Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.1, 2015

Application of Elzaki Transform Method on Some Fractional Differential Equations

M. Khalid*, Mariam Sultana, Faheem Zaidi and Uroosa Arshad
Department of Mathematical Sciences,
Federal Urdu University of Arts, Sciences & Technology, Karachi-75300
*E-mail of corresponding author: khalidsiddiqui@fuuast.edu.pk

Abstract In the paper, we begin by introducing the origin of fractional calculus and the consequent application of the Elzaki transform on fractional derivatives. The Elzaki transformation may be used to solve mathematical problems without resorting to a new frequency domain. Once we establish this connection firmly in the general setting, we turn our attention to the application of the Elzaki transform method to some non-homogeneous fractional, ordinary differential equations. Ultimately, we acquire the graphical solution of the problem by using Matlab 2013a, developed by MathWorks
Key Words: Elzaki Transform, Fractional Differential Equation, Linear and Non-linear, Initial Value Problem, Non-homogenous

1. Introduction

In the literature, there are many integral transforms being used in engineering and applied sciences. It is undoubtedly an effective tool for solving differential equations, integral equations. The fact that makes the integral transform so effective is that it can convert systems of differential equations and integral equations into algebraic equations.

Initially, the Elzaki transform was introduced by Elzaki [1] as a modification of the classical Sumudu Transform. The author [1-5] derived this transform for ordinary and partial derivatives. The main purpose of the presentation of this paper is to demonstrate how applicable the Elzaki transform is in solving fractional differential equation.

2 Fundamental Properties of ETM and Fractional Calculus

In this section, we will shed light on some properties of Elzaki Transformation and Fractional Calculus.

2.1 Fundamental Facts of the Elzaki Transformation Method

The Elzaki transform of the function's belonging to a class B, where $B=\left\{f(t) \mid \exists M, k_1, k_2>0 \text{ such that } |f(t)|<Me^{|t|/k_j} \text{ if } t\in(-1)^j\times[0,\infty)\right\}$, where $f(t)$ is denoted by $E[f(t)]=T(u)$ and is defined [1, 2] as

$$T(u)=u^2\int_0^\infty f(ut)e^{-t}dt\;;\quad k_1,k_2>0 \tag{1}$$

Or equivalently

$$T(u)=u\int_0^\infty f(t)e^{-t/u}dt\quad;\quad u\in(k_1,k_2) \tag{2}$$

The following results can be obtained from the definition and simple calculations

1) $$E[f'(t)]=\frac{T(u)}{u}-uf(0) \tag{3}$$

2) $$E[f''(t)]=\frac{T(u)}{u^2}-f(0)-uf'(0) \tag{4}$$

3) $$E[t\,f(t)]=u^2\frac{d}{du}\frac{T(u)}{u}-uT(u) \tag{5}$$

4) $E\left[t^2 f(t)\right] = u^4 \frac{d^2}{du^2} T(u)$ (6)

5) $E[t f'(t)] = u^2 \frac{d}{du}\left[\frac{T(u)}{u} - uf(0)\right] - u\left[\frac{T(u)}{u} - uf(0)\right]$ (7)

6) $E\left[t^2 f'(t)\right] = u^4 \frac{d^2}{du^2}\left[\frac{T(u)}{u} - uf(0)\right]$ (8)

7) $E[t f''(t)] = u^2 \frac{d}{du}\left[\frac{T(u)}{u^2} - f(0) - uf'(0)\right] - u\left[\frac{T(u)}{u^2} - f(0) - uf'(0)\right]$ (9)

8) Song, Y. & Kim,H. (2014|0, Legendre's Equation Expressed by the Initial Value by Using Integral Transforms, Applied Mathematical Sciences, 8 (2014) 531 – 540. For $E[f(t)] = T(u)$ (10)

9) $E\left[t^n\right] = n!u^{n+2}$ (11)

2.2 Fundamental Facts of the Fractional Calculus:

Firstly, we mention some of the fundamental properties of the fractional calculus. Fractional derivatives as well as integral definition may different, but the most widely used definitions are those of Abel-Riemann [6]. Following the nomenclature in [7], a derivative of fractional order in the Abel-Riemann [6] is defined by

$$D^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma[m-\alpha]} \frac{d^m}{dt^m} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau & m-1 < \alpha \le m \\ \frac{d^m}{dt^m} f(t) & \alpha = m \end{cases} \quad (12)$$

Where $m \in Z^+$ and $\alpha \in R^+$ [6]. D^{α} Is a derivative operator here and

$$D^{-\alpha}[f(t)] = \frac{1}{\Gamma[\alpha]} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \quad , \ t > 0 \quad , \ 0 < \alpha \le 1 \quad (13)$$

On the other hand, according to Abel-Riemann, an integral of fractional order is defined by implementing the integration operator J^{α} in the following manner

$$J^{\alpha}[f(t)] = \frac{1}{\Gamma[\alpha]} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau \ , \ t > 0 \quad , \ 0 < \alpha \quad (14)$$

When it come to some of the fundamental properties of fractional integration and fractional differentiation, then have been introduced to the literature by Podlubny [8]. Among these, we mention

$$J^{\alpha}\left[t^n\right] = \frac{\Gamma[1+n]}{\Gamma[1+n+\alpha]} t^{n+\alpha} \quad (15)$$

$$D^{\alpha}\left[t^n\right] = \frac{\Gamma[1+n]}{\Gamma[1+n-\alpha]} t^{n-\alpha} \quad (16)$$

Another main definition of the fractional derivative is that of Caputo [8, 9] who defined it by

$${}^{c}D^{\alpha}[f(t)] = \begin{cases} \frac{1}{\Gamma[m-\alpha]} \int_0^t \frac{f^m(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau \ , & m-1 < \alpha < m \\ \frac{d^m}{dt^m} f(t) \ , & \alpha = m \end{cases} \quad (17)$$

A fundamental feature of the Caputo fractional derivative [10] is that

$$J^{\alpha}\left[{}^{c}D^{\alpha} f(t)\right] = f(t) - \sum_{k=0}^{\infty} f^{(k)}\left(0^+\right) \frac{t^k}{k!} \quad (18)$$

Theorem 1:

If $T(u)$ is the Elzaki transformation of $f(t)$, we knows that the Elzaki transformation of the derivatives with integral order is given as follows [11]

$$E\left[\frac{d}{dt}f(t)\right]=\frac{T(u)}{u}-uf(0) \tag{19}$$

Proof:

Let us take the Elzaki transform [11] of $f'(t)=\frac{d}{dt}f(t)$ as follows

$$E\left[\frac{d}{dt}f(t)\right]=u\int_0^{\infty}e^{-t/u}\frac{d}{dt}f(t)dt=\lim_{p\to\infty}u\int_0^{p}e^{-t/u}\frac{d}{dt}f(t)dt$$

$$=\lim_{p\to\infty}u\left[e^{-t/u}\int_0^{p}\frac{d}{dt}f(t)dt-\int_0^{p}e^{-t/u}\left(-\frac{1}{u}\right)f(t)dt\right]$$

$$=\lim_{p\to\infty}u\left[e^{-t/u}\left|f(t)\right|_0^p+\frac{1}{u}\int_0^{p}e^{-t/u}f(t)dt\right]$$

$$=\lim_{p\to\infty}u\left[-f(0)+\frac{1}{u^2}\left\{u\int_0^{p}e^{-t/u}f(t)dt\right\}\right]$$

$$=\frac{T(u)}{u}-uf(0) \tag{20}$$

Equation (20) gives us the proof of theorem 1. When we continue in the same manner, we get the Elzaki transform of the second order derivative as follows [11];

$$E\left[\frac{d^2}{dt^2}f(t)\right]=E\left[\frac{d}{dt}\left\{\frac{d}{dt}f(t)\right\}\right]$$

$$E\left[\frac{d^2}{dt^2}f(t)\right]=\frac{1}{u}E\left[\frac{d}{dt}f(t)\right]-u\frac{d}{dt}f(t)\Big|_{t=0}$$

$$=\frac{1}{u}\left[\frac{T(u)}{u}-uf(0)\right]-u\frac{d}{dt}f(t)\Big|_{t=0}$$

$$=\frac{T(u)}{u^2}-f(0)-u\frac{d}{dt}f(t)\Big|_{t=0} \tag{21}$$

If we go on the same way, we get the Elzaki transform of the nth order derivative as follows:

$$T^{(n)}(u)=E\left[\frac{d^n}{dt^n}f(t)\right]$$

$$=\frac{T(u)}{u^n}-\sum_{k=0}^{n-1}u^{2-n+k}f^{(k)}(t)\Big|_{t=0} \qquad \text{for } n\geq 1$$

Or

$$E\left[\frac{d^n}{dt^n}f(t)\right]=u^{-n}\left[T(u)-\sum_{k=0}^{n-1}u^{k+2}\frac{d^n}{dt^n}f(t)\Big|_{t=0}\right] \tag{22}$$

Theorem 2:

If $T(u)$ is the Elzaki transformation of $f(t)$, one can take into consideration the Elzaki transform of the Riemann-Liouville derivative [10] as follow:

$$E\left[D^{\alpha}f(t)\right]=u^{-\alpha}\left[T(u)-\sum_{k=1}^{n}u^{\alpha-\mu+2}\left[D^{\alpha-k}f(t)\right]_{t=0}\right];\ -1<n-1\leq\alpha<n \qquad (23)$$

Proof: Let us take the Laplace transformation of $f'(t)=\frac{d}{dt}f(t)$ as follows:

$$L\left[D^{\alpha}f(t)\right]=S^{\alpha}T(s)-\sum_{k=0}^{n-1}S^{k}\left[D^{\alpha-k-1}(f(t))\Big|_{t=0}\right]$$

$$=S^{\alpha}T(s)-\sum_{k=0}^{n}S^{k-1}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]$$

$$=S^{\alpha}T(s)-\sum_{k=1}^{n}S^{k-2}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]$$

$$=S^{\alpha}T(s)-\sum_{k=1}^{n}\frac{1}{S^{-k+2}}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]$$

$$=S^{\alpha}T(s)-\sum_{k=1}^{n}\frac{1}{S^{\alpha-k+2-\alpha}}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]$$

$$=S^{\alpha}T(s)-\sum_{k=1}^{n}S^{\alpha}\frac{1}{S^{\alpha-k+2}}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]$$

$$L\left[D^{\alpha}f(t)\right]=S^{\alpha}\left[T(s)-\sum_{k=1}^{n}\left(\frac{1}{S}\right)^{\alpha-k+2}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]\right] \qquad (25)$$

Therefore, when we substitute $\frac{1}{u}$ for S, we get the Elzaki transformation of fractional order of $f(t)$ as follows:

$$E\left[D^{\alpha}f(t)\right]=\left(\frac{1}{u}\right)^{\alpha}\left[T(u)-\sum_{k=1}^{n}u^{\alpha-k+2}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]\right]$$

$$E\left[D^{\alpha}f(t)\right]=u^{-\alpha}\left[T(u)-\sum_{k=1}^{n}u^{\alpha-k+2}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]\right] \qquad (26)$$

3. Elzaki Transform Method on General Linear Fractional Differential Equation:

We will now apply ETM (Elzaki Transform Method) for solving Fractional Ordinary Differential Equations. We take into consideration a general linear ordinary differential equation with fractional order as follows:

$$\frac{\partial^{\alpha}U(t)}{\partial t^{\alpha}}=\frac{\partial^{2}U(t)}{\partial t^{2}}+\frac{\partial U(t)}{\partial t}+U(t)+c \qquad (27)$$

Being subject to the initial condition

$$U(0)=f(0) \qquad (28)$$

Then, we will obtain the analytical solutions of some of the fractional ordinary differential equation by using ETM. When we take the Elzaki Transformation of (27) under the terms of (22) and (26), we obtain the Elzaki transformation of (27) as follows:

$$E\left[\frac{\partial^{\alpha}U(t)}{\partial t^{\alpha}}\right]=E\left[\frac{\partial^{2}U(t)}{\partial t^{2}}\right]+E\left[\frac{\partial U(t)}{\partial t}\right]+E[U(t)]+E[c]$$

$$u^{-\alpha}\left[T(u)-\sum_{k=1}^{n}u^{\alpha-k+2}\left[D^{\alpha-k}(f(t))\Big|_{t=0}\right]\right]=\frac{1}{u^{2}}\left[T(u)-u^{2}f(0)-u^{3}\frac{\partial f(t)}{\partial t}\Big|_{t=0}\right]+$$

$$\frac{1}{u}\left[T(u)-u^{2}f(0)\right]+T(u)+c$$

$$T(u)-\sum_{k=1}^{n}u^{\alpha-k+2}\left[D^{\alpha-k}\left(f(t)\right)\Big|_{t=0}\right]=u^{\alpha-2}\left[T(u)-u^{2}f(0)-u^{3}\frac{\partial f(t)}{\partial t}\Big|_{t=0}\right]+$$
$$u^{\alpha-1}\left[T(u)-u^{2}f(0)\right]+u^{\alpha}T(u)+u^{\alpha}c$$

$$T(u)=u^{\alpha-2}T(u)-u^{\alpha}U(0)-u^{\alpha+1}\frac{\partial U(t)}{\partial t}\Big|_{t=0}+u^{\alpha-1}T(u)-u^{\alpha+1}U(0)+$$
$$u^{\alpha}T(u)+u^{\alpha}c+\sum_{k=1}^{n}u^{\alpha-u+2}\left[D^{\alpha-k}\left(U(t)\right)\Big|_{t=0}\right]$$

$$T(u)-u^{\alpha-2}T(u)-u^{\alpha-1}T(u)-u^{\alpha}T(u)=-u^{\alpha}U(0)-u^{\alpha+1}U(0)+\sum_{k=1}^{n}u^{\alpha-u+2}\left[D^{\alpha-k}\left(U(t)\right)\Big|_{t=0}\right]-$$
$$u^{\alpha+1}\frac{\partial U(t)}{\partial t}\Big|_{t=0}+u^{\alpha}c$$

$$T(u)\left[1-u^{\alpha-2}-u^{\alpha-1}-u^{\alpha}\right]=\sum_{k=1}^{n}u^{\alpha-u+2}\left[D^{\alpha-k}\left(U(t)\right)\Big|_{t=0}\right]-$$
$$u^{\alpha+1}\frac{\partial U(t)}{\partial t}\Big|_{t=0}-u^{\alpha}U(0)-u^{\alpha+1}U(0)+u^{\alpha}c$$

$$T(u)\left[1-u^{\alpha-2}-u^{\alpha-1}-u^{\alpha}\right]=t(u)-u^{\alpha}U(0)-u^{\alpha+1}U(0)+u^{\alpha}c$$

$$T(u)=\frac{1}{1-u^{\alpha-2}-u^{\alpha-1}-u^{\alpha}}\left\{t(u)-u^{\alpha}U(0)-u^{\alpha+1}U(0)+u^{\alpha}c\right\} \qquad (29)$$

Where $t(u)$ is defined by $\sum_{k=1}^{n}u^{\alpha-k+2}\left[D^{\alpha-k}\left(U(t)\right)\Big|_{t=0}\right]-u^{\alpha+1}\frac{\partial U(t)}{\partial t}\Big|_{t=0}$. When we take the inverse Elzaki transformation of (29) by using the inverse transform table in [1-5], we get the solution of (27) by using ETM as follows:

$$U(t)=E^{-1}\left[\frac{1}{1-u^{\alpha-2}-u^{\alpha-1}-u^{\alpha}}\left\{t(u)-u^{\alpha}U(0)-u^{\alpha+1}U(0)+u^{\alpha}c\right\}\right] \qquad (30)$$

4. Applications of ETM to Non-homogeneous Fractional Ordinary Differential Equation's:

In this section, we have applied Elzaki Transform Method to the non-homogeneous fractional ordinary differential equations as follows:

Example 1: Firstly, we consider the non-homogeneous fractional ordinary differential equation as follows [12]:

$$D^{\alpha}\left[U(t)\right]=-U(t)+\frac{2}{\Gamma[3-\alpha]}t^{2-\alpha}-\frac{1}{\Gamma[2-\alpha]}t^{1-\alpha}+t^{2}-t \quad ; \quad t>0 \quad ; \quad 0<\alpha\leq 1 \qquad (31)$$

With the initial condition being $U(0)=0$ (32)

In order to solve (31) by using ETM, when we take the Elzaki transform of both sides of (31), we get the Elzaki transform of (31) as follows:

$$E\left[D^{\alpha}\left[U(t)\right]+U(t)\right]=E\left[\frac{2}{\Gamma[3-\alpha]}t^{2-\alpha}-\frac{1}{\Gamma[2-\alpha]}t^{1-\alpha}+t^{2}-t\right] \qquad (32)$$

$$E\left[D^{\alpha}U(t)\right]+E\left[U(t)\right]=E\left[\frac{2}{\Gamma[3-\alpha]}t^{2-\alpha}\right]-E\left[\frac{1}{\Gamma[2-\alpha]}t^{1-\alpha}\right]+E\left[t^{2}\right]-E[t]$$

$$\frac{T(u)}{u^{\alpha}}-\frac{D^{\alpha-1}U(t)}{u}\Big|_{t=0}+T(u)=\frac{2}{\Gamma[3-\alpha]}E\left[t^{2-\alpha}\right]-\frac{1}{\Gamma[2-\alpha]}E\left[t^{1-\alpha}\right]+E\left[t^{2}\right]-E[t] \qquad (33)$$

$$\frac{T(u)}{u^{\alpha}}+T(u)=\frac{2}{\Gamma[3-\alpha]}u^{3-\alpha}\Gamma[3-\alpha]-\frac{1}{\Gamma[2-\alpha]}u^{2-\alpha}\Gamma[2-\alpha]+2!u^{4}-u^{3}$$

$$\left(\frac{1}{u^{\alpha}}+1\right)T(u)=2u^{4-\alpha}-u^{3-\alpha}+2u^{4}-u^{3}$$
$$\left(1+u^{\alpha}\right)T(u)=2u^{4}-u^{3}+2u^{4+\alpha}-u^{3+\alpha}$$
$$\left(1+u^{\alpha}\right)T(u)=(2u-1)\left(u^{3}+u^{3}u^{\alpha}\right)$$
$$\left(1+u^{\alpha}\right)T(u)=u^{3}(2u-1)\left(1+u^{\alpha}\right)$$
$$T(u)=u^{3}(2u-1)$$
$$T(u)=2u^{4}-u^{3} \quad (34)$$

When we take the inverse Elzaki Transform of (34), we get the analytical solution of (31) by ETM as follows:

$$E[U(t)]=2u^{4}-u^{3}$$
$$U(t)=t^{2}-t \quad (35)$$

If we take the corresponding values for some parameters into consideration, then the solution of (31) is in full agreement with the solution mentioned in [14].

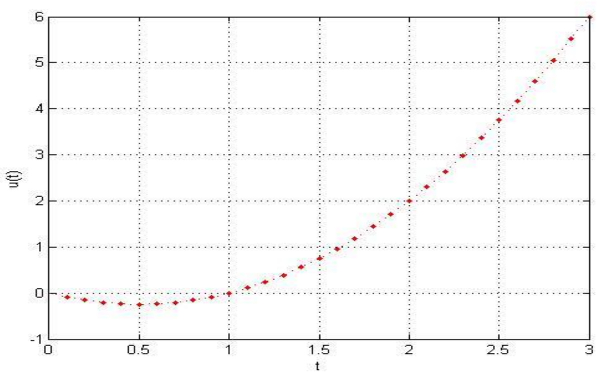

Fig 1: Analytical solution by Elzaki transform method of Example 1

Example 2:
Firstly, we consider the non-homogeneous fractional ordinary differential equation as follows [12]:

$$D^{0.5}[U(t)]+U(t)=t^{2}++\frac{\Gamma[3]}{\Gamma[2.5]}t^{1.5} \quad ; \quad t>0 \quad (36)$$

With the initial condition being $U(0)=0$ (37)

In order to solve (36) by using ETM, when we take the Elzaki transform of both sides of (36), we get the Elzaki transform of (36) as follows:

$$E\left[D^{0.5}U(t)\right]+E[U(t)]=E\left[t^{2}\right]+\frac{\Gamma[3]}{\Gamma[2.5]}E\left[t^{1.5}\right]$$
$$E\left[D^{0.5}U(t)\right]+E[U(t)]=E\left[t^{2}\right]+1.5045\,E\left[t^{1.5}\right]$$
$$\frac{T(u)}{u^{0.5}}-\left.\frac{D^{\alpha-1}[U(t)]}{u}\right|_{t=0}+T(u)=2!u^{4}+2u^{3.5}$$
$$\frac{T(u)}{u^{0.5}}+T(u)=2!u^{4}+2u^{3.5}$$
$$\left(\frac{1}{u^{0.5}}+1\right)T(u)=2u^{4}+2u^{3.5}$$
$$\left(1+u^{0.5}\right)T(u)=2u^{4.5}+2u^{4}$$

$$T(u) = \frac{2u^{4.5} + 2u^4}{1 + u^{0.5}}$$

$$T(u) = \frac{2u^4 \left(u^{0.5} + 1\right)}{1 + u^{0.5}}$$

$$T(u) = 2u^4 \qquad (38)$$

When we take the inverse Elzaki Transform of (38) by the inverse transform table, we get the analytical solution of (36) by using ETM as follows:

$$T(u) = E(U(t)) = 2u^4$$

$$U(t) = t^2 \qquad (39)$$

The solution (39) obtained by using the Elzaki transform method for (36) has been checked by the Matlab 2013.

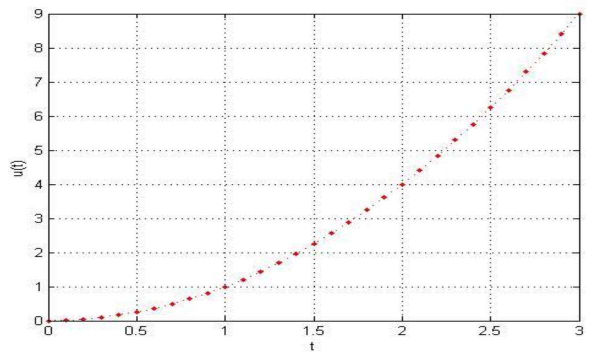

Fig 2: Analytical solution by Elzaki transform method of Example 2

4. Concluding Remarks

Various methods have been developed, preceding this study, in order to derive approximate solutions to a number of fractional differential equations. In the course of this paper, non-homogeneous, fractional and ordinary differential equations have been addressed and solved by using the Elzaki transform after yielding related formulae for fractional integrals, derivatives, and the Elzaki transform of Fractional Ordinary Differential Equations. The Elzaki technique may be applied to solve multiple types of problems, such as initial-value problems and boundary-value problems in applied sciences, engineering fields, mathematical physics, and aerospace sciences. In consequence, this newly hatched approach has been implemented successfully on fractional ordinary differential equations, which proves to be interesting. As such and practically so, it augments the library of integral transform approaches. There remains little doubt, based on our findings demonstrated in Figures 1 and 2, that the ETM technique remains a direct, strong and valuable tool for the solution of same fractional differential equations.

References

Tarig M. Elzaki and Salih M. Elzaki. (2011), “The new transform Elzaki Transform”, *Global Journal of Pure and Applied Mathematics*, 1: 57-64.

Tarig M. Elzaki and Salih M. Elzaki. (2011), “On the connections between Laplace and Elzaki transforms”, *Advances in Theorotical and Applied mathematics*, 1: 1-10.

Tarig M. Elzaki and Salih M. Elzaki. (2011), “On the Elzaki transforms and system of partial diffrential equations”, *Advances in Theoretical and Applid Mathematics*, 1: 115-123.

Tarig M. Elzaki and Salih M. Elzaki. (2011), “On the Elzaki transform and higher order ordinary diffrential equations”, *Advances in Theoretical and Applied Mathematics*, 1: 1-10.

Tarig M. Elzaki and Salih M. Elzaki. (2011), “Application of new transform Elzaki Transform”, *Global Journal of Pure and Applied mathematics*, 1: 65-70.

Bulut, H., Baskonus, H.M. and Belgacem, F.B.M. (2013), "The Analytical Solution of Some Fractional Ordinary Differential Equations by the Sumudu Transform Method", *Abstract and Applied Analysis*, Volume 2013, Article ID 203875

Mittal, R.C. and Nigam, R. (2008), "Solution of fractional integro-differential equations by adomian decomposition method", *The International Journal of Applied Mathematics and Mechanics*, 4: 87-94.

Podlubny, I. (1999), "Fractional Differential Equations, Mathematics in Science and Engineering", *Academic Press*, San Diego, Calif, USA, 198.

Caputo, M. (1967), "Linear model of dissipation whose Q is almost frequency independent-II", *Geophysical Journal of the Royal Astronomical Society*, 13: 529–539

Katatbeh, Q.D. and Belgacem, F.B.M. (2011), "Applications of the Sumudu transform to fractional differential equations", *Nonlinear Studies*, 18: 99–112.

Song, Y. and Kim,H. (2014), "Legendre's Equation Expressed by the Initial Value by Using Integral Transforms", *Applied Mathematical Sciences*, 8: 531 – 540.

Odibat, Z.M. and Momani, S. (2006), "Application of variational iteration method to nonlinear differential equations of fractional order", *International Journal of Nonlinear Sciences and Numerical Simulation*, 7: 27–34.

Ford, N.J. and Simpson, A.C. (2003), "The numerical solution of fractional differential equations: speed versus accuracy", *Numerical Analysis Report no. 385*, A Report in Association with Chester College.

Odibat, Z.M. and Momani, S. (2006), "An algorithm for the numerical solution of differential equations of fractional order", *Journal of Applied Mathematics and Informatics*, 26: 15–127.