

Some Results on Special Kind of Algebra

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Abstracts

In this paper we define a new class of algebra we call it a non associative seminear ring with BCK algebra and define a non associative sub seminear ring with BCK algebra , then we study and prove some properties of them .

1) Introduction

The notation of BCK-algebra was introduced first in 1966 by Y.Iami and K-Iseki [1] ,in in the same year, K-Iseki [2] introduced the notion of BCI- algebra which is a generalization of a BCK-algebra . In 1967 V. G. Van Hoorn and B. Van Root Selaar[9] introduced the concept of seminear-rings and discussed general theory of seminear-rings . We introduce a new class of algebra called a special kind of non associative seminear-ring with BCK algebra where we define as follows : Let (X, • , *) be a non-empty set with two binary operations '∗' and '•' satisfying the following conditions :

a.) (X, •) is a semigroup .

b.) (X, ∗, 0) is a BCK algebra.

c.) $(a \bullet b) * e = (a * e) \bullet (b * e)$ for all $a, b, e \in X$

e.) $0 \bullet x = x \bullet 0 = x$ for all $x \in X$

then we say that X is a special kind of non associative seminear-ring with BCK algebra**(** SNAK seminear-ring) ,then we define a special kind of non associative sub seminear-ring with BCK algebra we denoted by SNASK seminear-ring , then we study and prove some properties of them .

keywords : semigroup, BCK –algebra, seminear-ring , non associative seminear -ring

2) preliminary

In this section we view some concepts we needed in this paper .

Definition2.1 [5],[6] ,[7]

Let S be a non-empty set. S is said to be a **semigroup** if on S is defined a binary operation '•' such that for all $a, b \in S$, $a \bullet b \in S$ and $(a \bullet b) \bullet c = a \bullet (b \bullet c)$ for all $a, b, c \in S$.

Definition 2.2[4]

The **direct product** S ×T of two semigroups S and T is defined by

$(x_1, y_1) \bullet (x_2 , y_2) = (x_1 \bullet x_2 , y_1 \bullet y_2)$ where $x_1,x_2 \in S$, $y_1,y_2 \in T)$.

It is easy to show that the direct product is a semigroup

Definition (2.3) [4]

A semigroup S with finite number of elements is called a **finite semigroup** and its order is finite and it is denoted by o(S) = |S|. If |S| is infinite we say S is a semigroup of **infinite order**.

Definition 2.4 [5],[4]

Let (S , •) be a semigroup. P a non-empty proper subset of S is said to be a **subsemigroup** if (P, .) is a semigroup

Definition 2.5 [3]

Let X be a semigroup and x an element of X . An element e of X is a **left identity** of x if $e \bullet x = x$, a **right identity** of x if $x \bullet e = x$, an **identity** of x if $x \bullet e = e \bullet x = x$

Definition 2.6 [5] , [4]

A semigroup which has an identity element $e \in S$ is called **a monoid**, if e is such that $x \bullet e = e \bullet x = x$ for all $x \in S$.

Definition 2.7 [9]

A non empty set R with two binary operations + (addition) and • (multiplication) is called a **seminear-ring** , if it satisfies the following axioms:

(1) (R, +) and (R, •) are semigroups,

(2) $(x + y) \bullet z = x \bullet z + y \bullet z$ for all x, y, z ∈ R.

Precisely speaking, it is a right seminear-ring because it satisfies the right distributive law.

Definition 2.8 [5]

Let (N, + , •) be a non-empty set with two binary operation '+' and '•' satisfying the following conditions :

a. (N, +) is a semigroup.

b. (N, •) is a groupoid.

c. $(a + b) \bullet e= a \bullet e + b \bullet e$ for all a, b, c ∈ N; (N, +, •) is called the **right seminear-ring which is non-associative.**

If we replace (c) by $a \bullet (b + e) = a \bullet b + a \bullet e$ for all a , b, e ∈ N Then (N, +, •) is a non-associative left seminear-ring . In this text we denote by (X, +, •) a non-associative right seminear-ring and by default of notation call X just a non-associative seminear-ring

Definition 2.9 [5]

Let (N, +, •) be a seminear-ring which is not associative . A subset P of N is said to be a **subseminear-ring** if (P, +, •) is a seminear-ring.

Definition 2.10 [5]

Let (N, +, •) be a non-associative seminear-ring; we say N is a **P-seminear-ring** if

$(x \bullet y) \bullet x = x \bullet (y \bullet x)$ for all x, y ∈ N.

Definition 2.11 [5]

we call a non-associative seminear-ring N to be a **Bol seminear-ring** if

$((x \bullet y) \bullet z) \bullet y = (x \bullet (y \bullet z) \bullet y)$ for all x, y, z ∈ N .

Definition (2.12) [11],[12]

An algebraic system (X, ∗, 0) is called a **BCK algebra** if it satisfies the following conditions:

1) $((x * y) * (x * z)) * (z * y) = 0$,

2) $(x * (x * y)) * y = 0$,

3) $x * x = 0$,

4) $0 * x = 0$

5) if $x * y = 0$ and $y * x = 0$ then $x = y$, $\forall$ x, y, z $\in X$.

Remarks (2.13) [8]

Let X be a BCK algebra then :

A partial ordering " ≤ " on X can be defined by $x \le y$ if and only if $x * y = 0$**.**

A BCK-algebra X has the following properties:

1) $x * 0 = x$.

2) if $x*y=0$ and $y*z=0$ imply $x*z=0$.

3) if $x*y=0$ implies $(x*z)*(y*z)=0$ and $(z*y)*(z*x)=0$.

4) $(x*y)*z=(x*z)*y$.

5) $(x*y)*x=0$.

6) $x*(x*(x*y))=x*y$.

7) if $(x*y)*z=0$ implies $(x*z)*y = 0$.

8) $[(x*z)*(y*z)]*(x*y)=0$.

9) [((x*z)*z)*(y*z)]*[(x*y)*z] = 0 . for all x, y, z ∈ X

3) Main Results :

In this section we define a new class of algebra , we call it a special kind of non associative seminear-ring with BCK algebra then we study and prove some of properties .

Definition 3.1

Let X be a non-empty set with two binary operations '∗' and '•' satisfying the following conditions :

a.) (X, •) is a semigroup .

b.) (X, ∗, 0) is a BCK algebra.

c.) (x • y) ∗ z = (x ∗ z) • (y ∗ z) for all x, y, z ∈ X which is called the distributive law

e.) 0 • x = x • 0 = x for all x∈X

Then ; (X , • , ∗ , 0) is called **Special Kind Of Non Associative Seminear-Ring With BCK Algebra**, we denoted by **SNAK seminear-ring**

Example: 3.3

Let X={0,1,2,3}with two binary operation • and ∗ are defined by the following tables :

•	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	2
3	3	3	2	2

∗	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	0	0

Then by usual calculation we can prove that (X ,•, ∗,0) is a SNAK seminear-ring

Example:3.2

Let X={0,1,2,3}with two binary operation • and ∗ are defined by the following tables :

•	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	3	1
3	3	3	1	2

∗	0	1	2	3
0	0	0	0	0
1	1	0	0	3
2	2	2	0	2
3	3	0	0	0

Then (X , •, ∗,0) is not SNAK seminear-ring since 1, 2 , 3 ∈ X but (1 • 2) ∗3 = 2 ∗3 = 2 ≠ (1 ∗ 3) • (2 ∗ 3) = 3 • 2 = 1

Remark 3.3

Let X be a SNAK seminear-ring then $x^n \leq x$ ∀ x ∈ X

Proof

Let X be a SNAK seminear-ring since x ∗ x = 0 so x ≤ x

$x^2 * x = x * x \bullet x * x$ [by c of definition 3.1]

$= 0 \bullet 0 = 0$ [by e definition 3.1]

so $x^2 \leq x$ .By mathematical induction we have $x^n \leq x \ \forall x \in X, n \in N$

Proposition 3.4

Let X be a SNAK seminear-ring and x • y = 0 for some x , y ∈ X then x ∗ y = 0 and y ∗ x = 0 , the converse is not true

Proof

Let X be a SNAK seminear-ring and $x \bullet y = 0 \Rightarrow (x \bullet y) * x = 0 * x$

$\Rightarrow x * x \bullet y * x = 0 \bullet (y * x) = (y * x) \Rightarrow y * x = 0$ by similar way we have $x * y = 0$, we will show that the converse is not true by the following example:

Let X={0,1,2,3}with two binary operation $\bullet$ and $*$ are defined by the following tables :

•	0	1	2	3
0	0	1	2	3
1	1	1	1	1
2	2	1	2	3
3	3	3	3	3

*	0	1	2	3
0	0	0	0	0
1	1	0	3	0
2	2	0	0	0
3	3	0	3	0

Then, since $1,3 \in X$ and $1 * 3 = 0$ and $3 * 1 = 0$ but $1 \bullet 3 = 1 \neq 0$

proposition (3.5)

let X be SNAK seminear-ring then

1) $0 \bullet 0 = 0$

2) $((a \bullet b) \bullet c) * b = (a \bullet c) * b$ for all $a,b,c \in X$

3) $((a \bullet b) \bullet c) * a = (b \bullet c) * a$ for all $a,b,c \in X$

4) $((a \bullet b) \bullet c) * c = (a \bullet b) * c$ for all $a,b,c \in X$

proof

let X is a SNAK seminear-ring

1) since $0 \in X$ and $0 \bullet x = x \bullet 0 = x \quad \forall x \in X$ so $0 \bullet 0 = 0$

2) let $a , b , c \in X$ then

$((a \bullet b) \bullet c) * b = (a \bullet (b \bullet c) * b$ [since $(x , \bullet)$ is semigroup]

$= a * b \bullet ((b \bullet c) * b$

$= (a * b) \bullet [(b * b) \bullet (c * b)]$ [by c of definition 3.1]

$= a * b \bullet [0 \bullet (c * b)]$ [by e of definition 3.1]

$= a * b \bullet c * b = (a \bullet c) * b$

4)In a similar way we can prove 3,4.

Proposition 3.6

Let X be a SNAK seminear-ring then X is not P-seminear-ring

Proof

Let X be a SNAK seminear-ring and suppose that X is a P-seminear-ring

$\Rightarrow (x * y) * x = x * (y * x) \quad \forall\ x , y \in X$

$\Rightarrow \quad x * (y * x) = (x * x) * y$ [by 4 of Remarks 2.13]

$= 0 * y = 0 \quad \forall\ x , y \in X$ [by 1 of Remarks 2.13]

since it is true $\forall\ x , y \in X$ so if $y = x \Rightarrow x * 0 = 0$

$\Rightarrow x = 0$ contradiction $\Rightarrow$ X is not P-seminear-ring

Proposition 3.7

Let X be a SNAK seminear-ring then X is not Bol seminear-ring

Proof

Let X be a SNAK seminear-ring and suppose that X is Bol seminear-ring

$\Rightarrow ((x*y)*z)*y = x*((y*z)*y) \quad \forall\ x , y , z \in X$ [X is SNAK seminear-ring]

$\Rightarrow ((x*y)*y)*z = (x*(0*z))$

$= x * 0 = x$

Since it is true $\forall$ x , y $\in$ X so it is true if x = y $\Rightarrow$ (0 $*$ y) $*$ z= x $\Rightarrow$ x = 0 contradiction

$\Rightarrow$ X is not Bol seminear-ring

Proposition 3.8:

Let S , T be a SNAK seminear-ring then S×T = {(s , t) : s$\in$S , t$\in$T} is a SNAK seminear-ring and , where the binary operations '•' and '*' defined by the following :

$(a_1 , b_1) \bullet (a_2 , b_2) = (a_1 \bullet a_2 , b_1 \bullet b_2)$

$(a_1 , b_1) * (a_2 , b_2) = (a_1 * a_2 , b_1 * b_2)$

for all (a_1,b_1) , $(a_2,b_2) \in S\times T$

Proof

Let S and T a SNAK seminear-ring so [by 2.1.1] S and T are semigroup then S × T are semigroup [by Definition 2.2] since S and T are BCK algebra then S × T it is easy to prove S × T a BCK where

(0 , 0) $\in$ S × T since 0 $\in$ S and 0 $\in$ T also for each x = (a , b) $\in$ S × T we have

(a , b) $*$ (a , b) = (a $*$ a , b $*$ b) = (0 , 0) [by 4 of definition 2.1 2]

(a , b) $*$ (0 , 0) = (a $*$ 0) , (b $*$ 0) = (a , b) [by 1 0f Remark 2.13]

and (0 , 0) $*$ (a , b) = (0 $*$ a , 0 $*$ b) = (0 , 0) [by 4 of definition 2. 12]

and all condition of BCK algebra are satisfied .

Now , to proof that (x • y) $*$ z = (x $*$ z) • (y $*$ z) for all x, y, z $\in$ S × T

Let x = (a_1 , b_1) , y = (a_2 , b_2) , z = $(a_3 , b_3) \in S \times T$

Where $a_1, a_2, a_3 \in S$ and $b_1 , b_2 , b_3 \in T$ then

$(x \bullet y) * z =[(a_1 , b_1) \bullet (a_2 , b_2)] * (a_3 , b_3)$

$= ((a_1 \bullet a_2) , (b_1 \bullet b_2)) * (a_3 , b_3)$

$= ((a_1 \bullet a_2) * a_3 , (b_1 \bullet b_2) * b_3)$ [since S and, T are SNAK seminear-ring]

$= ((a_1*a_3) \bullet (a_2*a_3) , (b_1 *b_3) \bullet (b_2 *b_3))$

$= (((a_1*a_3) , (b_1 *b_3)) \bullet ((a_2*a_3) , (b_2 *b_3)))$

$= ((a_1 , b_1) * (a_3 , b_3) \bullet (a_2 , b_2) * (a_3 , b_3)) = (x *z) \bullet (y *z)$

Now,,it is clear that

(0 , 0) • (a , b) = (0 • a , 0 • b) = (a , b) [by e Definition 3.1]

= (a , b) • (0 , 0)

Then S × T is a SNAK seminear-ring .

Proposition 3.9

If (X , • , $*$, 0) be a SNAK seminear-ring there is no (P $\neq$ Ø) $\subseteq$ X such that (P , • , $*$, 0) is a seminear-ring

Proof

Suppose that (P $\neq$ Ø) $\subseteq$ X is a seminear-ring and

Let 0$\neq$x $\in$ P Then (x $*$ x) $*$ x = 0 $*$ x = 0 [by 4 definition 2.12]

but x $*$ (x $*$ x) = x $*$ 0 = x [by 1 of Remark 2.13]

So (x $*$ x) $*$ x $\neq$ x $*$ (x $*$ x) for each x $\in$ P

So (P , $*$) will not be a semigroup so (P , • , $*$, 0) not a seminear-ring

Example 3.10:

Let X={0,1,2}with two binary operations '•' and '$*$' are defined by the following tables :

•	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	1	0

Then by usual calculation we can prove that (X, • , ∗ ,0) is a SNAK seminear-ring .

Let P = { 0 , 1 } ⊆ X then P is not a seminear-ring since it is not associative where (1∗ 1) ∗ 1 = 0 ≠ 1 ∗ (1 ∗ 1) = 1 .

Definition 3.11 :

Let (X , •, ∗ ,0) is a SNAK seminear-ring a non empty subset P of X is said to be **a Special Kind of Non Associative Sub Seminear-Ring With BCK Algebra** if (P, •, ∗,0) is a SNAK seminear-ring we denoted by **SNASK** seminear-ring.

Example 3.12 :

Let X={0,1,2,3}with two binary operations '•' and '∗' are defined by the following tables :

•	**0**	**1**	**2**	**3**
0	0	1	2	3
1	1	1	1	1
2	2	1	2	1
3	3	1	1	1

∗	**0**	**1**	**2**	**3**
0	0	0	0	0
1	1	0	2	0
2	2	0	0	0
3	3	0	2	0

Then by usual calculation we can prove that (X , • , ∗, 0) is a SNAK seminear-ring

Let P = { 0 , 1 , 2 } ⊆ X then by usual calculation we can prove that P is a SNASK seminear –ring from above tables.

Proposition 3.12

Let (X_1, • , ∗ , 0) , (X_2 , • , ∗ , 0) be a SNASK seminear-ring of X such that $X_1 \cap X_2 \neq \emptyset$ then The following are SNASK seminear-ring

1) ($X_1 \cap X_2$, • , ∗ , 0)

2) ($X_1 \cup X_2$, • , ∗ , 0) such that $X_1 \subseteq X_2$ or $X_2 \subseteq X_1$

Proof :

1) a) Let X_1 , X_2 be a SNASK seminear-ring Since (X_1 , •) , (X_2 , •) are semigroup so it is easy to prove that ($X_1 \cap X_2$, •) is a semigroup

b) To prove that ($X_1 \cap X_2$, ∗) is a BCK algebra

since 0 ∈ X_1 and X_2 so 0 ∈ $X_1 \cap X_2$ [since X_1 , X_2 are SNASK seminear-ring]

Let x , y , z ∈ $X_1 \cap X_2$

⇒ x , y , z ∈ X_1 and x , y , z ∈ X_2 since X_1 , X_2 is a BCK algebra

So it is easy to prove that all the conditions of definition BCK satisfies for all x , y , z ∈ X_1 , and X_2 then satisfies to $X_1 \cap X_2$

⇒ ($X_1 \cap X_2$, ∗ , 0) is a BCK algebra

c) Let x , y , z ∈ $X_1 \cap X_2$ then x , y , z ∈ X_1 and x , y , z ∈ X_2 since X_1 , X_2 are SNASK seminear-ring

⇒ (x • y) ∗ z = (x ∗ z) • (y ∗ z) for all x , y , z ∈ $X_1 \cap X_2$

d) let x ∈ $X_1 \cap X_2$ then x ∈ X_1 and x ∈ X_2

so x • 0 = 0 • x = x [since X_1 , X_2 are a SNASK seminear-ring] Then ($X_1 \cap X_2$, • , ∗ , 0) is a SNASK seminear-ring

2) let X_1 and X_2 be a SNASK seminear-ring such that $X_1 \subseteq X_2$

Since $X_1 \subseteq X_2 \Rightarrow$ $X_1 \cup X_2 = X_2$ and X_2 is a SNASK seminear-ring so ($X_1 \cup X_2$, • , ∗ , 0) is a SNAK seminear-ring .If $X_2 \subseteq X_1 \Rightarrow$ $X_2 \cup X_1 = X_1$ and X_1 is SNASK seminear-ring so

($X_1 \cup X_1$, • , ∗ , 0) is a SNASK seminear-ring .

Proposition 3.13:

Let X be a SNAK seminear-ring and let $a \in X$ then $£_a = \{ x \in X : x * a = 0\}$ is a SNASK seminear-ring **Proof**

1) let X be a SNAK seminear-ring. It is clear that $£_a \subseteq X$ and $£_a \neq \emptyset$

since $a \in £_a$ where $a * a = 0$. To prove that $(£_a , \bullet)$ is a semigroup

a) Let $x , y \in £_a \Rightarrow x * a = 0$ and $y * a = 0$

$\Rightarrow (x \bullet y) * a = (x * a) \bullet (y * a)$ [since X be a SNAK seminear-ring]

$= 0 \bullet 0 = 0$

$\Rightarrow x \bullet y \in £_a$ then $£_a$ is closed under $(\bullet)$

b) let $x , y , z \in £_a$ but $£_a$ subset of X so $x , y , z \in X$ and X is a SNAK seminear-ring

so $(x \bullet y) \bullet z = x \bullet (y \bullet z)$ for all $x , y , z \in X$ so $£_a$ is associative for all $x, y, z \in £_a$ then $£_a$ is semigroup.

2) to prove that $£_a$ is a BCK algebra

Since $0 * a = 0 \Rightarrow 0 \in £_a$. Now ,since $£_a \subseteq X$ and $0 \in £_a$ so it is easy to prove that $(£_a , * , 0)$ is a BCK algebra. Now , let $x, y, z \in £_a$ so $x, y, z \in X$

$\Rightarrow (x \bullet y) * z = (x * z) \bullet (y * z)$ [by c of definition 3.1]

and $x \bullet 0 = 0 \bullet x = x$ for all $x \in £_a$ [by e of definition 3.1]

Then $£_a$ is SNASK seminear-ring

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