

On Applications of Transitive Permutation Groups To Wreath Product

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Abstract

In this article we investigate and examine some of our results from transitive permutation groups which have some bearing to wreath product

Keywords: transitive, permutation, group, wreath product

I. INTRODUCTION

Here we prove some properties of transitive groups that bear consequences upon the Wreath Product of two permutation groups.

II. RESULTS

1.1 Lemma

Let $C \leq Sym\ (n)$, then C acts transitively on a set Δ if and only if C is cyclic, where
$|\Delta| \leq n$.

Proof

Suppose C acts transitively on the set $\Delta = \{1, 2, \ldots, r\}$ and $|C| = m$. Let $\alpha \in C$, $\alpha \neq (1)$, then $\alpha^s = (1)$ for some positive integer s and by Lagrange's theorem $s|m$. If $s < m$, then some elements of Δ are fixed by α. Consequently C moves elements of a proper subset of Δ, contrary to assumption that C acts transitively on Δ. Thus $s = m$ and C is cyclic generated by α.
Conversely suppose C is cyclic, say $C = \{(1), \alpha, \alpha^2, \ldots, \alpha^{m-1}\}$, where $\alpha^m = (1)$,
Then $r \leq m$ and $\alpha = (1, 2, 3, \ldots, m)$. We observe that for each j $(j = 1, 2, \ldots, r)$,
$\alpha^i (j) \equiv i + j \pmod m$, $i = 0, 1, 2, \ldots, m - 1$. Here $\alpha^0 (j) = (1)$ denotes the identity permutation. Clearly for each $i \neq 0$, $i + j \neq j$ and $\alpha^i (j) \in \Delta$. Thus $\alpha^i (j) = k$, for some
$j, k \in \Delta$ and $\alpha^i \in C$. This shows that C is transitive on Δ as required

1.2 Lemma

Let $G \leq Sym\ (n)$ and $|G| = m$. Then G is the unique permutation group acting on a set Δ with $|\Delta| = r$ if and only if $r = m$.

Proof

Suppose G is the unique permutation group acting transitively on Δ, then $r|m$ otherwise G will not be transitive on Δ by [1].
If $r < m$, consider the permutation $\alpha = (r, r-1, \ldots, 1)$ and let $H = \{(1), \alpha, \ldots, \alpha^{r-1}\}$, with $\alpha^r = (1)$. Since for each i $(i = 1, 2, \ldots, r-1)$, we have

$$\alpha^i(j) \equiv \begin{cases} r - i + j (\mathrm{mod}\, r) & if\ i \neq 0, \\ j & if\ i = 0 \end{cases}$$

holds for j = 1, 2, . . . , r and r - i + j ≠ j for i ≠ 0, it follows that H is transitive on Δ, contradicting the uniqueness of C as the only transitive group acting on Δ, hence r = m.

Conversely if $C^/$ is another permutation group acting transitively on Δ with $|\Delta| = r = m$, let $|C^/| = n$, then by Lemma 1.1, $G^/$ is cyclic and let $C^/ = \{(1), \beta, \beta^2, \dots, \beta^{n-1}\}$ where

$\beta^n = (1)$. As $C^/$ acts transitively on Δ, $\beta^r = (1)$.

Thus $r \leq n$ (1.21)

But n being the order of β means that n ≤ r (1.2.2)

Hence from (1.2.1) and (1.2.2), we see that n = r = m and so $C \cong C^/$.

1.3 Proposition

Let $G \leq Sym$ (n), (n > 1) be a permutation group of exponent n such that for 1 ≤ k < n,

$|G|$ = kn. Then G is transitive on the set Ω with $|\Omega|$ = n.

Proof

If k = 1, then $|G|$ = n and as G is of exponent n, G is cyclic. Let $\alpha \in G$ with

$G = \langle \alpha \rangle$, $\alpha^n = 1$. Then G is transitive on the set Ω = {1, 2, … , n }.

If 1< k < n, as $|G|$ = kn and G is of exponent n, G contains an element a, say of order n. Let $H = \langle a \rangle$ with a^n =1, then the normalizer of H in G, $N_G(H)$ is of order multiple >1 of n and a divisor of kn, thus $|N_G(H)|$ = kn $= |G|$ and $H \trianglelefteq G$. Let $b \in G - H$ such that

$b^k = (1)$ and set $K = \langle b \rangle$, then $H \cap K = \{1\}$ and G = HK. As $H \triangleleft G$, $b^{-1} a b \in H = \langle a \rangle$, we obtain $G = \langle a, b: a^n = 1, b^k = 1, b a = a^r b$ for some r such that $0 < r < n \rangle$, a group which contains a cyclic subgroup of order n that is a transitive subgroup of order n and by [1], G is transitive on Ω ={1, 2, … , n } as required.

1.4 Proposition

Let C and D be transitive permutation groups on sets Γ and Δ respectively such that $|\Delta| = |D|$ and $|C| < |D|$, then the wreath product of C and D, W = C wr D with base group $P = C^{\Delta}$ is the unique group acting transitively on the set Γ x Δ.

Proof

Let $C^/$ and $D^/$ be other permutation groups acting transitively on the sets Γ and Δ respectively with $|\Delta| = |D^/|$ and consider $W^/ = C^/$ wr $D^/$ with base group $P = C^{/\,\Delta}$, then $|W^/| = |C^/|^{|\Delta|}|D^/|$. Since $|\Delta| = |D|$ and $|\Delta| = |D^/|$, then $|D| = |D^/|$, also by

Lemma 1.1, D and $D^/$ are cyclic and so $D \cong D^/$. Also since $|P| = |C^/|^{|\Delta|} = |C|^{|\Delta|}$ then $|C^/| = |C|$ and again by Lemma 1.1, C and $C^/$ are cyclic, thus $C \cong C^/$. Hence

$|W^/| = |C^/|^{|\Delta|}|D^/| = |C|^{|\Delta|}|D| = |W|$, and so $W \cong W^/$.

1.5 Proposition

Let Ω be a set of size *p*, *p* a prime. If G is Sylow *q* – subgroup of *Sym* (Ω), where *q* is a prime with $p \neq q$, then G is not transitive on Ω.

Proof

Suppose G acts transitively on the set Ω, and let $|G| = q^r$, q^r the highest power of *q* dividing p!. As $q \neq p$, we may assume $q < p$. Since G is transitive on Ω, G contains at least one element of order *p*, say a. Let $H = \langle a \rangle$, with $a^p = (1)$, then $|H|$ = p and by Lagrange's theorem, $p \mid q^r$ which is impossible since *p* and *q* are prime and *q*

$< p$. Hence G is not transitive on Ω.

1.6 Proposition

Let C be a permutation group acting transitively on a set Γ and let $D \leq Sym$ (n) be a cyclic permutation group acting transitively on a set Δ such that $|D| = n$, $|\Delta| = r$, then there exists a unique Wreath Product of C and D, W = C wr D if and only if n = r.

Proof

Suppose $n \neq r$, then $n > r$ (by [1]), let $\Delta = \{1, 2, \ldots, r\}$ and consider the base group $P = C^{\Delta} = \{f_i : \Delta \to C, i = 1, 2, \ldots, |C|^{|\Delta|}\}$. Since D is cyclic, we consider the element $d = (1, 2, 3, 4, \ldots, r, r + 1, \ldots, n)$ of D, then for $(\alpha, r) \in \Gamma \times \Delta$ and
$i, j \in \{1, 2, \ldots, |C|^{|\Delta|}\}$, we have
$(\alpha, r)(f_i d)(f_j d) = (\alpha f_i (r) f_j (rd), d^2) = (\alpha f_i (r) f_j (r + 1), d^2)$ is not defined since
$r + 1 \notin \Delta$, thus W = C wr D does not exist.
Also if $n < r$, consider the generator $\alpha = (1, 2, 3, \ldots, n)$ of the cyclic group D. Then
$2, r \in \Delta$, but we cannot find $\alpha^i \in D$ (for any $i \in \{1, 2, 3, \ldots, n - 1\}$) such that $\alpha^i(r) = 2$ since $n > r$. This means that D is not transitive on Δ, thus W = C wr D exist implies n = r.
Conversely if n = r, then the elements of the base group
$P = C^{\Delta} = \{f_i: \Delta \to C, i = 1, 2, \ldots, |C|^{|\Delta|}\}$ and the elements of $D = < \alpha >$,
$\alpha = (1, 2, 3, \ldots, n)$ are all defined and the groups $P = C^{\Delta}$ and D are finite, so is
$|C|^{|\Delta|}|D| = |W|$, hence, W = C wr D is a Wreath Product of C and D. The uniqueness of W = C wr D follows from the fact that any two cyclic groups of the order are isomorphic.

1.7 Lemma

Let C acts transitively on a group Γ and $D \leq Sym$ (n) acting transitively on a set Δ, such that D is not cyclic, $|D| = n$ and $|\Delta| = r$. Then there exists a Wreath product if and only $r|n$, $r \neq n$.

Proof

Let W = C wr D, be a Wreath product of C and D, then $|W| = |C|^{|\Delta|}|D| = |C|^r n$. Since D is not cyclic, D contains no elements of order n. Let $\alpha \in D$ such that $\alpha^m = (1)$, then $m|n$. For transitivity of D on Δ, we must have m = r, thus and $r \neq n$ and $r|n$.
Conversely if $r \neq n$ and $r|n$, then the cyclic group H generated by the element
$\alpha = (1, 2, \ldots, r)$ of D is a subgroup of D transitive on the set $\Delta = \{1, 2, 3, \ldots, r\}$. Thus $|H| = r = |\Delta|$ and so by Proposition 1.6, a Wreath product of C and H, W = C wr H exists and is unique.

1.8 Lemma

Let C be a transitive permutation groups on the set Γ and let D and $D^{/}$ be transitive permutation groups on the set Δ. Then the Wreath products W = C wr D and
$W^{/}$ = C wr $D^{/}$ are isomorphic if and only if D and $D^{/}$ are isomorphic.

Proof

If D and $D^{/}$ are not isomorphic, then $|D| \neq |D^{/}|$, hence
$|W| = |C|^{|\Delta|}|D| \neq |C|^{|\Delta|}|D^{/}| = |W^{/}|$, W and $W^{/}$ do not have the same order and hence cannot be isomorphic.
Conversely if W and $W^{/}$ are not isomorphic then $|W| \neq |W^{/}|$, that is,
$|C|^{|\Delta|}|D| \neq |C|^{|\Delta|}|D^{/}|$, hence $|D| \neq |D^{/}|$ and so D is not isomorphic to $D^{/}$.

1.9 Theorem

For every prime number p, there is a non – abelian transitive p – group of degree p^2 isomorphic to a unique

Wreath product W = C wr D transitive on the set Γ x Δ with
$|\Gamma \times \Delta| = p^2$, where $|C| = |\Gamma| = p$, $|D| = |\Delta| = p$.

Proof

Let G be a p – group acting transitively on a set Ω with $|\Omega| = p^2$, p an arbitrary but fixed prime. Then the size of any of its orbits is of cardinality $p^2 > 1$ and it is readily seen that the order of G is at most p^{p+1}. Consequently G is non – abelian and
$|G| = p^{p+1} = p^p p = |C|^{|\Delta|}|D| = |C \text{ wr } D|$. Clearly C and D are cyclic and by Proposition 1.6, the Wreath product C wr D is the unique group acting transitively on set ΓxΔ, of size p^2. Since Wreath products are non – abelian, it follows that G ≅ C wr D.

From Theorem 1.9, we deduce the following:

1.10 Corollary

There is, up to isomorphism, only one non – abelian transitive p – group of degree p^2 and order p^{p+1}, namely the Wreath product C_p wr C_p, for every prime number p.

1.11 Corollary

Every transitive p – group of degree p^2 and order p^{p+1} is isomorphic to a unique transitive p – group of degree p^3.

Proof:

We consider the transitive p – group $G^/$ of degree p^3 and order $|G^/| = p^{p+1} = |G|$, where G is transitive p – group of degree p^2. Then by Corollary 1.10, such group G is unique and whence $G^/ \cong G$.

1.12 Remark

We draw our attention here to the fact that a similar result to Corollary 1.10 was obtained by Audu, M. S. in [8].

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