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Existence and Uniqueness of Continuous Solutions of Fractional Mixed Type Integrodifferential Equations in Cone-Metric Spaces

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Abstract:
In this paper we investigate the existence and uniqueness for the fractional mixed type integro-differntial equations and the existence of unique common solution of the Urysohn integral equations in cone metric spaces. The result is obtained by using the some extensions of Banach's contraction principle, common fixed points for two self-mappings in complete cone metric space and the theory of cosine family.

Keywords: Cone metric space, Cosine family, fixed point, Contractive mapping, Ordered Banach space.

Introduction: The purpose of this paper is study the existence and uniqueness of solutions for the fractional mixed type integro-differntial equations of the forms[1],[9].[10],[13]:

$$\begin{bmatrix} d^{\alpha}x(t)/d^{\alpha}t = Ax(t) + \int_0^t k(t,s,x(s))ds + \int_0^T h(t,s,x(s))ds \\ x(0) = x_0, x_0 \in X \text{ and } J \in [0,T] \end{bmatrix} \qquad (1)$$

where J = [0, T], 0 $\leq \alpha < 1$,. −A(t, .) is a closed linear operator defined on a dense domain D(A) in X into X such that D(A) is independent of t. It is assumed also that −A(t, .) generates resolvent operator in the Banach space X .The nonlinear functions k, h: J × J × X→Z are continuous and x0 is element of X, X is Banach space. The operator d^{α} x(t)/d^{α} t denotes the Caputo fractional derivative of order α. properties The objective of the present paper is to study the existence and uniqueness of solutions the system under condition in respect of cone metric space and fixed point theory[3],[7] .In this research, The paper is organized as follows: Section 2, we discuss the preliminaries of cone metric space. Section 3, we dealt with study of the the fractional mixed type integro-differntial equations and in Section 4, we consider an Urysohn Volterra-Fredholm type equations. Finally in Section 5, we give examples to illustrate the application of our results. we study the existence of unique common solution of the Urysohn integral equations of Volterra-Fredholm type:

$$x(t) = \int_0^t k_1(t,s,x(s))ds + \int_0^T h_1(t,s,x(s))ds + g_1(t) \qquad (2)$$

$$x(t) = \int_0^t k_2(t,s,x(s))ds + \int_0^T h_2(t,s,x(s))ds + g_2(t) \qquad (3)$$

where x, g1, g2 : [a, T] → X; the functions ki, hi : [a, T] × [a, T] × X → X, (i = 1, 2), are continuous functions. The objective of the present paper is to study the existence and uniqueness of solution of the system (1) under the conditions in respect of the cone metric space, fixed point theory and the cosine family. Hence we extend and improve some results reported in [11]. We are motivated by the work of P. Raja and S. M. Vaezpour in [12] and influenced by the work of M. Arshad [9].

2. Preliminaries Let X is a Banach space with norm. Let B = C(J, X) be the Banach space of all continuous functions from J into X endowed with supremum norm

$\|x\|_{\infty} = \sup\{\|x(t)\| : t \in J\}$.

Let P = {(x, y): x, y ≥ 0} ∈ E = R^2 be a cone and define

$d(f,g) = (\|f-g\|_{\infty}, \alpha\|f-g\|_{\infty})$,

for every f, g ∈ B. Then it is easily seen that (B, d) is a cone metric space .

Let us recall the concepts of the cone metric and refer the reader [3], [7], [8], [9] for the more details. Let X be a real Banach space and P is a subset of X. Then P is called a cone if and only if,

1. P is closed ,nonempty and P≠0;
2. a, b ∈ R, a, b ≥ 0, x, y ∈ P→ ax +by ∈ P;
3. x ∈ P and –x ∈ P→ x=0,

For a given cone P subset of E, we define a partial ordering relation ≤ with respect to P by x≤ y if and only if y-x € P. We shall write x < y to indicate that x ≤ y but x ≠ y, while x << y will stand for y − x ∈ intP , where intP denotes the interior of P. The cone P is called normal if there is a number K >0 such that 0 ≤ x ≤ y implies $\|x\| \le K\|y\|$, for every x, y ∈ E. The least positive number satisfying

above is called the normal constant of P. In the following we always suppose E is a real Banach space , P is a cone in E with intP ≠ φ, and ≤ is partial ordering with respect to P. Definition 2.1 Let X be a nonempty set. Suppose that the mapping d : X × X → E satisfies:
(d1) 0 ≤ d(x, y) for all x, y ∈ X and d(x,y)=0 if and only if x = y;
(d2) d(x, y) = d(y, x), for all x, y ∈ X;
(d3) d(x, y) ≤ d(x, z) + d(z, y), for all x, y, z ∈ X.
Then d is called a cone metric on X and (X, d) is called a cone metric space.
The concept of cone metric space is more general than that of metric space.
Example 2.2 Let E = R2, P = {(x, y) ∈ E: x, y ≥ 0}, X = R, and
D : X × X → E such that d(x, y) = (|x − y|, α|x − y|), where α ≥ 0 is a constant. Then (X, d) is a cone metric space.

3 Existence and uniqueness of solutions

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

Definition3.1[5],[6]. The Riemann–Liouville fractional integral operator of order β > 0 of a function x : [0,1) → R is defined as

$$I^{\beta} x(t) = 1/\Gamma(\beta)\int_0^t (t-s)^{\beta-1} x(s)ds, where\ \Gamma\ is\ gamma\ functions$$

Definition3.2[5],[6]. The Caputo derivative of order α, for a function x : [0,1) → R can be written as

$$d^{\alpha} x(t)/d^{\alpha} t = 1/\Gamma(1-\alpha)\int_0^t x'(s)/(t-s)^{\alpha}, 0 \prec \alpha \le 1 \qquad (4)$$

$$where \quad x'(s) = dx(s)/ds$$

If x is an abstract function with values in X, then the integrals and derivatives which appear in (1) are taken in Bochner's sense.

Definition 3.3 (Compare[13] with [1]). A family of bounded linear operators Rz(t, s) ∈ B(X), 0 ≤ s ≤ t ≤ T is called resolvent operator for equations (1)) if the following conditions hol (a) Rz(t, s) is strongly continuous in t and s, Rz(t, t) = I, t ∈ J. Next we introduce the so-called "Mild Solution" and "Classical Solution" for (1).

Definition3.4(Compare[13] with [1]). A continuous solution x of the integral equation

$$x(t) = R_x(t,0)x_0 + 1/\Gamma(\alpha)\int_0^t (t-s)^{\alpha-1} R_x(t,s)(\int_0^s k(s,\iota,x(\iota))d\iota + \int_0^T h(s,\iota,x(\iota))d\iota)ds \quad (5)$$

with t ∈ J, is said to be a mild solution of (1) on J.

Definition3.5 ([1],[2]). By a classical solution of (1) on J, we mean a function x with values in X such that:
(i) x is continuous function on J,
(ii) d^{α} x(t)/d^{α} t exists and is continuous on (0, T), and satisfying (1)on J.

Lemma 3.6 [12] Let (X, d) be a complete cone metric space, where P is a normal cone with normal constant K. Let T : X → X be a function such thatthere exists a comparison function Φ : P → P such that d(T(x), T(y)) ≤ Φ(d(x, y)), for every x, y ∈ X. Then T has a unique fixed point.

We list the following hypotheses for our convenience:

(**H1**) There exist continuous functions p1, p2: J × J → R+ and a comparison Function Φ : R2 → R2 such that

$$(\|[(k(t,s,u)\text{- } k(t,s,v)]\|,\ \alpha\|[(k(t,s,u)\text{- } k(t,s,v)]\|) \le p_1(t,\ s)\Phi(d(u,\ v)),$$

$$(\|[(h(t,s,u)\text{- } h(t,s,v)]\|,\ \alpha\|[(h(t,s,u)\text{- } h(t,s,v)]\|) \le p_2(t,\ s)\Phi(d(u,\ v)),$$

For every t, s ∈ J and u, v ∈ Z.

(**H2**) $N\int_0^b [p1(t,s) + p2(t,s)]dsdt \le 1.$

Theorem3.7: Assume that hypotheses (H1)-(H2) hold. Then the abstract integral equation (1)–(2) has a unique solution x on J.

Proof: The operator F: B → B is defined by

$$Fx(t) = R_x(t,0)x_0 + 1/\Gamma(\alpha)\int_0^t (t-s)^{\alpha-1} R_x(t,s)(\int_0^t k(t,s,x(s))ds + \int_0^T h(t,s,x(s))ds)ds \quad (6)$$

By using the hypotheses (H1) − (H2), we have

$$(\|F_x(t) - F_y(t)\|, \alpha\|F_x(t) - F_y(t)\| \le \int_0^t N(\| \int_0^s k(s,\tau,x(\tau))d\tau + \int_0^T h(s,\tau,x(\tau))d\tau$$

$$-[\int_0^s k(s,\tau,x(\tau))d\tau + \int_0^T h(s,\tau,x(\tau))d\tau \ \|, \alpha(\| \int_0^s k(s,\tau,x(\tau))d\tau$$

$$+\int_0^T h(s,\tau,x(\tau))d\tau - \int_0^s k(s,\tau,x(\tau))d\tau - \int_0^T h(s,\tau,x(\tau))d\tau \ \|)ds$$

$$\le \int_0^t N[(\int_0^s \|k(s,\tau,x(\tau)) - k(s,\tau,y(\tau))\| d\tau, \alpha(\int_0^s \| k(s,\tau,x(\tau)) + k(s,\tau,y(\tau))\| d\tau$$

$$+(\int_0^T h(s,\tau,x(\tau)) - h(s,\tau,y(\tau))\| d\tau, \alpha(\int_0^T \| h(s,\tau,x(\tau))d\tau + h(s,\tau,y(\tau))\| d$$

$$\le \int_0^t N[\int_0^T p_1(s,\tau)\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)d\tau + \int_0^T p_2(s,\tau)\Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)d\tau]ds$$

$$\le \Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty)N\int_0^T\int_0^T [p_1(s,\tau)+p_2(s,\tau)]d\tau ds$$

$$= \Phi(\|x-y\|_\infty, \alpha\|x-y\|_\infty),$$

for every x, y ∈ B. This implies that d(Fx, Fy) ≤ Φ(d(x, y)), for every x, y ∈ B. Now an application of Lemma 3.6, the operator has a unique point in B. This means that the equation (1) has unique solution. This completes the proof of the Theorem 3.7.

4 Existence of Common solutions[8],[9]

Let X is a Banach space with norm. Let Z = C([a, b],X) be the Banach space of all continuous functions from J into X endowed with supremum norm

$$\|x\|_\infty = \sup\{\|x(t)\| : t \in J\}.$$

Let P = {(x, y) : x, y ≥ 0} ∈ E = R2 be a cone and define

$d(f, g) = (\|f - g\|_\infty, \alpha\|f - g\|_\infty)$

for every f, g ∈ Z. Then it is easily seen that (Z, d) is a cone metric space.

Definition4.1 [4] A pair (S, T) of self-mappings X is said to be weakly compatible if they commute at their coincidence point (i. e. ST x = TS x whenever S x = T x). A point y ∈ X is called point of coincidence of a family Tj, j = 1, 2, …,of self-mappings on X if there exists a point x ∈ X such that y = T j x for all j = 1, 2,… We need the following lemma for further discussion:

Lemma 4.2 [9] Let (X, d) be a complete cone metric space and P be an order cone. Let S, T, F: X → X be such that S(X) ∪ T(X) ⊂ F(X). Assume that the following conditions hold:

(i) d(Sx,Ty) ≤ αd(Fx,Sx)+βd(Fy, Ty)+γd(Fx, Fy), for all x, y ∈ X, with x ≠y, where α, β, γ are non-negative real numbers with α + β +γ < 1.

(ii) d(Sx, Tx) < d(Fx,Sx) + d(Fx,Tx), for all x ∈ X, whenever Sx ≠ Tx.

If F(X) or S(X) ∪ T(X) is a complete subspace of X, then S, T and f have a unique point of coincidence. Moreover, if (S, F) and (T, F) are weakly compatible, then S, T and F have a unique common fixed point.

We list the following hypotheses for our convenience:

(**H3**)Assume that for all t, s $\in$ [a,T].

$$Fx(t)=\int_0^t \mathrm{k}_1(\mathrm{t},s,\mathrm{x}(s))\mathrm{ds}+\int_0^T \mathrm{h}_1(\mathrm{t},s,\mathrm{x}(s))\mathrm{ds}$$

$$Gx(t)=\int_0^t \mathrm{k}_2(\mathrm{t},s,\mathrm{x}(s))\mathrm{d}\tau+\int_0^T \mathrm{h}_2(\mathrm{t},s,\mathrm{x}(s))\mathrm{ds}$$

$$(\left|Fx(t)-Gy(t)+g_1(t)-g_2(t)\right|,\alpha\left|Fx(t)-Gy(t)+g_1(t)-g_2(t)\right|)$$

(**H4**) $\leq \alpha(\left|Fx(t)+g_1(t)-x(t)\right|,p(\left|Fx(t)+g_1(t)-x(t)\right|$

$$+\beta\left| Gy(t)+g_2(t)-y(t)\right|,p(\left|Gy(t)+g_2(t)-y(t)\right|+\gamma(\left|x(t)-y(t)\right|,p\left|x(t)-y(t)\right|),$$

where $\alpha+\beta+\gamma<1$, for every x, y $\in$ Z with $\neq$ y and t $\in$ J.

H5) Whenever F x + g1 $\neq$ G x + g2

$\sup_{t\in J}$ (|Fx(t) – Gx(t) + g1(t) − g2(t)|, α|Fx(t) – Gx(t) +g1(t) −g2(t)|) ,

< $\sup_{t\in J}$ α(|Fx(t) + g1(t) − x(t)|, p|Gx(t) + g1(t) − x(t)|)+β(|Fx(t) + g2(t) − x(t)|, p|Gx(t) + g2(t) − x(t)|), for every x $\in$ Z.

Theorem 4.3 Assume that hypotheses (H3)−(H5) hold. Then the integral equations (3)–(4) have a unique common solution x on [a, b].

Proof: Define S, T : Z → Z by S(x) = Fx+g1 and T(x) = Gx+g2. Using hypotheses, we have

(|S x(t) − Ty(t)|, α |S x(t) − Ty(t)|) ≤ α (|S x(t) − x(t)|, p |S x(t) − x(t)|)
+ β(|Ty(t) − y(t)|, p |Ty(t) − y(t)|) + γ(|x(t) − y(t)|, p |x(t) − y(t)|)

For every x,y $\in$ Z and x≠y. Hence

$$(\|S-T\|_\infty,\|S-T\|_\infty\leq\alpha(\|Sx-x\|_\infty\leq\alpha(\|Sx-x\|_\infty,p\|Sx-x\|_\infty)$$

$$\beta(\|Ty-y\|_\infty,p\|Tx-x\|_\infty)$$

For every x $\in$ Z.By lemma 4.2 , if f is the identity map on Z, the Urysohn integral equations (2), (3) have a unique comm solution. This completes the proof of theorem 4.3.

5 Application

In order to illustrate the applications of some of our result established in previous section, we consider the fractional mixed Volterra–Fredholm partial integro -differential equation

$$\frac{\partial^\alpha w(t,u)}{\partial t^\alpha}=\frac{\partial^2 w(t,u)}{\partial t^2}+\int_0^t[\frac{ts+w(s,u)s}{2}]ds+\int_0^1[(ts)^2+\frac{tsw^2(s,u)}{2}]ds,$$

$t\in[0,1],\ u\in I=[0,\pi],$ (7)

$w(t,0)=w(t,\pi)=0, t\in[0,1],$ (8)

$\left.\frac{\partial w(t,u)}{\partial t}\right|_{t=0}=x_0(u), u\in I,$ (9)

$let\ X=L^2([0,1])\, and\, w(t,u)=x(t)(u)$

where a : (0, 1)×[0, T]×R → R, , k1, h1 : [0, T]×[0, T]× R → R are continuous functions.

First, we reduce the equations (7)–(9) into (1) by making suitable choices of A, k and h. Let X = L^2[0, 1] be the space of square integrable functions. Define the operator

A(t, .) : X → X by (A z)(u) = a(u, t, .)z'' with dense domain D(A(t, ・)) = {z $\in$ X : z, z' are absolutely continuous, z'' $\in$ X and z(0) = z(1) = 0}, generates an evolution system and Rx(t, s) can be extracted from evolution system, such that Rx(t, s)≤1, for s < t .Define the functions

$$k(t,s,x(s))=ts+\frac{x}{2}\, and\ k(t,s,x(s))=(ts)^2+\frac{tsx^2}{2}.Now\ we\ have$$

(|k(t, s, x(s)) − k(t, s, y(s))|, α|k(t, s, x(s)) − k(t, s, y(s))|)

=(|ts +xs/2− ts – ys/2|, α|ts +xs/2− ts – ys/2|)

$= (|xs/2 - ys/2|, \alpha|xs/2 - ys/2|)$
$= (s/2(|x - y|, \alpha\, s/2|x - y|) = s/2\, (|x - y|, \alpha|x - y|) \le s/2\, (\|x - y\|_\infty, \alpha \|x - y\|_\infty)$
$= p^*_1\Phi^*((\|x - y\|_\infty, \alpha \|x - y\|_\infty)$.
where $p_1{*}1(t, s) = s$, which is continuous function of $[0, 1]\times[0, 1]$ into R+ and acomparison function $\Phi^* : R2 \to R2$ such that $\Phi^*(x, y) = 1/2\, (x, y)$. Similarly, we can show that
$(|h(t, s, x(s)) - h(t, s, y(s))|, \alpha|h(t, s, x(s)) - h(t, s, y(s))|) \le p^*_2\Phi^*((\|x - y\|_\infty, \alpha \|x - y\|_\infty)$.
where $p_2^*(t, s) = st$, which is continuous function of $[0, 1] \times [0, 1]$ into R+. Moreover,

$$\int_0^1 [p^*_1(s,t) + p_2^*(s,t)]ds = \int_0^1 [s + st]ds = 1/2(1+t)$$

$$\sup_{t\in[0,1]} \{1/2(1+t)\} = 1.$$

$$\int_0^1 \int_0^1 [p^*_1(s,t) + p_2^*(s,t)]dsdt = \int_0^1 \int_0^1 [s + st]dsdt = \int_0^1 1/2(1+t)dt \le \frac{3}{4} \le 1.$$

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