Note on Omega -closed sets in topological spaces

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Abstract.

In this paper we introduce a new class of sets called $\hat{\Omega}$- closed sets in topological spaces and we study some of its basic properties.It turns out that this class lies between the class of δ -open sets and the class of δg (resp. ω)-closed sets.Unique feature is, this new class of sets forms a topology.

Key words and Phrases: semi open sets, δ -open sets, δ -closure, sker, $\hat{\Omega}$ -closed sets.

1.Introduction

Levine [11] initiated the study of generalized closed sets (briefly g -closed) in general topology. The concept of g -closed set has been studied further by weaker forms of open sets such as α-open, semi open, pre open, and semi-pre open. By using δ -closure operator, Donham and Ganster [8] introduced and studied the concept of δg -closed set, strong than g -closed set. We introduce and study a new class of sets known as $\hat{\Omega}$ -closed set, slightly stronger than the class of δg (resp. ω)-closed sets. Also it properly lies between δ -closedness and δg (resp. ω)-closedness.

2. Preliminaries

Throughout this paper (X, τ) and (Y, σ) represent topological spaces on which no separation axioms are assumed unless explicitly stated. For a subset A of (X, τ) , we denote the closure of A, the interior of A and the complement of A as cl(A), int(A) and A^c respectively.

Let us recall the following definitions, which are useful in the sequel.

Definition 2.1. A subset A of a topological space (X, τ) is called a

(i) α-open set [1] if A $\subseteq$ int(cl(int(A)).

(ii) semi-open set [10] if A $\subseteq$ cl(int(A)).

(iii) pre-open set [13] if A $\subseteq$ int(cl(A)).

(iv) β -open (or semi pre open) set[1] if A $\subseteq$ cl(int(cl(A)).

(v) regular open set [14] if A = int(cl(A)).

(vi) b -open set [5] if A $\subseteq$ cl(int(A)) $\cup$ int(cl(A)).

The complement of the above sets are called α-closed, semi-closed, pre-closed, β -closed regular closed and b -closed sets respectively. The α-closure (resp.semi-closure,pre-closure, β -closure)

of a subset A of (X, τ) is the intersection of all α-closed (resp.semi-closed ,pre-closed, β -closed,) sets containing A and is denoted by αcl(A) (resp. scl(A), pcl(A), βcl(A)). The intersection of all semi open subsets of (X, τ) containing A is called the semi kernel of A and is denoted by sker(A).

Definition 2.2. [17] A subset **A** of X is called δ **-**closed in a topological space (X, τ) if A = δcl(A) , where δcl(A) = {x $\in$ X : int(cl(U))∩A $\neq \Phi$,U$\in$ τ, x $\in$ U }**.**The complement of δ –closed set in (X, τ) is called δ -open set in (X, τ). From [9],lemma 3, δcl(A) = ∩{F $\in$δC(X) : A $\subseteq$ F} and from corollary 4, δcl(A) is a δ -closed for a subset A in a topological space (X, τ).

Definition 2.3. [17] A subset A of X is called θ -closed in a topological space (X, τ) if A = θcl (A) , where **θcl(A) = {x $\in$ X : cl(U) ∩ A $\neq\Phi$,U $\in$ τ, x $\in$ U}**.The complement of θ -open set in (X, τ) is called θ -closed set in (X, τ).

Definition 2.4. A subset A of a topological space (X, τ) is called

(i) a generalized closed (briefly g -closed) set [11] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in (X, τ).

(ii) a generalized α- closed (briefly gα-closed) set [12] if αcl(A) $\subseteq$ U whenever A$\subseteq$ U and U is α-open in (X, τ).

(iii) a α- generalized closed (briefly αg -closed) set[12] if αcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

(iv) a generalized semi-closed (briefly gs -closed) set [2] if scl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

(v) a generalized semi-closed (briefly sg -closed) set [3] if scl(A) ⊆ U whenever A ⊆ U and U is semi open in (X, τ).

(vi) a generalized semi-pre closed (briefly gsp -closed) set [7] if spcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

(vii) a δ generalized closed (briefly δg -closed) set [8] if δcl(A) ⊆ U whenever A ⊆ U and U is open in (X, τ).

(viii) $\hat{g}$ (or) ω -closed set [15] if cl(A) ⊆ U whenever A ⊆ U and U is semi open in (X, τ).

The complement of g -closed (resp. gα-closed, αg -closed, gs -closed, sg -closed, gsp -closed, δg -Closed, ω -closed) set is called g -open (resp. gα-open, αg -open, gs -open, sg -open, gα-open, gsp -open, δg –open, ω -open).

3. $\hat{\Omega}$ -Closed Sets

In this section we introduce a basic definition of new class of sets known as $\hat{\Omega}$ -closed sets in topological spaces.

Definition 3.1. A subset A of a space (X, τ) is called $\hat{\Omega}$ -closed if δcl(A) ⊆ U whenever A ⊆ U and U is semi open set in (X, τ). The complement of $\hat{\Omega}$ -closed set in (X, τ) is called $\hat{\Omega}$ - open set in (X, τ).

Theorem 3.2. Every δ -closed set is $\hat{\Omega}$ -closed in (X, τ).

Proof. Let A be any δ -closed and U be any semi open set in (X, τ) such that A ⊆ U. Since A is δ -closed set in (X, τ) , δcl(A) ⊆U. Thus A is $\hat{\Omega}$ -closed set in (X, τ).

Remark 3.3. The reversible implication is not true in general from the following example.

Example 3.4. Let X = {a, b, c} and τ = { Φ ,{a}, {b, c},X}. Here {b} is $\hat{\Omega}$ -closed set in (X, τ) but not , δ -closed in (X, τ).

Theorem 3.5. In a topological space (X, τ) ,every $\hat{\Omega}$ -closed set is

(i) $\hat{g}$ (or ω) -closed set in (X, τ).

(ii) g (resp. gα,αg , sg , gs)-closed set in (X, τ).

(iii) δg -closed set in (X, τ).

Proof. (i) Suppose that A is a $\hat{\Omega}$ -closed and U be any semi open set in (X, τ) such that A ⊆ U. By hypothesis, δcl(A) ⊆ U. Then cl(A) ⊆ U and hence A is ˆg -closed set in (X, τ).

(ii) By [16], every $\hat{g}$ -closed set is g (resp. gα,αg , sg , gs)-closed set in (X, τ). Therefore.it holds.

(iii) Suppose that A is a $\hat{\Omega}$ -closed and U be any open sets in (X, τ) such that A ⊆ U.Then, U is semi open in (X, τ) and by hypothesis, δcl(A) ⊆ U. Hence A is δg -closed set in (X, τ).

Remark 3.6. The reversible implications are not true in general from the following example.

Example 3.7. Let X = {a, b, c, d} and τ = { Φ , {a}, {a, b},X}. Then the set {b, c} is g -closed,gα-closed, sg -closed, δg -closed but not $\hat{\Omega}$ -closed in (X, τ). Also {c, d} is $\hat{g}$ -closed but not $\hat{\Omega}$ -closed in (X, τ).

Remark 3.8. The following examples show that $\hat{\Omega}$ -closed set is independent of closed,α-closed,semi closed, and δ -semi-closed sets.

Example 3.9. Let X = {a, b, c, d} and τ = { Φ , {a}, {a, b},X}. Then the set {c, d} is closed,semi closed and α-closed but not $\hat{\Omega}$ -closed set in (X, τ).

Example 3.10. Let X = {a, b, c} and τ = { Φ , {a, b},X}. Then the set {a, c} is $\hat{\Omega}$ -closed,but not closed or semi closed or α-closed in (X, τ).

Example 3.11. Let X = {a, b, c, d} and τ = { Φ , {a}, {b}, {a, b},X}. Then the set {c} is δ -semi-closed but not $\hat{\Omega}$ -closed set in (X, τ).

Example 3.12. Let X = {a, b, c, d} and τ = { Φ , {c}, {a, d}, {a, c, d},X}. Then the set {a, b, c} is $\hat{\Omega}$ -closed but not δ -semi-closed in (X, τ).

Remark 3.13. The pictorial representation of the above discussions and existing results is shown in Figure-1.The reversible implication is not possible in general.

4.Characterizations.

In this section we characterize $\hat{\Omega}$ -closed sets by giving three necessary and sufficient conditions.

Theorem 4.1. If A is $\hat{\Omega}$ -closed subset in (X, τ) ,then δcl(A) \ A does not contain any nonempty closed set in (X, τ).

Proof. Let F be any closed set in (X, τ) such that F $\subseteq$ δcl(A) \ A. Then A $\subseteq$ X \ F and X \ F is open in (X, τ). Since A is $\hat{\Omega}$-closed and X \ F is semi open, δcl(A) $\subseteq$ X \ F. Hence F $\subseteq$ X \ δcl(A). Thus F $\subseteq$ (δcl(A) \ (A)) ∩ (X \ δcl(A)) = Φ .

Remark 4.2. The converse may not be true in general from the following example.

Example 4.3. Let X = {a, b, c} and τ = { Φ , {a},X}. Let A = {b}. Then δcl(A) \ A = X \ {b} = {a, c} does not contain any non-empty closed set and A is not a $\hat{\Omega}$ -closed subset of (X, τ).

Theorem 4.4. If A is $\hat{\Omega}$ -closed subset in (X, τ) iff δcl(A)\A does not contain any non-empty semi closed set in (X, τ).

Proof. Necessity- Let F be any semi closed such that F $\subseteq$ δcl(A) \ A. Then A $\subseteq$ X \ F and X \ F is semi open in (X, τ). Since A is $\hat{\Omega}$ -closed set in (X, τ) , δcl(A) $\subseteq$ X \ F, F $\subseteq$X \ δcl(A). Thus, F $\subseteq$ (δcl(A) \ A) ∩ (X \ δcl(A)) = Φ .

Sufficiency- Suppose that A $\subseteq$U and U is any semi open set in (X, τ). If A is not $\hat{\Omega}$-closed set,then δcl(A) U and hence δcl(A) ∩ (X \ U) $\neq \Phi$ .We have a nonempty semi closed set δcl(A)∩(X \ U) such that δcl(A)∩(X \ U) $\subseteq$ δcl(A)∩(X \ A) = δcl(A) \A which contradicts the hypothesis.

Theorem 4.5. Let A be any $\hat{\Omega}$ -closed set in (X, τ). Then A is δ -closed in (X, τ) iff δcl(A)\A is semi closed set in (X, τ).

Proof. Necessity- Since A is δ -closed set in (X, τ) , δcl(A) = A. Then δcl(A) \ A = Φ is semi closed set in (X, τ).

Sufficiency- Since A is $\hat{\Omega}$ -closed set (X, τ) ,by theorem 4.4, δcl(A) \ A does not contain any non-empty semi closed set. Therefore, δcl(A) \ A = Φ .Hence δcl(A) = A. Thus, A is δ -closed in (X, τ).

Notations 4.6. In a topological space (X, τ) ,Xs = {x $\in$X : {x} is semi closed in (X, τ) } and $X_{\hat{\Omega}}$ = {x $\in$X : {x} is $\hat{\Omega}$ - open in (X, τ) }.

Proposition 4.7. In a topological space (X, τ) ,for each x $\in$X, either {x} is semi closed or {x}c is $\hat{\Omega}$ -closed set in (X, τ). That is, X = Xs $\cup$ $X_{\hat{\Omega}}$

Proof. Suppose that {x} is not a semi closed set in (X, τ). Then $\{x\}^c$ is not a semi open set and the only semi open set containing $\{x\}^c$ is X. Therefore δcl($\{x\}^c$) $\subseteq$ X and hence $\{x\}^c$ is $\hat{\Omega}$-closed set in (X, τ).

Theorem 4.8. Let A be any $\hat{\Omega}$ -closed set in (X, τ). If A$\subseteq$B $\subseteq$$\delta$cl(A) , then B is also a $\hat{\Omega}$-closed set in (X, τ).

Proof. Let B $\subseteq$ U where U is any semi open set in (X, τ). Then A $\subseteq$U. Since A is $\hat{\Omega}$-closed set, δcl(A) $\subseteq$ U. Since δcl(B) $\subseteq$ δcl(δcl(A)) = δcl(A) $\subseteq$ U, B is a $\hat{\Omega}$-closed set in (X, τ).

Definition 4.9. The intersection of all $\hat{\Omega}$ -open subsets of (X, τ) containing A is called the $\hat{\Omega}$-kernel of A and is denoted by $\hat{\Omega}$ker^(A).

Theorem 4.10. A subset A of a topological space (X, τ) is $\hat{\Omega}$-closed in (X, τ) if and only if δcl(A) $\subseteq$ sker(A)

Proof. Necessity. Suppose that A is $\hat{\Omega}$-closed set in (X, τ) and x $\in$ δcl(A) and x $\notin$ sker(A). Then there exists a semi open set U in (X, τ) such that A $\subseteq$ U and x $\notin$ U. Since A is $\hat{\Omega}$-closed set in (X, τ) , δcl(A) $\subseteq$ U which is a contradiction to x $\in$ δcl(A) and x $\notin$ U.

Sufficiency. Suppose that δcl(A) $\subseteq$ sker(A) and U is any semi open set in (X, τ) such that A $\subseteq$ U. Then sker(A) $\subseteq$ U and hence δcl(A) $\subseteq$ U. Thus A is $\hat{\Omega}$ -closed set in (X, τ).

Justification 4.11. By the following results,we justify that the original axioms for the topology are preserved by the class of $\hat{\Omega}$ -closed sets in a topological space (X, τ). It is denoted by $\tau_{\hat{\Omega}}$ which is weaker than $\tau\delta$ and stronger than the topology formed by the class of ω -closed sets.

Theorem 4.12. If A and B are $\hat{\Omega}$ -closed sets in a topological space (X, τ) ,then A $\cup$ B is $\hat{\Omega}$-closed set in (X, τ).

Proof. Suppose that A $\cup$B $\subseteq$U where U is any semi open in (X, τ). Then A $\subseteq$U and B $\subseteq$U. Since A and B are $\hat{\Omega}$-closed sets in (X, τ) , δcl(A) $\subseteq$ U and δcl(B) $\subseteq$U. Always δcl(A $\cup$ B) = δcl(A) $\cup$ δcl(B). Therefore δcl(A $\cup$ B) $\subseteq$ U Thus A $\cup$ B is $\hat{\Omega}$ -closed in (X, τ).

Lemma 4.13. [6] Let x be any point in a topological space (X, τ). Then {x} is either nowhere dense or pre-open in (X, τ). Also,X = X_1 $\cup$ X_2 ,where X1 = {x $\in$ X : {x} is nowhere dense in (X, τ) } and X_2 = {x $\in$ X : {x} is pre-open in (X, τ) } is known as Jankovic-Reilly decomposition.

Theorem 4.14. In a topological space (X, τ) , X2 $\cap$ δcl(A) $\subseteq$sker(A) for any subset A of (X, τ).

Proof. Suppose that x $\in$ X_2 $\cap$ δcl(A) and x $\notin$ sker(A) .Since x $\in$ X_2 , scl({x}) = int(cl({x})).

Moreover,x $\notin$ X_1 implies scl({x}) $\neq\Phi$. Since x $\in$ δcl(A),A$\cap$int(cl(U)) $\neq\Phi$ where U = int(cl({x})). Thus A $\cap$ (int(cl({x}))) $\neq\Phi$. Choose y $\in$ A $\cap$ (int(cl({x}))). Since x $\notin$ sker(A) ,there exists a semi open set V in (X, τ) such that A $\subseteq$ V and x $\notin$ V . If F = X \ V ,then x $\in$ F $\subseteq$ X \ A. Also int(cl({x}))$\subseteq$ int(cl(F)) $\subseteq$ F and hence y $\in$ A $\cap$ F, a contradiction. Thus, x $\in$ sker(A) .

Theorem 4.15. A subset A is $\hat{\Omega}$-closed set in a topological space in (X, τ) if and only if X_1 $\cap$ δcl(A) $\subseteq$ A.

Proof. Necessity- Suppose that A is $\hat{\Omega}$ -closed set in (X, τ) and x $\in$ X_1 $\cap$ δcl(A) but not in A. Therefore, {x} is semi closed set in (X, τ) and hence X \ {x} is semi open set in (X, τ).

Since X\{x} is the semi open set in (X, τ) containing A and by hypothesis, δcl(A) $\subseteq$ X \ {x},a contradiction to x $\in$ δcl(A). Thus X_1 $\cap$ δcl(A) $\subseteq$ A.

Sufficiency- Suppose that X_1 $\cap$ δcl(A) $\subseteq$ A. Since A $\subseteq$ sker(A),X $\cap$ δcl(A) $\subseteq$sker(A) . By theorem 4.14,X_2 $\cap$ δcl(A) $\subseteq$ sker(A) . Therefore, δcl(A) = (X_1 $\cup$ X_2) $\cap$ δcl(A) = (X_1 $\cap$ δcl(A)) $\cup$ (X_2 $\cap$ δcl(A)) $\subseteq$ sker(A) .By theorem 4.10,A is $\hat{\Omega}$-closed set in X.

Theorem 4.16. Arbitrary intersection of $\hat{\Omega}$ -closed sets in a topological space (X, τ) is $\hat{\Omega}$-closed set in (X, τ).

Proof. Let {Ai : i $\in$ I} be any family of $\hat{\Omega}$ -closed sets in (X, τ) and A = $\bigcap_{i\in I} A_i$. Therefore,$X_1$$\cap$$\delta$cl(Ai) $\subseteq$Ai for each i $\in$ I and hence $X_1$$\cap$$\delta$cl(A) $\subseteq$ $X_1$$\cap$$\delta$cl(Ai) $\subseteq$ Ai for each i $\in$ I .Thus X_1 $\cap$ δcl(A) $\subseteq \bigcap_{i\in I} A_i$ = A. By theorem 4.15,A is $\hat{\Omega}$ -closed set in (X, τ). Thus, arbitrary intersection of $\hat{\Omega}$-closed sets in a topological space (X, τ) is $\hat{\Omega}$ -closed set in (X, τ).

Notations 4.17. In a topological space (X, τ) ,the set of all semi (resp.pre, $\hat{\Omega}$) open sets are denoted by SO(X) (resp.PO(X), $\hat{\Omega}$O(X)). The set of all δ -closed sets are denoted by δC(X).

Lemma 4.18. If A is $\hat{\Omega}$ -closed and B is δ -closed sets in (X, τ) then A $\cap$ B is $\hat{\Omega}$ -closed in (X, τ) because of arbitrary intersection of $\hat{\Omega}$ -closed sets is a $\hat{\Omega}$ -closed set.

Let us characterize partition space via $\hat{\Omega}$ -closed sets.

Remark 4.19. [8] A partition space is a topological space (X, τ) where every open set is closed.Also a topological space is partition space if and only if every subset is pre open.

Theorem 4.20. In a topological space (X, τ) ,

(i) SO(X) $\subseteq$ δC(X) if and only if $\hat{\Omega}$O(X) = P(X) .

(ii) (X, τ) is a partition space if and only if $\hat{\Omega}$O(X) = P(X) .

Proof. (i) Necessity- Let A be arbitrary subset of (X, τ) such that A $\subseteq$ U where U $\in$ SO(X) .By hypothesis, δcl(A) $\subseteq$ δcl(U) = U Therefore, A is $\hat{\Omega}$ -closed set in (X, τ).

Sufficiency- Let U be any semi open set in (X, τ). By hypothesis, U is $\hat{\Omega}$ -closed set in (X, τ). Since every $\hat{\Omega}$ -closed set is pre closed set, U is a pre closed set in (X, τ). It is clear that if U is both semi open and pre closed, then U is regular closed and hence it is δ -closed in (X, τ).

(ii) Necessity- Let A be arbitrary subset of (X, τ) and suppose that x $\in$ $X_1 \cap$ δcl(A), x $\notin$ A. We have {x} is a semi closed set and hence it is a closed set in (X, τ). Therefore X\{x} is an open set in (X, τ) and by hypothesis, it is a closed set in (X, τ). Now X\{x} is a clopen set in (X, τ) and then δ -closed set in (X, τ). Therefore δcl(A) $\subseteq$ δcl(X \{x}) = X \{x},a contradiction to x $\in$ δcl(A). $X_1 \cap$Thus X1 ∩ δcl(A) $\subseteq$ A. By 4.15,A is $\hat{\Omega}$ -closed set in (X, τ).

Sufficiency- Let U be any open and hence semi open set in (X, τ). By hypothesis,U is $\hat{\Omega}$-closed set in (X, τ). Since every $\hat{\Omega}$ -closed set is pre closed set,U is a pre closed set in (X, τ). It is clear that if U is both semi open and pre closed,then U is a regular closed and hence it is a δ -closed in (X, τ). Therefore U is a closed set in (X, τ). Thus every open set is closed in (X, τ).

Remark 4.21. From the above discussions, a topological space is partition space if and only if $\hat{\Omega}$O(X) = PO(X) = P(X) .

5. $\hat{\Omega}$ -closure

In this section we define the closure of $\hat{\Omega}$ -closed sets and prove that it is a "Kuratowski closure operator."

Definition 5.1. Let A be a subset of a topological space (X, τ). Then the $\hat{\Omega}$ -closure of A is defined to be the intersection of all $\hat{\Omega}$ -closed sets containing A and it is denoted by $\hat{\Omega}$cl(A). That is $\hat{\Omega}$cl(A) = ⋂{F /A $\subseteq$F and F is a $\hat{\Omega}$ - closed set in (X, τ) } Always A $\subseteq$ $\hat{\Omega}$cl(A) .

Remark 5.2. From the definition and 4.16, $\hat{\Omega}$cl(A) is the smallest $\hat{\Omega}$ -closed set containing A.

Theorem 5.3. Let A and B be subsets of a topological space (X, τ). Then

(i) $\hat{\Omega}$cl(Φ) = Φ and $\hat{\Omega}$cl(X) = X.

(ii) If A $\subseteq$B, then $\hat{\Omega}$cl(A) $\subseteq$ $\hat{\Omega}$cl(B).

(iii) $\hat{\Omega}$cl(A ∩ B) $\subseteq$ $\hat{\Omega}$cl(A) ∩ $\hat{\Omega}$cl(B).

(iv) $\hat{\Omega}$cl(A ∪ B) = $\hat{\Omega}$cl(A) ∪ $\hat{\Omega}$cl(B).

(v) A is $\hat{\Omega}$-closed in (X, τ) if and only if A = $\hat{\Omega}$cl(A).

(vi) $\hat{\Omega}$cl($\hat{\Omega}$cl(A)) = $\hat{\Omega}$cl(A).

(vii) $\hat{\Omega}$cl(A) $\subseteq$δcl(A).

Proof. (i) Obvious.

(ii) A $\subseteq$B $\subseteq$ $\hat{\Omega}$cl(B). But $\hat{\Omega}$cl(A) is the smallest $\hat{\Omega}$ -closed set containing A. Hence $\hat{\Omega}$cl(A) $\subseteq$ $\hat{\Omega}$cl(B).

(iii) A ∩ B $\subseteq$ A and A ∩ B $\subseteq$ B. By (ii), $\hat{\Omega}$cl(A∩B) $\subseteq$ $\hat{\Omega}$cl(A) and $\hat{\Omega}$cl(A ∩ (B) $\subseteq$ $\hat{\Omega}$cl(B).Hence $\hat{\Omega}$cl(A∩B) $\subseteq$ $\hat{\Omega}$cl(A) ∩ $\hat{\Omega}$cl(B).

(iv) A $\subseteq$ A ∪B and B $\subseteq$ A ∪ B. By(ii), $\hat{\Omega}$cl(A) $\subseteq$ $\hat{\Omega}$cl(A ∪B) and $\hat{\Omega}$cl(B) $\subseteq$ $\hat{\Omega}$cl(A ∪ (B).Hence $\hat{\Omega}$cl(A) ∪ $\hat{\Omega}$cl(B) $\subseteq$ $\hat{\Omega}$cl(A ∪B). Also A $\subseteq$ $\hat{\Omega}$cl(A) and B $\subseteq$ $\hat{\Omega}$cl(B) which implies A∪B $\subseteq$ $\hat{\Omega}$cl(A∪B) .But $\hat{\Omega}$cl(A∪B) is the smallest $\hat{\Omega}$ -closed set containing A∪B. Hence $\hat{\Omega}$cl(A∪B) $\subseteq$ $\hat{\Omega}$cl(A) ∪ $\hat{\Omega}$cl(B).

(v) Necessity- Suppose that A is $\hat{\Omega}$ -closed in (X, τ). By 5.2, A $\subseteq$ $\hat{\Omega}$cl(A). By the definition of $\hat{\Omega}$ closure and hypothesis, $\hat{\Omega}$cl(A) $\subseteq$ A. Therefore A = $\hat{\Omega}$cl(A).

Sufficiency-Suppose that A = $\hat{\Omega}$cl(A). By the definition of $\hat{\Omega}$ closure, $\hat{\Omega}$cl(A) is a $\hat{\Omega}$- closed set and hence A is a $\hat{\Omega}$ -closed set in (X, τ).

(vi) Since arbitrary intersection of $\hat{\Omega}$ -closed sets in a topological space (X, τ) is $\hat{\Omega}$ -closed set in (X, τ) , $\hat{\Omega}$cl(A) is a $\hat{\Omega}$-closed set in (X, τ). By previous division, $\hat{\Omega}$cl($\hat{\Omega}$cl(A)) = $\hat{\Omega}$cl(A).

(vii) Suppose that x $\notin$ δcl(A) .Then there exists a δ -closed set F such that A $\subseteq$ F and x $\notin$ F .Since every δ -closed set is $\hat{\Omega}$ -closed set, x $\notin$ $\hat{\Omega}$cl(A) . Thus $\hat{\Omega}$cl(A) $\subseteq$ δcl(A).

Remark 5.4. The reversible implication of (iii) is not true in general from the following example.

Example 5.5. Let X = {a, b, c, d} and τ = { Φ , {a}, {b}, {a, b},X}. If A = {a} and B = {b}, then $\hat{\Omega}$cl(A) = {a, c, d} , $\hat{\Omega}$cl(B) = {b, c, d},A ∩ B = $\hat{\Omega}$cl(A ∩ B) = ;.But $\hat{\Omega}$cl(A) ∩ $\hat{\Omega}$cl(B) = {c, d}.

Remark 5.6. From $\hat{\Omega}$cl(Φ) = Φ ,A$\subseteq$ $\hat{\Omega}$cl(A), $\hat{\Omega}$cl(A $\cup$ B) = $\hat{\Omega}$cl(A) $\cup$ $\hat{\Omega}$cl(B) ,and $\hat{\Omega}$cl($\hat{\Omega}$cl(A)) = $\hat{\Omega}$cl(A) we can say that $\hat{\Omega}$-closure is the Kuratowski closure operator on (X, τ).

Definition 5.7. A point x of a space (X, τ) is called a $\hat{\Omega}$ -limit point of a subset A of (X, τ).if for each $\hat{\Omega}$ -open set U containing x intersects A other than x. That is A ∩ (U $-$\{x\}) $\neq \Phi$ .The set of all limit points of A is denoted by $D_{\hat{\Omega}}$ (A) and is called the $\hat{\Omega}$-derived set of A.

Theorem 5.8. Let A and B be any two subsets of a space (X, τ). Then

(i) $D_{\hat{\Omega}}$(Φ) = Φ and $D_{\hat{\Omega}}$ (X) = X.

(ii) If A $\subseteq$B, then $D_{\hat{\Omega}}$ (A) $\subseteq$ $D_{\hat{\Omega}}$ (B).

(iii) $D_{\hat{\Omega}}$ (A $\cup$ B) = $D_{\hat{\Omega}}$ (A) $\cup$ $D_{\hat{\Omega}}$ (B).

(iv) $D_{\hat{\Omega}}$ (A ∩ B) $\subseteq$ $D_{\hat{\Omega}}$ (A) ∩ $D_{\hat{\Omega}}$ (B).

(v) A subset A is $\hat{\Omega}$ -closed iff $D_{\hat{\Omega}}$ (A) $\subseteq$A.

(vi) $\hat{\Omega}$cl(A) = A $\cup$ $D_{\hat{\Omega}}$ (A).

Proof.Follows from the definition and similar to theorem 5.3.

Remark 5.9. The reversible implication of (iv) is not true in general from the following example.

Example 5.10. Let X = {a, b, c, d} and τ = { Φ , {a}, {b}, {a, b},X} If A = {a} and B = {b}, $D_{\hat{\Omega}}$ (A) = {c, d} and $D_{\hat{\Omega}}$ (B) = {c, d},A ∩ B = $D_{\hat{\Omega}}$ (A ∩ B) = ;. But $D_{\hat{\Omega}}$ (A) ∩ $D_{\hat{\Omega}}$ (B) = {c, d}.

Theorem 5.11. In a topological space (X, τ) ,for x $\in$ X, x $\in$ $\hat{\Omega}$cl(A) if and only if U ∩A $\neq \Phi$ for every $\hat{\Omega}$ -open set U containing x.

Proof. Necessity- Suppose that x $\in$ $\hat{\Omega}$cl(A) and there exists a $\hat{\Omega}$ -open set U containing x such that U ∩A = Φ. Then A $\subseteq U^c$ and U^c is a $\hat{\Omega}$ -closed set and so $\hat{\Omega}$cl(A) $\subseteq$ U^c .This shows that x $\notin$ $\hat{\Omega}$cl(A), a contradiction.

Sufficiency- Suppose that x $\notin$ $\hat{\Omega}$cl(A) Then there exists $\hat{\Omega}$ -closed set F containing A such that $\notin$F. Hence F^c is a $\hat{\Omega}$ -open set containing x such that. $F^c \subseteq A^c$ .Therefore F^c ∩A = Φ which contradicts hypothesis.

Definition 5.12. A point x in a topological space (X, τ) is called a $\hat{\Omega}$ -interior point of a subset A of (X, τ) if there exists some $\hat{\Omega}$ -open set U containing x such that U $\subseteq$A. The set of all $\hat{\Omega}$-interior points of A is called the $\hat{\Omega}$ -interior of A and is denoted by $\hat{\Omega}$int(A).

Remark 5.13. $\hat{\Omega}$int(A) is the union of all $\hat{\Omega}$-open sets contained in A and by 4.16, $\hat{\Omega}$int(A) is the largest $\hat{\Omega}$ -open set contained in A.

Theorem 5.14. A subset A of (X, τ) is $\hat{\Omega}$ -open if and only if F $\subseteq$ δint(A) whenever F is semi closed and F $\subseteq$ A.

Proof. obvious.

Theorem 5.15. (i) $\hat{\Omega}$cl(X \ A) = X \ $\hat{\Omega}$int(A).

(ii) $\hat{\Omega}$int(X \ A) = X \ $\hat{\Omega}$cl(A).

Proof. (i) $\hat{\Omega}$int(A) $\subseteq$ A $\subseteq$ $\hat{\Omega}$cl(A). Hence X \ $\hat{\Omega}$cl(A) $\subseteq$ X \ A $\subseteq$ X \ $\hat{\Omega}$int(A). Therefore X \ $\hat{\Omega}$cl(A) is the $\hat{\Omega}$ -open set contained in X \A. But ˆ $\hat{\Omega}$int(X −A) is the largest $\hat{\Omega}$ -open setcontained in X\A. Thus X\$\hat{\Omega}$cl(A) $\subseteq$ $\hat{\Omega}$int(X\A). On the other hand, if x $\in$ $\hat{\Omega}$int(X\A),there exists a $\hat{\Omega}$ -open set U containing x such that U $\subseteq$ X \ (A). Hence U $\cap$ A = Φ.Therefore, x $\notin$ $\hat{\Omega}$cl(A) and hence x $\in$(X \ $\hat{\Omega}$cl(A)) . Thus $\hat{\Omega}$int(X \ A) $\subseteq$ X \ $\hat{\Omega}$cl(A).

(ii) Similar to the proof of (i).

6.Applications in Subspace Topology

Notations 6.1. For any set A $\subseteq$ X (A, τ |A) represents subspace topological space with respective to τ. Let A and B be any two subsets in a topological space (X, τ) such that B $\subseteq$ A ,then δcl_X(B) (resp. $\hat{\Omega} cl_X$(B)) represents δ (resp. $\hat{\Omega}$) closure of B in (X, τ) and δcl_A(B) (resp. $\hat{\Omega} cl_A$(B)) represents δ (resp. $\hat{\Omega}$) closure of B in the subspace (A, τ |A) . Also $sker_X$ (B) (resp. {$\hat{\Omega} ker_X$ (B)) represents semi (resp. $\hat{\Omega}$) kernel of B in (X, τ) and $sker_A$ (B) (resp. $\hat{\Omega} ker_A$ (B)) represents semi (resp. $\hat{\Omega}$) kernel of B in the
subspace (A, τ |A) .

Remark 6.2. Let A be open set in a topological space (X, τ). Let B $\subseteq$ A Then δclA(B) = A $\cap$ δcl_X(B)

Remark 6.3. Let A be pre open set in a topological space (X, τ). Let B $\subseteq$ A
Then $Sker_A$ (B) = A $\cap$ $sker_X$ (B).

Theorem 6.4. If A is both semi open and pre closed set in a topological space (X, τ) , then A is $\hat{\Omega}$ -closed in (X, τ).

Proof. It is clear that if A is both semi open and pre closed, then A is regular closed and hence it is δ -closed in (X, τ). Therefore it is $\hat{\Omega}$closed in (X, τ).

Theorem 6.5. Let B $\subseteq$ A $\subseteq$ X where A is open in (X, τ). If B is $\hat{\Omega}$ -closed set in (X, τ) ,then B is $\hat{\Omega}$ -closed set in the subspace (A, τ |A).

Proof. Suppose that B is $\hat{\Omega}$-closed set in (X, τ). By 4.10, δcl_X(B) $\subseteq$ $sker_X$ (B) and hence A$\cap\delta cl_X$ (B) $\subseteq$ A$\cap sker_X$ (B) .By 6.2 and 6.3, δcl_A (B) $\subseteq sker_A$ (B) .Again by 4.10,B is $\hat{\Omega}$-closed set in the
subspace (A, τ |A) .

Theorem 6.6. Let B $\subseteq$ A $\subseteq$ X where A is both open and pre closed set in (X, τ). If B is $\hat{\Omega}$-closed set in the subspace (A, τ |A), then B is $\hat{\Omega}$-closed set in (X, τ).

Proof. Suppose that B is $\hat{\Omega}$-closed set in the subspace (A, τ |A). By 4.10, δcl_A(B) $\subseteq sker_A$(B) and hence by 6.2 and 6.3, A $\cap$ δcl_X(B) $\subseteq$ A $\cap$ $sker_X$ (B). Since A is δ -closed in (X, τ) , δcl_X (B) = δcl_X(A) $\cap$ δcl_X(B) = A $\cap$ δcl_X(B) $\subseteq$ A $\cap$ $sker_X$(B) $\subseteq$ $sker_X$(B). Therefore, δcl_X(B) $\subseteq$ $sker_X$(B). By 4.10, B is $\hat{\Omega}$ -closed set in (X, τ).

Theorem 6.7. If F is $\hat{\Omega}$ -closed set in (X, τ) ,then F $\cap$ A is $\hat{\Omega}$-closed set in the subspace
(A, τ |A) provided that A is both open and pre closed set in a topological space (X, τ).

Proof. Since F $\cap$ A is $\hat{\Omega}$-closed set in (X, τ) , by 4.10, δcl_X(F $\cap$ A) $\subseteq$ $sker_X$(F $\cap$ A) .Then A$\cap$ δcl_X(F $\cap$A) $\subseteq$ A $\cap$ $sker_X$(F $\cap$ A) and hence by 6.2 and 6.3, δcl_A(F $\cap$A) $\subseteq$ $sker_A$(F $\cap$ A) Again by theorem 4.10,F $\cap$ A is $\hat{\Omega}$ -closed set in the subspace (A, τ |A) .

Theorem 6.8. Let U $\subseteq$ A $\subseteq$ X where A is both open and pre closed set in (X, τ). If U is $\hat{\Omega}$-open set in (X, τ) , then U is $\hat{\Omega}$ -open in the subspace (A, τ |A) .

Proof. Suppose that U is $\hat{\Omega}$ -open set in (X, τ). Then X \U is $\hat{\Omega}$ -closed set in (X, τ). By theorem 6.7,(X \U)∩A is $\hat{\Omega}$ -closed set in the subspace (A, τ |A) .That is A\ (A ∩ U) is $\hat{\Omega}$ -closed set in the subspace (A, τ |A) . Then A \ U is $\hat{\Omega}$ -closed set in the subspace (A, τ |A) .Thus U is $\hat{\Omega}$ -open set in the subspace (A, τ |A)

Theorem 6.9. Let U ⊆ A ⊆ X where A is both δ -open and pre closed set in (X, τ). If U is $\hat{\Omega}$-open set in the subspace (A, τ |A) ,then U is $\hat{\Omega}$ -open in (X, τ).

Proof. Suppose that U is $\hat{\Omega}$ -open set in the subspace (A, τ |A) . Then A\ U is $\hat{\Omega}$-closed set in the subspace (A, τ |A) . By 6.6, A \ U is $\hat{\Omega}$ -closed set in (X, τ). That is A \ U = (X \ U) ∩ A is $\hat{\Omega}$-closed set in (X, τ). Then U = [X \ ((X \ U) ∩ A)] ∩ A is $\hat{\Omega}$-open set in (X, τ).

Theorem 6.10. Let A be both open and pre closed set in a topological space (X, τ). If U is $\hat{\Omega}$-open set in (X, τ) ,then U ∩ A is $\hat{\Omega}$ -open set in a subspace (A, τ |A) .

Proof. Suppose that U is $\hat{\Omega}$ -open set in (X, τ) ,then X \ U is $\hat{\Omega}$ -closed set in (X, τ). By theorem 6.7,(X \ U) ∩ A is $\hat{\Omega}$ -closed set in a subspace (A, τ |A) . Then A \ (U ∩ A) is $\hat{\Omega}$ -closed set in a subspace (A, τ |A) . Thus U ∩ A is $\hat{\Omega}$ -open set in a subspace (A, τ |A) .

Theorem 6.11. Let A be both open and pre closed set in a topological space (X, τ). If E is any subset of X such that E ⊆ A ⊆X, then $\hat{\Omega}\,cl_A(E) \subseteq A \cap \hat{\Omega}\,cl_X(E)$.

Proof. Suppose that x ∈ $\hat{\Omega}\,cl_A(E)$ and F be an arbitrary $\hat{\Omega}$-closed set in (X, τ) such that E⊆ F By theorem 6.7, F ∩ A is $\hat{\Omega}$ -closed set in a subspace (A, τ |A) such that E ⊆F ∩ A. Therefore, $\hat{\Omega}\,cl_A$ (E) ⊆F ∩ A and hence x ∈ F ∩ A ⊆ F By the definition of closure, x ∈ $\hat{\Omega}\,cl_X(E)$ and hence x ∈ A ∩ $\hat{\Omega}\,cl_X(E)$ .Thus $\hat{\Omega}\,cl_A$ (E) ⊆A ∩ $\hat{\Omega}\,cl_X(E)$.

Theorem 6.12. Let A be both open and pre closed set in a topological space (X, τ). If E is any subset of X such that E ⊆A ⊆ X .then A ∩ $\hat{\Omega}\,cl_X(E) \subseteq \hat{\Omega}\,cl_A(E)$.

Proof. Suppose that x ∈ A∩ $\hat{\Omega}\,cl_X(E)$ and F is an arbitrary $\hat{\Omega}$ -closed set in the subspace (A, τ |A) such that E ⊆ F ⊆ A. By 6.6, F is $\hat{\Omega}$ -closed set in (X, τ). Therefore, $\hat{\Omega}\,cl_X(E) \subseteq \hat{\Omega}\,cl_X(F)$ = F. Therefore, x ∈ F. By the definition of $\hat{\Omega}$ -closure in subspace, x ∈ $\hat{\Omega}\,cl_A(E)$. Thus A ∩ $\hat{\Omega}\,cl_X(E) \subseteq \hat{\Omega}\,cl_A(E)$.

Theorem 6.13. Let A be both open and pre closed set in a topological space (X, τ). If E is any subset of X such that E⊆A ⊆ X, then $\hat{\Omega}\,ker_A(E)$ ⊆A ∩ $\hat{\Omega}\,ker_X(E)$.

Proof. Similar to 6.11.

Theorem 6.14. Let A be both δ -open and pre closed set in a topological space (X, τ) and E ⊆A⊆ X .Then A ∩ $\hat{\Omega}\,ker_X(E) \subseteq \hat{\Omega}\,ker_A(E)$.

Proof. Similar to 6.12.

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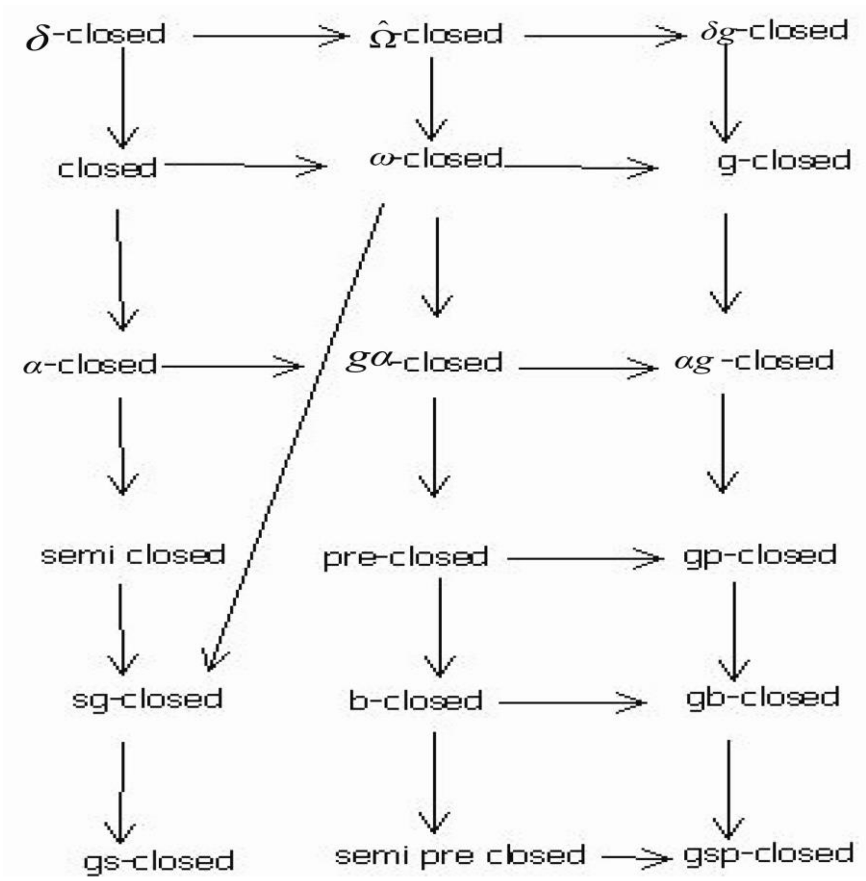

Figure-1