(X,\tau)$ be a topological space and Y be an open subspace of X. If A is an s*g-α-closed set in X, then $A \cap Y$ is an s*g-α-closed set in Y.

**1.9 Proposition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. If $A \subseteq X$ and $B \subseteq Y$. Then if $A \times B$ is an s*g-α-closed set in $X \times Y$, then A and B are s*g-α-closed sets in X and Y respectevely.

**Proof:** It is obvious.

**1.10 Definition [7]:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A function $f : X \to Y$ is called s*g-α-irresolute if the inverse image of every s*g-α-open subset of Y is an s*g-α-open subset of X.

**1.11 Proposition [7]:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A function $f : X \to Y$ is s*g-α-irresolute if the inverse image of every s*g-α-closed subset of Y is an s*g-α-closed subset of X.

**1.12 Definition [4]:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A function $f : X \to Y$ is called compact if the inverse image of every compact set in Y is a compact set in X.

**1.13 Definition:** A family $\{U_\alpha\}_{\alpha\in\Lambda}$ of s*g-α-open sets in a topological space $(X,\tau)$ is called an s*g-α-open cover of a subset A of X if $A \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$.

**1.14 Definition:** A topological space $(X,\tau)$ is called an s*g-α-compact space if every s*g-α-open cover of X has a finite subcover.

**1.15 Definition:** A subset A of a topological space $(X,\tau)$ is called s*g-α-compact if every cover of A by s*g-α-open subsets of X has a finite subcover.

**1.16 Proposition:** Every s*g-α-compact space is a compact space.

The converse of proposition (1.16) is not true in general as shown by the following example:

**1.17 Example:** Let X be any infinite set and $p \in X$, then $\tau = \{X, \phi, \{p\}\}$ is a topology on X. Notice that $(X,\tau)$ is a compact space. However, it is not an s*g-α-compact space, because $\{\{p,x\} : x \in X\}$ is an s*g-α-open cover of X which has no finite subcover.

**1.18 Proposition:** The s*g-α-irresolute image of an s*g-α-compact space is s*g-α-compact.

**Proof:** It is obvious.

**1.19 Definition:** A subset F of a topological space $(X,\tau)$ is called compactly s*g-α-closed if $F \cap K$ is a compact set in X for each s*g-α-compact set K in X.

Clearly every s*g-α-closed subset of a topological space $(X,\tau)$ is compactly s*g-α-closed. But the converse is not true in general as shown by the following example:

 

**1.20 Example:** Let $X=\{a,b,c\}$ and $\tau=\{\phi,X,\{a\}\}$ be a topology on X. Therefore the sets in $\{X,\phi,\{b\},\{c\},\{b,c\}\}$ are s*g-α-closed in X. Thus $\{a\}$ is a compactly s*g-α-closed set in X, but is not s*g-α-closed.

**1.21 Definition:** A topological space $(X,\tau)$ is called an s*g-α-K-space if every compactly s*g-α-closed subset of X is s*g-α-closed.

**1.22 Definition:** Let $(x_d)_{d\in D}$ be a net in a topological space $(X,\tau)$. Then $(x_d)_{d\in D}$ s*g-α-converges to $x\in X$ (written $x_d \xrightarrow{s*g-\alpha} x$) if for each s*g-α-neighborhood U of x, there is some $d_0\in D$ such that $d\geq d_0$ implies $x_d\in U$. This is sometimes said $(x_d)_{d\in D}$ s*g-α-converges to x if $(x_d)_{d\in D}$ is eventually in every s*g-α-neighborhood of x. The point x is called an s*g-α-limit point of $(x_d)_{d\in D}$.

**1.23 Proposition:** Let $(X,\tau)$ be a topological space and $A\subseteq X$. If x is a point of X, then $x\in \overline{A}^{s*g-\alpha}$ if and only if there exists a net $(x_d)_{d\in D}$ in A such that $x_d \xrightarrow{s*g-\alpha} x$.

**Proof:** $\Leftarrow$ Suppose that $\exists$ a net $(x_d)_{d\in D}$ in A such that $x_d \xrightarrow{s*g-\alpha} x$. To prove that $x\in \overline{A}^{s*g-\alpha}$. Let $U\in N_{s*g-\alpha}(x)$, since $x_d \xrightarrow{s*g-\alpha} x \Rightarrow \exists\ d_0\in D$ such that $x_d\in U$, $\forall\ d\geq d_0$. But $x_d\in A, \forall d\in D \Rightarrow U\cap A\neq\phi$, $\forall\ U\in N_{s*g-\alpha}(x)$. Hence by theorem ((1.7),(vi)), we get $x\in \overline{A}^{s*g-\alpha}$.

Conversely, suppose that $x\in \overline{A}^{s*g-\alpha}$. To prove that $\exists$ a net $(x_d)_{d\in D}$ in A such that $x_d \xrightarrow{s*g-\alpha} x$. Since $x\in \overline{A}^{s*g-\alpha}$, then by theorem ((1.7),(vi)), we get $N\cap A\neq\phi, \forall\ N\in N_{s*g-\alpha}(x)$. Hence $D=N_{s*g-\alpha}(x)$ is a directed set by inclusion. Since $N\cap A\neq\phi, \forall\ N\in N_{s*g-\alpha}(x) \Rightarrow \exists\ x_N\in N\cap A$.
Define $x: N_{s*g-\alpha}(x)\to A$ by: $x(N)=x_N$, $\forall\ N\in N_{s*g-\alpha}(x)$. Thus $(x_N)_{N\in N_{s*g-\alpha}(x)}$ is a net in A. To prove that $x_N \xrightarrow{s*g-\alpha} x$. Let $U\in N_{s*g-\alpha}(x)$ to find $d_0\in D$ such that $x_d\in U, \forall\ d\geq d_0$. Let $d_0=U \Rightarrow \forall d\geq d_0 \Rightarrow d=M\in N_{s*g-\alpha}(x)$ i.e. $M\geq U \Leftrightarrow M\subseteq U \Rightarrow x_d=x(d)=x(M)=x_M\in M\cap A\subseteq M \subseteq U \Rightarrow x_M\in U \Rightarrow x_d\in U$, $\forall\ d\geq d_0$. Thus $x_N \xrightarrow{s*g-\alpha} x$.

**1.24 proposition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A function $f:X\to Y$ is s*g-α-irresolute iff whenever $(x_d)_{d\in D}$ is a net in X such that $x_d \xrightarrow{s*g-\alpha} x$, then $f(x_d) \xrightarrow{s*g-\alpha} f(x)$ in Y.

**Proof:** It is obvious.

**1.25 Definition [3]:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces, and $f:X\to Y$ be a function. Then f is called a proper function if:
**i)** f is a continuous function.
**ii)** $f\times I_Z: X\times Z\to Y\times Z$ is closed for every topological space Z.

## 2. Properties of s*g-α-Closed Functions

In this section we introduce a new definition (to the best of our knowledge), namely, s*g-α-closed functions which is weaker than closed functions, and prove some of the results which relate to this concept. Also, we explain the relationship between an s*g-α-closed function and an s*g-α-compact function.

**2.1 Definition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A function $f:X\to Y$ is called an s*g-α-closed (resp. s*g-α-open) function if the image of every closed (resp. open) subset of X is an s*g-α-closed (resp. s*g-α-open) set in Y.

**2.2 Examples:**

**i)** Let $f:(\Re,\mu)\to(\Re,\mu)$ be a function which is defined by: $f(x)=0, \forall\ x\in\Re$. Then f is an s*g-α-closed function.

 

**ii)** If F is an s*g-α-closed (not closed) set in X, then the inclusion function $\iota_F : F \to X$ is s*g-α-closed, but is not a closed function.

Since every closed set is an s*g-α-closed set, then we have the following proposition.

**2.3 Proposition:** Every closed function is an s*g-α-closed function.

The converse of proposition (2.3) may not be true in general as shown by the following example.

**2.4 Example:** Let $X = \{a,b,c,d\}$ and $Y = \{x,y,z\}$ be sets and let $\tau = \{\phi, X, \{a,b,c\}, \{b,c\}, \{a\}\}$ and $\tau' = \{\phi, Y, \{x\}\}$ be topologies on X and Y, respectively. So the sets in $\{X, \phi, \{d\}, \{a,d\}, \{b,c,d\}\}$ are closed in X. Also, the sets in $\{Y, \phi, \{y,z\}, \{z\}, \{y\}\}$ are s*g-α-closed sets in Y. Define the function $f : X \to Y$ by: $f(a) = f(c) = z$, $f(b) = x$ and $f(d) = y$. Notice that f is an s*g-α-closed function. But f is not a closed function, since $\{d\}$ is a closed set in X, but $f(\{d\}) = \{y\}$ is not a closed set in Y.

**2.5 Theorem:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A function $f : X \to Y$ is s*g-α-closed if and only if for each subset B of Y and each open subset U of X containing $f^{-1}(B)$, there exists an s*g-α-open set V in Y containing B such that $f^{-1}(V) \subseteq U$.

**Proof:** $\Rightarrow$ Suppose that B is an arbitrary subset of Y and U is an arbitrary open subset of X containing $f^{-1}(B)$. Put $V = Y - f(X - U)$. Then by definition (2.1), V is an s*g-α-open set in Y. Since $f^{-1}(B) \subseteq U$ $\Rightarrow X - U \subseteq f^{-1}(Y - B) \Rightarrow f(X - U) \subseteq Y - B \Rightarrow B \subseteq Y - f(X - U) \Rightarrow B \subseteq V$ and $f^{-1}(V) \subseteq U$.

Conversely, Let F be any closed set in X. Put $B = Y - f(F)$, then we have $f^{-1}(B) \subseteq X - F$. Since $X - F$ is an open set in X, then by hypothesis there exists an s*g-α-open set V in Y such that $B \subseteq V$ and $f^{-1}(V) \subseteq X - F$. Therefore, we obtain $f(F) = Y - V$ and hence $f(F)$ is an s*g-α-closed set in Y. This shows that f is an s*g-α-closed function.

**2.6 Proposition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A function $f : X \to Y$ is s*g-α-closed if and only if $\overline{f(A)}^{s*g-\alpha} \subseteq f(\overline{A})$ for each $A \subseteq X$.

**Proof:** $\Rightarrow$ suppose that $f : X \to Y$ is an s*g-α-closed function. Since $f(A) \subseteq f(\overline{A})$ and $\overline{A}$ is a closed set in X, then $f(\overline{A})$ is s*g-α-closed in Y. Therefore $\overline{f(A)}^{s*g-\alpha} \subseteq \overline{f(\overline{A})}^{s*g-\alpha} = f(\overline{A})$. Hence $\overline{f(A)}^{s*g-\alpha} \subseteq f(\overline{A})$ for each $A \subseteq X$. Conversely, assume that $\overline{f(A)}^{s*g-\alpha} \subseteq f(\overline{A})$ for each $A \subseteq X$. Let F be a closed subset of X, thus by hypothesis $\overline{f(F)}^{s*g-\alpha} \subseteq f(\overline{F}) = f(F)$. But $f(F) \subseteq \overline{f(F)}^{s*g-\alpha}$, then $f(F) = \overline{f(F)}^{s*g-\alpha}$. Hence $f(F)$ is an s*g-α-closed set in Y. Thus $f : X \to Y$ is an s*g-α-closed function.

**2.7 Proposition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A bijective function $f : X \to Y$ is an s*g-α-closed function if and only if f is an s*g-α-open function.

**Proof:** $\Rightarrow$ Let $f : X \to Y$ be a bijective s*g-α-closed function and U be an open subset of X, thus $U^c$ is closed. Since f is s*g-α-closed, then $f(U^c)$ is s*g-α-closed in Y, thus $(f(U^c))^c$ is s*g-α-open. Since f is a bijective function, then $(f(U^c))^c = f(U)$, hence $f(U)$ is an s*g-α-open set in Y. Therefore f is an s*g-α-open function.

Conversely, let $f : X \to Y$ be a bijective s*g-α-open function and F be a closed subset of X, thus $F^c$ is open. Since f is s*g-α-open, then $f(F^c)$ is s*g-α-open in Y, thus $(f(F^c))^c$ is s*g-α-closed. Since f is a bijective function, then $(f(F^c))^c = f(F)$, hence $f(F)$ is an s*g-α-closed set in Y. Therefore f is an s*g-α-closed function.

**2.8 Proposition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces and $f : X \to Y$ be a function. If $\overline{f(A)}^{s*g-\alpha} = f(\overline{A})$

 

for each $A \subseteq X$, then f is a continuous s*g-α-closed function.

**Proof:** To prove that $f: X \to Y$ is an s*g-α-closed function. Let F be a closed subset of X, then $\overline{F} = F$. By hypothesis $\overline{f(F)}^{s*g-\alpha} = f(\overline{F}) = f(F)$, hence $f(F)$ is an s*g-α-closed set in Y. Therefore $f: X \to Y$ is an s*g-α-closed function. Now, to prove that f is a continuous function. Since $f(\overline{A}) = \overline{f(A)}^{s*g-\alpha} \subseteq \overline{f(A)}$ for each $A \subseteq X$, thus by ([10], theorem (7.2)), $f: X \to Y$ is a continuous function.

**2.9 Theorem:** Let $(X, \tau)$, $(Y, \tau')$ and $(Z, \tau'')$ be three topological spaces and $f: X \to Y$, $g: Y \to Z$ be two functions. Then:
**i)** If f is closed and g is s*g-α-closed, then $g \circ f$ is s*g-α-closed.
**ii)** If $g \circ f$ is s*g-α-closed and f is continuous and onto, then g is s*g-α-closed.
**iii)** If $g \circ f$ is s*g-α-closed and g is one-to-one and s*g-α-irresolute, then f is s*g-α-closed.

**Proof:**
**i)** To prove that $g \circ f: X \to Z$ is an s*g-α-closed function. Let F be a closed subset of X. Since f is closed, then $f(F)$ is a closed set in Y. But g is an s*g-α-closed function, then $g(f(F))$ is an s*g-α-closed set in Z, hence $(g \circ f)(F)$ is an s*g-α-closed set in Z. Thus $g \circ f: X \to Z$ is an s*g-α-closed function.
**ii)** To prove that $g: Y \to Z$ is an s*g-α-closed function. Let F be a closed subset of Y, since f is continuous, then $f^{-1}(F)$ is a closed set in X. Since $g \circ f$ is s*g-α-closed, then $(g \circ f)(f^{-1}(F)) = g(f \circ f^{-1}(F))$ is an s*g-α-closed set in Z. Since f is onto, then g(F) is an s*g-α-closed set in Z. Thus $g: Y \to Z$ is an s*g-α-closed function.
**iii)** To prove that $f: X \to Y$ is an s*g-α-closed function. Let F be a closed subset of X, since $g \circ f$ is s*g-α-closed, then $(g \circ f)(F)$ is s*g-α-closed in Z. Since g is s*g-α-irresolute, then $g^{-1}(g \circ f(F)) = (g^{-1} \circ g)(f(F))$ is an s*g-α-closed set in Y. Since g is one-to-one, then $f(F)$ is an s*g-α-closed set in Y. Thus $f: X \to Y$ is an s*g-α-closed function.

**2.10 Corollary:** Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces. If $f: X \to Y$ is an s*g-α-closed function, then the restriction of f to a closed subset F of X is an s*g-α-closed function of F into Y.
**Proof:** Since F is a closed set in X, then the inclusion function $\iota_F: F \to X$ is a closed function. Since $f: X \to Y$ is an s*g-α-closed function, then by theorem ((2.9),(i)), $f \circ \iota_F: F \to Y$ is an s*g-α-closed function. But $f \circ \iota_F = f \mid F$, thus the restriction function $f \mid F: F \to Y$ is an s*g-α-closed function.

**2.11 Proposition:** Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces, and $f: X \to Y$ be an s*g-α-closed function. Then for each open subset T of Y, the function $f_T: f^{-1}(T) \to T$ which agrees with f on $f^{-1}(T)$ is also s*g-α-closed.

**Proof:** Let F be a closed subset of $f^{-1}(T)$, then there is a closed subset $F_1$ of X such that $F = F_1 \cap f^{-1}(T)$. Since $f_T(F) = f(F_1) \cap T$ and $f(F_1)$ is s*g-α-closed in Y and T is an open subset of Y, then by proposition (1.8), $f(F_1) \cap T$ is an s*g-α-closed set in T .Thus $f_T$ is an s*g-α-closed function.

**2.12 Remark:** If $f: X \to Y$ is an s*g-α-closed function and $T \subseteq Y$ is not an open set. Then $f_T: f^{-1}(T) \to T$ is not necessarily an s*g-α-closed function as the following example shows.

**2.13 Example:** In example (2.4), let $T = \{y, z\}$, notice that T is not open in Y and $\tau'_T = \{\phi, T\}$, then $f^{-1}(T) = \{a, c, d\}$ and $\tau_{f^{-1}(T)} = \{f^{-1}(T), \phi, \{a\}, \{c\}, \{a, c\}\}$. Define the function $f_T: f^{-1}(T) \to T$ by:

$f_T(x) = f(x)$, $\forall x \in f^{-1}(T)$. Notice that the subset $\{d\}$ of $f^{-1}(T)$ is closed in $f^{-1}(T)$, but $f_T(\{d\}) = \{y\}$ is not an s*g-α-closed set in T, since $\overline{(((\overline{\{y\}})_T)_T}^{s*go})_T = T \not\subset \{y\}$. Thus $f_T$ is not an s*g-α-closed function.

The product of two s*g-α-closed functions is not necessarily an s*g-α-closed function as shown by the following example:

**2.14 Example:** Let $f_1 : (\Re, \mu) \to (\Re, \mu)$ be a function which is defined by: $f_1(x) = 0$, $\forall\ x \in \Re$. And let $I_\Re : (\Re, \mu) \to (\Re, \mu)$ be a function which is defined by: $I_\Re(x) = x$, $\forall\ x \in \Re$ where $I_\Re$ is the identity function on $\Re$. Clearly $f_1$ and $I_\Re$ are s*g-α-closed functions, but $f_1 \times I_\Re : \Re \times \Re \to \Re \times \Re$ such that $(f_1 \times I_\Re)(x, y) = (0, y)$ for each $(x, y) \in \Re \times \Re$ is not an s*g-α-closed function, since the set $A = \{(x, y) \in \Re \times \Re : x\,y = 1\}$ is closed in $\Re \times \Re$, but $(f_1 \times I_\Re)(A) = \{0\} \times \Re / \{0\}$ is not s*g-α-closed in $\Re \times \Re$.

**2.15 Theorem:** Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two functions. If $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is s*g-α-closed, then $f_1$ and $f_2$ are also s*g-α-closed functions.

**Proof:** Suppose that $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is an s*g-α-closed function. To prove that $f_1 : X_1 \to Y_1$ is s*g-α-closed. Let F be a closed subset of $X_1$, to prove that $f_1(F)$ is an s*g-α-closed set in $Y_1$. Suppose that $G = f_1(F)$ $\Rightarrow$ $F \times X_2$ is a closed set in $X_1 \times X_2$. Since $f_1 \times f_2$ is s*g-α-closed, then $(f_1 \times f_2)(F \times X_2) = f_1(F) \times f_2(X_2) = G \times f_2(X_2)$ is s*g-α-closed in $Y_1 \times Y_2$ i.e. $\overline{G \times f_2(X_2)}^{s*go} \subseteq G \times f_2(X_2)$. But by proposition (1.9), we have: $\overline{G}^{s*go} \times \overline{f_2(X_2)}^{s*go} \subseteq \overline{G \times f_2(X_2)}^{s*go} \subseteq G \times f_2(X_2) \Rightarrow \overline{G}^{s*go} \subseteq G$. Therefore by proposition (1.5), $G = f_1(F)$ is an s*g-α-closed set in $Y_1$. Thus $f_1$ is an s*g-α-closed function. By the same way we can prove that $f_2$ is an s*g-α-closed function. Thus $f_1$ and $f_2$ are s*g-α-closed functions.

**2.16 Definition:** Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces. A function $f : X \to Y$ is called s*g-α-compact if the inverse image of every s*g-α-compact set in Y is a compact set in X.

**2.17 Proposition:** Let $(X, \tau)$, $(Y, \tau')$ and $(Z, \tau'')$ be three topological spaces and $f : X \to Y$, $g : Y \to Z$ be two functions. Then:

**i)** If f is compact and g is s*g-α-compact, then $g \circ f$ is s*g-α-compact.

**ii)** If $g \circ f$ is s*g-α-compact and f is continuous and onto, then g is s*g-α-compact.

**iii)** If $g \circ f$ is s*g-α-compact and g is s*g-α-irresolute and one-to-one, then f is s*g-α-compact.

**Proof:** The proof is similar of theorem (2.9).

**2.18 Remark:** s*g-α-closed function and s*g-α-compact function are in general independent. Consider the following examples:

**2.19 Examples:(i)** Let $X = Y = \{a, b, c\}$ and $\tau = \{\phi, X, \{a, c\}\}$ and $\tau' = \{\phi, Y, \{b\}\}$, and let $f : (X, \tau) \to (Y, \tau')$ be a function which is defined by: $f(a) = f(c) = a$ and $f(b) = b$. Since X and Y are finite spaces, then $f^{-1}(K)$ is a compact set in X for each s*g-α-compact subset K of Y. Hence f is an s*g-α-compact function, but f is not an s*g-α-closed function, since $\{b\}$ is a closed set in X, but $f(\{b\}) = \{b\}$ is not an s*g-α-closed set in Y, since $\overline{\{b\}}^{s*go} = Y \not\subset \{b\}$.

**(ii)** Let $(\Re, \mu)$ be the usual topological space and let $f : (\Re, \mu) \to (\Re, \mu)$ be a function which is defined by: $f(x) = 0$, $\forall\ x \in \Re$. Then f is an s*g-α-closed function, but f is not an s*g-α-compact function, since $\{0\}$ is an s*g-α-compact set in $\Re$, but $f^{-1}(\{0\}) = \Re$ is not compact in $\Re$.

 

**2.20 Proposition:** Let $(X,\tau)$ be a topological space and $(Y,\tau')$ be an s*g-α-K-space. Then every continuous s*g-α-compact function $f: X \to Y$ is an s*g-α-closed function.

**Proof:** Let F be a closed set in X, to prove that $f(F)$ is an s*g-α-closed set in Y. Let K be an s*g-α-compact set in Y. Since f is an s*g-α-compact function, then $f^{-1}(K)$ is a compact set in X. Since $F \cap f^{-1}(K)$ is a compact set in X and f is continuous, then by ([10], theorem (17.7)), $f(F \cap f^{-1}(K))$ is a compact set in Y. But $f(F \cap f^{-1}(K))$ $= f(F) \cap K$, thus $f(F) \cap K$ is a compact set in Y. Therefore by definition (1.19), $f(F)$ is a compactly s*g-α-closed set in Y. Since Y is an s*g-α-K-space, then by definition (1.21), $f(F)$ is an s*g-α-closed set in Y. Hence f is an s*g-α-closed function.

**2.21 Proposition:** Any one-to-one s*g-α-closed function is an s*g-α-compact function.

**Proof:** Let $f:(X,\tau) \to (Y,\tau')$ be a one-to-one s*g-α-closed function and K be an s*g-α-compact set in Y. To prove that $f^{-1}(K)$ is a compact set in X. Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any open cover of $f^{-1}(K)$, then $f^{-1}(K) \subseteq \bigcup_{\alpha\in\Lambda} U_\alpha$ and $U_\alpha$ is an open set in X for each $\alpha \in \Lambda$. Hence $(\bigcup_{\alpha\in\Lambda} U_\alpha)^c \subseteq X - f^{-1}(K)$, therefore $\bigcap_{\alpha\in\Lambda} U_\alpha^c \subseteq f^{-1}(Y-K)$. Since f is a one-to-one function, then $\bigcap_{\alpha\in\Lambda} f(U_\alpha^c) = f(\bigcap_{\alpha\in\Lambda} U_\alpha^c) \subseteq f(f^{-1}(Y-K)) \subseteq Y-K \Rightarrow K \subseteq \bigcup_{\alpha\in\Lambda}(Y - f(U_\alpha^c)$.
Since f is an s*g-α-closed function and $U_\alpha^c$ is a closed set in X for each $\alpha \in \Lambda$, then $f(U_\alpha^c)$ is an s*g-α-closed set in Y for each $\alpha \in \Lambda$. Thus $\{Y - f(U_\alpha^c)\}_{\alpha\in\Lambda}$ is an s*g-α-open cover of K. Since K is s*g-α-compact, then $\exists \{Y - f(U_{\alpha_i}^c)\}_{i=1}^n$ is a finite subcover of $\{Y - f(U_\alpha^c)\}_{\alpha\in\Lambda}$ i.e. $K \subseteq \bigcup_{i=1}^n (Y - f(U_{\alpha_i}^c)$ $\Rightarrow$ $f^{-1}(K) \subseteq$ $\bigcup_{i=1}^n (X - f^{-1}(f(U_{\alpha_i}^c))) \subseteq \bigcup_{i=1}^n U_{\alpha_i}$. So, $\{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $\{U_\alpha\}_{\alpha\in\wedge}$. Hence $f^{-1}(K)$ is a compact set in X. Thus $f: X \to Y$ is an s*g-α-compact function.

**2.22 Corollary:** Let $(X,\tau)$ be a topological space and $(Y,\tau')$ be an s*g-α-K-space. Then a one-to-one continuous function $f: X \to Y$ is an s*g-α-closed function if and only if f is an s*g-α-compact function.

**Proof:** It is obvious.

**2.23 Definition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. A function $f: X \to Y$ is called an s*g-α-homeomorphism if:
**i)** f is bijective.
**ii)** f is continuous.
**iii)** f is s*g-α-closed (resp. s*g-α-open).

## 3. Properties of s*g-α-Proper Functions

In this section we introduce a new definition (to the best of our knowledge), namely, s*g-α-proper functions. Also, we study the basic properties and characterizations of these functions. Moreover we study the relation between s*g-α-proper functions and certain types of functions such as proper functions, s*g-α-perfect functions, closed functions, s*g-α-closed functions and s*g-α-compact functions.

**3.1 Definition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces, and $f: X \to Y$ be a function. Then f is called an s*g-α-proper function if:
**i)** f is a continuous function.
**ii)** $f \times I_Z : X \times Z \to Y \times Z$ is s*g-α-closed for every topological space Z.

**3.2 Examples:**
**i)** Let $f:(\Re,\mu) \to (\Re,\mu)$ be a function which is defined by: $f(x) = 0$, $\forall\ x \in \Re$. Notice that f is an s*g-α-

 

closed function, but f is not s*g-α-proper, since for the usual topological space $(\Re,\mu)$, the function $f\times I_{\Re}:\Re\times\Re\to\Re\times\Re$ such that $(f\times I_{\Re})(x,y)=(0,y)$ for each $(x,y)\in\Re\times\Re$ is not an s*g-α-closed function.

**ii)** An inclusion function $\iota_F:F\to X$ is s*g-α-proper if and only if F is an s*g-α-closed set in X.

Since every closed function is an s*g-α-closed function, then we have the following proposition:

**3.3 Proposition:** Every proper function is an s*g-α-proper function.

The converse of proposition (3.3) may not be true in general as shown by the following example:

**3.4 Example:** Let $X=Y=\{a,b,c\}$ and let $\tau=\{\phi,\{a\},\{a,c\},X\}$ and $\tau'=\{\phi,\{a\},Y\}$ be topologies on X and Y, respectively. Define the function $f:X\to Y$ by: $f(a)=a$, $f(b)=b$ and $f(c)=c$. Therefore f is an s*g-α-proper function, but f is not a proper function, since f is not a closed function.

**3.5 Proposition:** Every s*g-α-proper function is an s*g-α-closed function.

**Proof:** Let $f:X\to Y$ be an s*g-α-proper function, then the function $f\times I_Z:X\times Z\to Y\times Z$ is s*g-α-closed for each topological space Z. Let $Z=\{t\}$, then $X\times Z=X\times\{t\}\cong X$ and $Y\times Z=Y\times\{t\}\cong Y$ and we can replace $f\times I_Z$ by $f$. Thus $f:X\to Y$ is an s*g-α-closed function.

**3.6 Remark:** The converse of proposition (3.5) may not be true in general. Observe that in examples ((3.2),(i)) , $f:(\Re,\mu)\to(\Re,\mu)$ is an s*g-α-closed function, but is not an s*g-α-proper function.

**3.7 Theorem:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces, and $f:X\to Y$ be a continuous, one-to-one function. Then f is an s*g-α-proper function if and only if f is an s*g-α-closed function.

**Proof:** $\Rightarrow$ By proposition (3.5) .

Conversely, assume that $f:X\to Y$ is an s*g-α-closed function. To prove that f is s*g-α-proper i.e. to prove that $h=f\times I_Z:X\times Z\to Y\times Z$ is s*g-α-closed for every topological space Z. Let C be any closed set in $X\times Z$. To prove that $h(C)=D$ is an s*g-α-closed set in $Y\times Z$. Let $(y,s)\in D^c \Rightarrow h^{-1}(y,s)\in h^{-1}(D^c)\Rightarrow (f\times I_Z)^{-1}(y,s)\in h^{-1}(D^c)\Rightarrow (f^{-1}\times I_Z{}^{-1})(y,s)\in h^{-1}(D^c)\Rightarrow f^{-1}(y)\times\{s\}\subseteq C^c$,
where $C^c$ is an open set in $X\times Z$. Since f is a one-to-one s*g-α-closed function, then by proposition (2.21), $f^{-1}(y)$ is a compact set in X. Hence by ([10], theorem (17.6)) there are open sets U in X and V in Z such that $f^{-1}(y)\times\{s\}\subseteq U\times V\subseteq C^c \Rightarrow f^{-1}(y)\subseteq U$ and $\{s\}\subseteq V$. Since f and $I_Z$ are s*g-α-closed, then by theorem (2.5), there are s*g-α-open sets $U'$ in Y and $V'$ in Z such that $\{y\}\subseteq U'$, $\{s\}\subseteq V'$, $f^{-1}(U')\subseteq U$ and $I_Z{}^{-1}(V')\subseteq V \Rightarrow (y,s)\in U'\times V'\subseteq D^c \Rightarrow D^c$ is an s*g-α-open set in $Y\times Z \Rightarrow$ D is an s*g-α-closed in $Y\times Z$. Hence $f\times I_Z:X\times Z\to Y\times Z$ is an s*g-α-closed function. Thus $f:X\to Y$ is an s*g-α-proper function.

**3.8 Corollary:** Every s*g-α-homeomorphism is an s*g-α-proper function.

The converse of corollary (3.8) may not be true in general as shown by the following example:

**3.9 Example:** Let $f:([0,1],\mu')\to(\Re,\mu)$ be a function which is defined by: $f(x)=x$, $\forall\, x\in[0,1]$ where $\mu'$ is the relative usual topology on [0,1]. Clearly that f is an s*g-α-proper function, but is not s*g-α-homeomorphism.

**3.10 Theorem:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces, and $f:X\to Y$ be a continuous, function. Then the following statements are equivalent:

**i)** f is an s*g-α-proper function.

**ii)** f is an s*g-α-closed function and $f^{-1}(y)$ is a compact set in X for each $y\in Y$.

**iii)** If $(x_d)_{d\in D}$ is a net in X and $y\in Y$ is an s*g-α-limit point of the net $(f(x_d))_{d\in D}$, then there is a cluster point $x\in X$ of $(x_d)_{d\in D}$ such that $f(x)=y$.

 

**Proof:** $(i \to ii)$. If f is an s*g-α-proper function, then by proposition (3.5), f is an s*g-α-closed function. Also, by ([1], theorem (3.1.12)), $f^{-1}(y)$ is a compact set in X for each $y \in Y$.
$(ii \to iii)$. Let $(x_d)_{d\in D}$ be a net in X and $y \in Y$ be an s*g-α-limit point of a net $(f(x_d))_{d\in D}$ in Y. To prove that there is a cluster point $x \in X$ of $(x_d)_{d\in D}$ such that $f(x) = y$. Claim $f^{-1}(y) \neq \phi$, if $f^{-1}(y) = \phi \Rightarrow y \notin f(X) \Rightarrow y \in (f(X))^c$, since X is a closed set in X and f is s*g-α-closed, then $f(X)$ is an s*g-α-closed set in Y. Thus $(f(X))^c$ is an s*g-α-open set in Y. Therefore $(f(x_d))_{d\in D}$ is eventually in $(f(X))^c$. But $f(x_d) \in f(X), \forall d \in D$, then $f(X) \cap (f(X))^c \neq \phi$, and this is a contradiction. Thus $f^{-1}(y) \neq \phi$.

Now, suppose that the statement (iii) is not true, that means, for all $x \in f^{-1}(y)$ there exists an open set $U_x$ in X contains x such that $(x_d)_{d\in D}$ is not frequently in $U_x$. Notice that $f^{-1}(y) = \bigcup_{x\in f^{-1}(y)} \{x\} \subseteq \bigcup_{x\in f^{-1}(y)} U_x$.
Therefore the family $\{U_x : x \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is a compact set, then there exists $x_1, x_2, \ldots, x_n$ such that $f^{-1}(y) \subseteq \bigcup_{i=1}^{n} U_{x_i} \Rightarrow f^{-1}(y) \cap (\bigcup_{i=1}^{n} U_{x_i})^c = \phi \Rightarrow f^{-1}(y) \cap (\bigcap_{i=1}^{n} U_{x_i}^c) = \phi$. But $(x_d)_{d\in D}$ is not frequently in $U_{x_i}, \forall\, i = 1,..n$, thus $(x_d)_{d\in D}$ is not frequently in $\bigcup_{i=1}^{n} U_{x_i}$. Since $\bigcup_{i=1}^{n} U_{x_i}$ is an open set in X, then $\bigcap_{i=1}^{n} U_{x_i}^c$ is a closed set in X. Thus $f(\bigcap_{i=1}^{n} U_{x_i}^c)$ is an s*g-α-closed set in Y. Claim $y \notin f(\bigcap_{i=1}^{n} U_{x_i}^c)$, if $y \in f(\bigcap_{i=1}^{n} U_{x_i}^c)$, then there exists $x \in \bigcap_{i=1}^{n} U_{x_i}^c$ such that $f(x) = y$, thus $x \notin \bigcup_{i=1}^{n} U_{x_i}$, but $x \in f^{-1}(y)$, therefore $f^{-1}(y)$ is not a subset of $\bigcup_{i=1}^{n} U_{x_i}$, and this is a contradiction. Hence $y \notin f(\bigcap_{i=1}^{n} U_{x_i}^c)$ and by theorem ((1.7),(vi)), there is an s*g-α-open set A in Y such that $y \in A$ and $A \cap f(\bigcap_{i=1}^{n} U_{x_i}^c) = \phi \Rightarrow$

$f^{-1}(A) \cap f^{-1}(f(\bigcap_{i=1}^{n} U_{x_i}^c)) = \phi \Rightarrow$

$f^{-1}(A) \cap (\bigcap_{i=1}^{n} U_{x_i}^c) = \phi \Rightarrow f^{-1}(A) \subseteq \bigcup_{i=1}^{n} U_{x_i}$. But $(f(x_d))_{d\in D}$ is eventually in A, then $(f(x_d))_{d\in D}$ is frequently in A, thus $(x_d)_{d\in D}$ is frequently in $f^{-1}(A)$ and then $(x_d)_{d\in D}$ is frequently in $\bigcup_{i=1}^{n} U_{x_i}$, this is a contradiction. Thus there is a cluster point $x \in f^{-1}(y)$ of $(x_d)_{d\in D}$ such that $f(x) = y$.
$(iii \to i)$. To prove that $f \times I_Z : X \times Z \to Y \times Z$ is an s*g-α-closed function for every topological space Z. Let F be a closed subset of $X \times Z$ and $(f \times I_Z)(F) = G$. To prove that G is an s*g-α-closed set in $Y \times Z$. Let $(y,z) \in \overline{G}^{s*g-\alpha}$, then by proposition (1.23), there exists a net $\{(y_d, z_d)\}_{d\in D}$ in G such that $(y_d, z_d) \xrightarrow{s*g-\alpha} (y,z)$. Thus there is a net $\{(x_d, z_d)\}_{d\in D}$ in F such that $(f \times I_Z)(x_d, z_d) = (y_d, z_d), \ \forall d \in D$.
Since $(f(x_d), I_Z(z_d)) \xrightarrow{s*g-\alpha} (y,z)$, then $f(x_d) \xrightarrow{s*g-\alpha} y$ and $z_d \xrightarrow{s*g-\alpha} z$, hence by hypothesis there is a point $x \in X$ such that $x_d \propto x$ and $f(x) = y$. Since $z_d \xrightarrow{s*g-\alpha} z$, then $z_d \to z$. Therefore $x_{d_u} \to x$ and $z_{d_u} \to z \Rightarrow (x_{d_u}, z_{d_u}) \to (x,z)$. Since $\{(x_{d_u}, z_{d_u})\}$ is a net in F and F is closed, thus by ([10], theorem (11.7)), $(x,z) \in \overline{F} = F \Rightarrow (y,z) = (f \times I_Z)(x,z) \in G$. Thus $\overline{G}^{s*g-\alpha} \subseteq G$

 

$\Rightarrow\ G = \overline{G}^{s*g-\alpha}$ . Hence G is an s*g-α-closed set in $Y \times Z$. Therefore $f \times I_Z : X \times Z \to Y \times Z$ is an s*g-α-closed function for every topological space Z. Thus $f : X \to Y$ is an s*g-α-proper function.

**3.11 Corollary:** Let $(X, \tau)$ be a topological space and $\{p\}$ be a space consisting of a single point. Then a function $f : X \to \{p\}$ is s*g-α-proper if and only if X is a compact space.

**Proof:** It is obvious.

**3.12 Definition:** If the function $f : (X, \tau) \to (Y, \tau')$ is s*g-α-proper and $(X, \tau)$ is a $T_2$-space, then f is called an s*g-α-perfect function.

**3.13 Corollary:** Every s*g-α-perfect function is an s*g-α-proper function.

**3.14 Remark:** The converse of corollary (3.13) may not be true in general. Consider the following example:

**3.15 Example:** Let $f : (\Re, \tau_{cof.}) \to (\Re, \tau_{cof.})$ be the identity function, where $\tau_{cof.}$ be the cofinite topology on $\Re$. Then f is an s*g-α-homeomorphism and by corollary (3.8), f is s*g-α-proper. Since $(\Re, \tau_{cof.})$ is not a $T_2$-space, then f is not an s*g-α-perfect function.

**3.16 Theorem:** Let $(X, \tau)$, $(Y, \tau')$ and $(Z, \tau'')$ be topological spaces, and $f : X \to Y$, $g : Y \to Z$ be continuous functions. Then:

**i)** If f is proper and g is s*g-α-proper, then $g \circ f$ is s*g-α-proper.

**ii)** If $g \circ f$ is s*g-α-proper and f is onto, then g is s*g-α-proper.

**iii)** If $g \circ f$ is s*g-α-proper and g is one-to-one and s*g-α-irresolute, then f is s*g-α-proper.

**Proof:**

**i)** It is clear that $g \circ f : X \to Z$ is a continuous function. Let $(x_d)_{d \in D}$ be a net in X such that $(g \circ f)(x_d) \xrightarrow{s*g-\alpha} z \in Z$. Since g is an s*g-α-proper function and $g(f(x_d)) \xrightarrow{s*g-\alpha} z$, then by theorem (3.10), there is a point $y \in Y$ such that $f(x_d) \propto y$ and $g(y) = z$. Since f is a proper function, then by [3], there is a point $x \in X$ such that $x_d \propto x$ and $f(x) = y$. Hence there is $x \in X$ such that $x_d \propto x$ and $(g \circ f)(x) = g(f(x)) = g(y) = z$. Thus $g \circ f : X \to Z$ is an s*g-α-proper function.

**ii)** Let $(y_d)_{d \in D}$ be a net in Y such that $g(y_d) \xrightarrow{s*g-\alpha} z \in Z$. Since $(y_d)_{d \in D}$ is a net in Y and f is onto, then there is a net $(x_d)_{d \in D}$ in X such that $f(x_d) = y_d$, $\forall\, d \in D$. Hence $g(f(x_d)) = (g \circ f)(x_d) \xrightarrow{s*g-\alpha} z$. Since $g \circ f$ is s*g-α-proper, then by theorem (3.10), there is a point $x \in X$ such that $x_d \propto x$ and $(g \circ f)(x) = z$. Since f is continuous, then by ([10], theorem (11.8)), $f(x_d) \propto f(x)$. Hence there is a point $f(x) \in Y$ such that $y_d \propto f(x)$ and $g(f(x)) = (g \circ f)(x) = z$. Thus $g : Y \to Z$ is an s*g-α-proper function.

**iii)** Let $(x_d)_{d \in D}$ be a net in X such that $f(x_d) \xrightarrow{s*g-\alpha} y \in Y$. Since g is s*g-α-irresolute, then by proposition (1.24), $g(f(x_d)) \xrightarrow{s*g-\alpha} g(y)$. But $g \circ f$ is s*g-α-proper, then by theorem (3.10), there is a point $x \in X$ such that $x_d \propto x$ and $(g \circ f)(x) = g(y)$. Since $(g \circ f)(x) = g(f(x)) = g(y)$ and since g is one-to-one, then $f(x) = y$. Thus $f : X \to Y$ is an s*g-α-proper function.

**3.17 Corollary:** Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces. If $f : X \to Y$ is an s*g-α-proper function, then the restriction of f to a closed subset F of X is an s*g-α-proper function of F into Y.

**Proof:** Since F is a closed set in X, then the inclusion function $\iota_F : F \to X$ is a proper function. Since $f : X \to Y$ is an s*g-α-proper function, then by theorem ((3.16),(i)), $f \circ \iota_F : F \to Y$ is an s*g-α-proper function. But $f \circ \iota_F = f \mid F$, thus the restriction function $f \mid F : F \to Y$ is an s*g-α-proper function.

**3.18 Corollary:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces. If $f: X \to Y$ is an s*g-α-perfect function, then the restriction of $f$ to a closed subset $F$ of $X$ is an s*g-α-perfect function of F into Y.

**Proof:** It is obvious.

**3.19 Proposition:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces and $f: X \to Y$ be an s*g-α-proper function. Then for each open subset $T$ of $Y$, the function $f_T : f^{-1}(T) \to T$ which agrees with f on $f^{-1}(T)$ is also s*g-α-proper.

**Proof:** Since $f: X \to Y$ is continuous, then so is $f_T$. To prove that $f_T \times I_Z : f^{-1}(T) \times Z \to T \times Z$ is s*g-α-closed for every topological space Z. Since $f$ is s*g-α-proper, then $f \times I_Z : X \times Z \to Y \times Z$ is s*g-α-closed for every topological space Z. Since $f_T \times I_Z = (f \times I_Z)_{T\times Z}$ and $T \times Z$ is an open subset of $Y \times Z$, then by proposition (2.11), $f_T \times I_Z$ is an s*g-α-closed function. Thus $f_T : f^{-1}(T) \to T$ is an s*g-α-proper function.

**3.20 Corollary:** Let $(X,\tau)$ and $(Y,\tau')$ be topological spaces and $f: X \to Y$ be an s*g-α-perfect function. Then for each open subset $T$ of $Y$, the function $f_T : f^{-1}(T) \to T$ which agrees with f on $f^{-1}(T)$ is also s*g-α-perfect.

**Proof:** It is obvious.

**3.21 Proposition:** If $f_1 : X_1 \to Y_1$ is a proper function and $f_2 : X_2 \to Y_2$ is an s*g-α-proper function. Then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is an s*g-α-proper function.

**Proof:** Let $Z$ be any topological space. We can write $f_1 \times f_2 \times I_Z$ by the composition of $I_{Y_1} \times f_2 \times I_Z$ and $f_1 \times I_{X_2} \times I_Z$. Since $f_1$ is proper, then $f_1 \times I_{X_2} \times I_Z$ is closed. Since $f_2$ is s*g-α-proper, then $I_{Y_1} \times f_2 \times I_Z$ is s*g-α-closed, hence by theorem ((2.9),(i)), $(I_{Y_1} \times f_2 \times I_Z) \circ (f_1 \times I_{X_2} \times I_Z)$ is s*g-α-closed. But $f_1 \times f_2 \times I_Z = (I_{Y_1} \times f_2 \times I_Z) \circ (f_1 \times I_{X_2} \times I_Z) \Rightarrow f_1 \times f_2 \times I_Z$ is s*g-α-closed. Thus $f_1 \times f_2$ is an s*g-α-proper function.

**3.22 Theorem:** Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be functions such that $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ is an s*g-α-proper function. Then $f_1$ and $f_2$ are s*g-α-proper.

**Proof:** Let Z be any topological space. To prove that $f_2 \times I_Z : X_2 \times Z \to Y_2 \times Z$ is s*g-α-closed. Let F be a closed set in $X_2 \times Z$ and $G = (f_2 \times I_Z)(F)$. To prove that G is s*g-α-closed in $Y_2 \times Z$. Since $X_1 \neq \phi$, then $X_1 \times F$ is closed in $X_1 \times X_2 \times Z$. Since $f_1 \times f_2$ is s*g-α-proper, then $(f_1 \times f_2 \times I_Z)(X_1 \times F) = f_1(X_1) \times G$ is s*g-α-closed in $Y_1 \times Y_2 \times Z$ i.e. $\overline{f(X_1) \times G}^{s*g0} \subseteq f_1(X_1) \times G$. But by proposition (1.9), we have $\overline{f_1(X_1)}^{s*g\circ} \times \overline{G}^{s*g\circ} \subseteq \overline{f_1(X_1) \times G}^{s*g\circ} \subseteq f_1(X_1) \times G \Rightarrow \overline{G}^{s*g\circ} \subseteq G$. Hence by proposition (1.5), $G = (f_2 \times I_Z)(F)$ is an s*g-α-closed set in $Y_2 \times Z$. Therefore $f_2 \times I_Z$ is an s*g-α-closed function. Thus $f_2$ is an s*g-α-proper function. By the same way we can prove that $f_1$ is an s*g-α-proper function.

**3.23 Proposition:** If X is any compact topological space and Y is any topological space, then the projection $pr_2 : X \times Y \to Y$ is an s*g-α-proper function.

**Proof:** $pr_2$ factorizes into $X \times Y \xrightarrow{h} Y \times X \xrightarrow{I_Y \times f} Y$, where $h(x,y) = (y,x)$. h is a homeomorphism, hence h is proper. Since X is a compact space, then by corollary (3.11), $f : X \to \{p\}$ is s*g-α-proper, since $I_Y : Y \to Y$ is proper, then by proposition (3.21), $Y \times X \xrightarrow{I_Y \times f} Y \times \{p\} \cong Y$ is s*g-α-proper. Therefore by theorem ((3.16),(i)), $pr_2 = (I_Y \times f) \circ h$ is an s*g-α-proper function.

Now, we shall explain the relationships between the s*g-α-proper functions and the s*g-α-compact functions.

**3.24 Proposition:** Every s*g-α-proper function is an s*g-α-compact function.

**Proof:** Let $f:(X,\tau)\to(Y,\tau')$ be an s*g-α-proper function. To prove that f is an s*g-α-compact function. Let K be an s*g-α-compact subset of Y and let $\{U_\alpha\}_{\alpha\in\Lambda}$ be any open cover of $f^{-1}(K)$. Since f is an s*g-α-proper function, then by theorem (3.10), $f^{-1}(k)$ is a compact set in X for each $k\in K$. But $f^{-1}(k)\subseteq f^{-1}(K)\subseteq\bigcup_{\alpha\in\Lambda}U_\alpha$ ,thus there exists $n_k$ such that $f^{-1}(k)\subseteq\bigcup_{i=1}^{n_k}U_{\alpha_i}$ . Let $U_k=\bigcup_{i=1}^{n_k}U_{\alpha_i}$ , thus $f^{-1}(k)\subseteq U_k$ . Notice that for each $k\in K$, $k\in(Y\setminus f(X\setminus U_k))$. Hence $K\subseteq\bigcup_{k\in K}(Y\setminus f(X\setminus U_k))$, but K is an s*g-α-compact set in Y and the sets $(Y\setminus f(X\setminus U_k))$ are s*g-α-open. Thus there exists $k_1,k_2,....,k_j$ such that $K\subseteq\bigcup_{\alpha=1}^{j}(Y\setminus f(X\setminus U_{k_\alpha}))$. Hence $f^{-1}(K)\subseteq\bigcup_{\alpha=1}^{j}U_{k_\alpha}$ . Therefore $f^{-1}(K)$ is a compact set in X. Hence the function $f:(X,\tau)\to(Y,\tau')$ is an s*g-α-compact function.

The converse of proposition (3.24) may not be true in general. Consider the following example:

**3.25 Example:** Let $f:(\Re,\mu)\to(\Re,\tau)$ be a function from the usual topological space $(\Re,\mu)$ to a topological space $(\Re,\tau)$, where $\tau=\{\phi,\Re,\{0\}\}$ such that $f(x)=x$ for each $x\in\Re$. Then f is not an s*g-α-proper function, since $\{0\}$ is a closed set in $(\Re,\mu)$, but $f(\{0\})=\{0\}$ is not an s*g-α-closed set in $(\Re,\tau)$. While f is an s*g-α-compact function.

**3.26 Proposition:** Let $f:(X,\tau)\to(Y,\tau')$ be a continuous function such that Y is an s*g-α-K-space. Then f is an s*g-α-proper function if and only if f is an s*g-α-compact function.

**Proof:** $\Rightarrow$ By proposition (3.24), every s*g-α-proper function is an s*g-α-compact function.

Conversely, since f is an s*g-α-compact function and $\{y\}$ is an s*g-α-compact set in Y, then by definition (2.16), $f^{-1}(y)$ is a compact set in X for each $y\in Y$. Now, to prove that f is an s*g-α-closed function. Let F be a closed set in X, to prove that $f(F)$ is an s*g-α-closed set in Y. Suppose that K is an s*g-α-compact set in Y, then $f^{-1}(K)$ is a compact set in X. But $F\cap f^{-1}(K)$ is a compact set in X and f is continuous, then by ([10], theorem (17.7)), $f(F\cap f^{-1}(K))$ is a compact set in Y. Since $f(F\cap f^{-1}(K))=f(F)\cap K$, then $f(F)\cap K$ is a compact set in Y. Therefore by definition (1.19), $f(F)$ is a compactly s*g-α-closed set in Y. Since Y is an s*g-α-K-space, then by definition (1.21), $f(F)$ is an s*g-α-closed set in Y. Thus by theorem (3.10), f is an s*g-α-proper function.

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