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A Fixed Point Theorem in Polish Spaces with W-Distance

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Abstract

In the present paper we establish a fixed point theorem for polish spaces for W-distance.

***Key words*: -** Polish space, Common fixed point, weakly commuting.

1. Introduction and Preliminaries

Definition 1.1: A metric space (X, d) is said to a polish space, if it is satisfying the following condition:

(i) X is complete,
(ii) X is separable

Definition 1.2: Let (X, d) be a Polish Space and let F and G be a mapping from $\Omega \times X \to X$ and $w \in \Omega$ be a selector. The mapping F and G will be called weakly commuting iff

$$d\big(FGx(w), GFx(w)\big) \le d\big(Fx(w), Gx(w)\big), \quad \text{for all } x \in X.$$

Definition 1.3: Let (X, d) be a Polish Space and let p be a mapping from $(\Omega \times X) \times (\Omega \times X) \to [0, \infty)$. The mapping p be called w-distance on X if

(i) $p\big(x(w), y(w)\big) \le p\big(x(w), z(w)\big) + p\big(z(w), y(w)\big)$ for $x, y, z \in X$
And $w \in \Omega$ be a selector.
(ii) For any $x \in X$, $p(x(w), .) \to [0, \infty)$ is a lower semi-continuous and
(iii) For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$p\big(z(w), x(w)\big) \le \delta \text{ and } p\big(z(w), y(w)\big) \le \delta \text{ imply } p\big(x(w), y(w)\big) \le \varepsilon$$

For any $x, y, z \in X, w \in \Omega$ be a selector.

In the present paper we prove a common fixed point theorem for three self mappings of a polish metric space with w-distance.

2. Main result

Theorem 2.1: Let F be a self mapping and G and H be continuous self mappings of Polish metric space (X, d) with a w-distance p satisfying the conditions:

(i) $FX \subset GX \cap HX$
(ii) $$p\big(Fx(w), Fy(w)\big) \le \alpha \frac{p^3(Gx(w),Fx(w)) - p^3(Hy(w),Fy(w))}{[p(Gx(w),Fx(w)) + p(Gx(w),Fy(w))].[p(Gx(w),Fx(w)) - p(Hy(w),Fy(w))]}$$
$$+\beta \frac{p^4(Gx(w),Fx(w)) - p^4(Hy(w),Fy(w))}{[p(Gx(w),Fx(w)) + p(Gx(w),Fy(w))].[p^2(Gx(w),Fx(w)) - p^2(Hy(w),Fy(w))]}$$
$$+\gamma \frac{p^2(Hy(w),Fx(w)) + p^2(Gx(w),Fy(w))}{p(Gx(w),Fx(w)) + p(Gx(w),Fy(w))}$$

For all $x, y \in X$ where $\alpha, \beta, \gamma > 0, \alpha + \beta + \gamma < \frac{1}{4}$ and $w \in \Omega$ be a selector.

(iii) $\{F, G\}$ and $\{F, H\}$ are weakly commuting pair.

Then F, G and H have a unique common fixed point in X.

Proof: Let $x_0(w)$ be an arbitrary point in X. Then $Fx_0(w) \in X$. Since $FX \subset GX$ there exist a point $x_1(w) \in X$ such that $Fx_0(w) = Gx_1(w)$. Since $FX \subset HX$ there exist a point $x_2(w) \in X$ such that $Fx_1(w) = Hx_2(w)$. In general one can choose point $x_{2n+1}(w)$ and $x_{2n+2}(w)$ such that

$$Fx_{2n}(w) = Gx_{2n+1}(w) \text{ and } Fx_{2n+1}(w) = Hx_{2n+2}(w), \text{ for } n = 0,1,2,3, \dots$$

$$p(Fx_{2n}(w), Fx_{2n+1}(w)) = p(Fx_{2n+1}(w), Fx_{2n}(w))$$

$$\leq \alpha \frac{p^3(Gx_{2n+1}(w),Fx_{2n+1}(w)) - p^3(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gx_{2n+1}(w),Fx_{2n+1}(w)) + p(Gx_{2n+1}(w),Fx_{2n}(w))].[p(Gx_{2n+1}(w),Fx_{2n+1}(w)) - p(Hx_{2n}(w),Fx_{2n}(w))]}$$

$$+\beta \frac{p^4(Gx_{2n+1}(w),Fx_{2n+1}(w)) - p^4(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gx_{2n+1}(w),Fx_{2n+1}(w)) + p(Gx_{2n+1}(w),Fx_{2n}(w))].[p^2(Gx_{2n+1}(w),Fx_{2n+1}(w)) - p^2(Hx_{2n}(w),Fx_{2n}(w))]}$$

$$+\gamma \frac{p^2(Hx_{2n}(w),Fx_{2n+1}(w)) + p^2(Gx_{2n+1}(w),Fx_{2n}(w))}{p(Gx_{2n+1}(w),Fx_{2n+1}(w)) + p(Gx_{2n+1}(w),Fx_{2n}(w))}$$

$$\leq \alpha \frac{p^3(Fx_{2n}(w),Fx_{2n+1}(w)) - p^3(Fx_{2n-1}(w),Fx_{2n}(w))}{[p(Fx_{2n}(w),Fx_{2n+1}(w)) + p(Fx_{2n}(w),Fx_{2n}(w))].[p(Fx_{2n}(w),Fx_{2n+1}(w)) - p(Fx_{2n-1}(w),Fx_{2n}(w))]}$$

$$+\beta \frac{p^4(Fx_{2n}(w),Fx_{2n+1}(w)) - p^4(Fx_{2n-1}(w),Fx_{2n}(w))}{[p(Fx_{2n}(w),Fx_{2n+1}(w)) + p(Fx_{2n}(w),Fx_{2n}(w))].[p^2(Fx_{2n}(w),Fx_{2n+1}(w)) - p^2(Fx_{2n-1}(w),Fx_{2n}(w))]}$$

$$+\gamma \frac{p^2(Fx_{2n-1}(w),Fx_{2n+1}(w)) + p^2(Fx_{2n}(w),Fx_{2n}(w))}{p(Fx_{2n}(w),Fx_{2n+1}(w)) + p(Fx_{2n}(w),Fx_{2n}(w))}$$

$$p^2(Fx_{2n}(w), Fx_{2n+1}(w)) \leq \alpha \frac{\begin{bmatrix} p(Fx_{2n}(w),Fx_{2n+1}(w)) \\ -p(Fx_{2n-1}(w),Fx_{2n}(w)) \end{bmatrix} . \begin{bmatrix} p^2(Fx_{2n}(w),Fx_{2n+1}(w)) + p^2(Fx_{2n-1}(w),Fx_{2n}(w)) \\ +p(Fx_{2n}(w),Fx_{2n+1}(w)).p(Fx_{2n-1}(w),Fx_{2n}(w)) \end{bmatrix}}{p(Fx_{2n}(w),Fx_{2n+1}(w)) - p(Fx_{2n-1}(w),Fx_{2n}(w))}$$

$$+\beta \frac{\begin{bmatrix} p^2(Fx_{2n}(w),Fx_{2n+1}(w)) \\ -p^2(Fx_{2n-1}(w),Fx_{2n}(w)) \end{bmatrix} . \begin{bmatrix} p^2(Fx_{2n}(w),Fx_{2n+1}(w)) \\ +p^2(Fx_{2n-1}(w),Fx_{2n}(w)) \end{bmatrix}}{p^2(Fx_{2n}(w),Fx_{2n+1}(w)) - p^2(Fx_{2n-1}(w),Fx_{2n}(w))} + \gamma p^2(Fx_{2n-1}(w), Fx_{2n+1}(w))$$

$$\leq \alpha \begin{bmatrix} p^2(Fx_{2n}(w), Fx_{2n+1}(w)) + p^2(Fx_{2n-1}(w), Fx_{2n}(w)) \\ +p(Fx_{2n}(w), Fx_{2n+1}(w)). p(Fx_{2n-1}(w), Fx_{2n}(w)) \end{bmatrix}$$

$$+\beta[p^2(Fx_{2n}(w), Fx_{2n+1}(w)) + p^2(Fx_{2n-1}(w), Fx_{2n}(w))] + \gamma p^2(Fx_{2n-1}(w), Fx_{2n+1}(w))$$

$$\leq \alpha \begin{bmatrix} \{p(Fx_{2n}(w), Fx_{2n+1}(w)) + p(Fx_{2n-1}(w), Fx_{2n}(w))\}^2 \\ -\{p(Fx_{2n}(w), Fx_{2n+1}(w)). p(Fx_{2n-1}(w), Fx_{2n}(w))\} \end{bmatrix}$$

$$+\beta[p(Fx_{2n}(w), Fx_{2n+1}(w)) + p(Fx_{2n-1}(w), Fx_{2n}(w))]^2$$

$$+\gamma[p(Fx_{2n}(w), Fx_{2n+1}(w)) + p(Fx_{2n-1}(w), Fx_{2n}(w))]^2$$

$$p^2(Fx_{2n}(w), Fx_{2n+1}(w)) \leq (\alpha + \beta + \gamma)[p(Fx_{2n}(w), Fx_{2n+1}(w)) + p(Fx_{2n-1}(w), Fx_{2n}(w))]^2$$

$$p(Fx_{2n}(w), Fx_{2n+1}(w)) \leq \sqrt{(\alpha + \beta + \gamma)}[p(Fx_{2n}(w), Fx_{2n+1}(w)) + p(Fx_{2n-1}(w), Fx_{2n}(w))]$$

$$p(Fx_{2n}(w), Fx_{2n+1}(w)) \leq \frac{\sqrt{(\alpha+\beta+\gamma)}}{1-\sqrt{(\alpha+\beta+\gamma)}} p(Fx_{2n-1}(w), Fx_{2n}(w))$$

$$p(Fx_{2n}(w), Fx_{2n+1}(w)) \leq k\, p(Fx_{2n-1}(w), Fx_{2n}(w))$$

Let $k = \frac{\sqrt{(\alpha+\beta+\gamma)}}{1-\sqrt{(\alpha+\beta+\gamma)}} < 1$ with $\sqrt{(\alpha+\beta+\gamma)} > 0$

Similarly,

$$p\big(Fx_{2n+1}(w), Fx_{2n+2}(w)\big) < k\, p\big(Fx_{2n}(w), Fx_{2n+1}(w)\big)$$

In general,

$$p\big(Fx_n(w), Fx_{n+1}(w)\big) < k^n\, p\big(Fx_0(w), Fx_1(w)\big)$$

Now, we shall prove that $\{Fx_n(w)\}$ is a Cauchy Sequence. Since $\lim_{n\to\infty} p\big(Fx_n(w), Fx_{n+1}(w)\big) = 0$, it is sufficient to show that the sequence $\{Fx_{2n}(w)\}$ is a Cauchy Sequence. Suppose that $\{Fx_{2n}(w)\}$ is not a Cauchy Sequence. Then there is $\epsilon > 0$ such that for each integer $2k, k = 0,1,2,\dots$ there even integers $2n(k)$ and $2m(k)$ with $2k \le 2n(k) \le 2m(k)$ such that

$$p\left(Fx_{2n(k)}(w), F_{2m(k)}(w)\right) > \epsilon \tag{2.1.1}$$

Let for each even integer $2k$, $2m(k)$ be the least exceeding $2n(k)$ and satisfying (2.1.1). Therefore

$$p\left(Fx_{2n(k)}(w), F_{2m(k)-2}(w)\right) \le \epsilon \tag{2.1.2}$$

Then for each even integer $2k$, we have from (2.1.1)

$$\epsilon < p\left(Fx_{2n(k)}(w), F_{2m(k)}(w)\right)$$

$$\le p\left(Fx_{2n(k)}(w), F_{2m(k)-2}(w)\right) + p\left(Fx_{2m(k)-2}(w), F_{2m(k)-1}(w)\right) + p\left(Fx_{2m(k)-1}(w), F_{2m(k)}(w)\right)$$

Using (2.1.2) and $\lim_{n\to\infty} p\big(Fx_n(w), Fx_{n+1}(w)\big) = 0$, we have

$$\lim_{k\to\infty} p\left(Fx_{2n(k)}(w), F_{2m(k)}(w)\right) = \epsilon \tag{2.1.3}$$

If follows immediately from the triangular inequality that

$$\left|p\left(Fx_{2n(k)+1}(w), Fx_{2m(k)-1}(w)\right) - p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)\right| \le p\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)$$
$$+p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)$$

Using (2.1.3), and $\lim_{n\to\infty} p\big(Fx_n(w), Fx_{n+1}(w)\big) = 0$, we get

$$p\left(Fx_{2n(k)+1}(w), Fx_{2m(k)-1}(w)\right) \to \epsilon \tag{2.1.4}$$

Now

$$p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right) \le p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) + p\left(Fx_{2n(k)+1}(w), Fx_{2m(k)}(w)\right)$$

$$\le p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)$$

$$+\alpha \frac{p^3\left(Gx_{2n(k)+1}(w),Fx_{2n(k)+1}(w)\right)-p^3\left(Hx_{2m(k)}(w),Fx_{2m(k)}(w)\right)}{\left[p\left(Gx_{2n(k)+1}(w),Fx_{2n(k)+1}(w)\right)+p\left(Gx_{2n(k)+1}(w),Fx_{2m(k)}(w)\right)\right].\left[p\left(Gx_{2n(k)+1}(w),Fx_{2n(k)+1}(w)\right)-p\left(Hx_{2m(k)}(w),Fx_{2m(k)}(w)\right)\right]}$$

$$+\beta \frac{p^4\left(Gx_{2n(k)+1}(w),Fx_{2n(k)+1}(w)\right)-p^4\left(Hx_{2m(k)}(w),Fx_{2m(k)}(w)\right)}{\left[p\left(Gx_{2n(k)+1}(w),Fx_{2n(k)+1}(w)\right)+p\left(Gx_{2n(k)+1}(w),Fx_{2m(k)}(w)\right)\right].\left[p^2\left(Gx_{2n(k)+1}(w),Fx_{2n(k)+1}(w)\right)-p^2\left(Hx_{2m(k)}(w),Fx_{2m(k)}(w)\right)\right]}$$

$$+\gamma \frac{p^2\left(Hx_{2m(k)}(w), Fx_{2n(k)+1}(w)\right)+p^2\left(Gx_{2n(k)+1}(w), Fx_{2m(k)}(w)\right)}{p\left(Gx_{2n(k)+1}(w), Fx_{2n(k)+1}(w)\right)+p\left(Gx_{2n(k)+1}(w), Fx_{2m(k)}(w)\right)}$$

$$\leq p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)$$

$$+\alpha \frac{p^3\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)-p^3\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)}{\left[p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)\right].\left[p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)-p\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)\right]}$$

$$+\beta \frac{p^4\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)-p^4\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)}{\left[p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)\right].\left[p^2\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)-p^2\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)\right]}$$

$$+\gamma \frac{p^2\left(Fx_{2m(k)-1}(w), Fx_{2n(k)+1}(w)\right)+p^2\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)}{p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)}$$

$$\leq p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)$$

$$+\alpha \frac{p^2\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)+p^2\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)+p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right).p\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)}{\left[p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)\right]}$$

$$+\beta \frac{p^2\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)+p^2\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)}{\left[p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)\right]}$$

$$+\gamma \frac{p^2\left(Fx_{2m(k)-1}(w), Fx_{2n(k)+1}(w)\right)+p^2\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)}{p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)}$$

Using (2.1.3), (2.1.4) and $\lim_{n\to\infty} p\left(Fx_n(w), Fx_{n+1}(w)\right) = 0$ we have

$\Longrightarrow \varepsilon \leq 2\gamma\epsilon$

This is a contradiction. Hence $\{Fx_n(w)\}$ is a Cauchy Sequence and then by completeness of X, there is a point $z(w) \in X$ such that $Fx_n(w) \to z(w)$.

Since the sequence $\{Gx_{2n+1}(w)\}$ and $\{Hx_{2n}(w)\}$ are subsequences of $\{Fx_n(w)\}$, they have the same limit $z(w)$. Since G and H are continuous, we have $GHx_{2n}(w) \to Gz(w)$ and $HGx_{2n+1}(w) \to Hz(w)$.

Now,

$$p\left(GHx_{2n}(w), HGx_{2n+1}(w)\right) = p\left(GFx_{2n-1}(w), HFx_{2n}(w)\right)$$

$$\leq p\left(GFx_{2n-1}(w), FGx_{2n-1}(w)\right) + p\left(FGx_{2n-1}(w), FHx_{2n}(w)\right) + p\left(FHx_{2n}(w), HFx_{2n}(w)\right)$$

Using condition (2.1-(iii)), the weak commutative pair of $\{F, G\}$ and $\{F, H\}$, we get,

$$p\left(GHx_{2n}(w), HGx_{2n+1}(w)\right) \leq p\left(Gx_{2n-1}(w), Fx_{2n-1}(w)\right) + p\left(FGx_{2n-1}(w), FHx_{2n}(w)\right)$$

$$+p\left(Fx_{2n}(w), Hx_{2n}(w)\right)$$

$$\leq p\left(Gx_{2n-1}(w), Fx_{2n-1}(w)\right)$$

$$+\alpha \frac{p^3\left(G^2x_{2n-1}(w), FGx_{2n-1}(w)\right)-p^3\left(H^2x_{2n}(w), FHx_{2n}(w)\right)}{[p(G^2x_{2n-1}(w), FGx_{2n-1}(w))+p(G^2x_{2n-1}(w), FHx_{2n}(w))].[p(G^2x_{2n-1}(w), FGx_{2n-1}(w))-p(H^2x_{2n}(w), FHx_{2n}(w))]}$$

$$+\beta \frac{p^4\left(G^2x_{2n-1}(w), FGx_{2n-1}(w)\right)-p^4\left(H^2x_{2n}(w), FHx_{2n}(w)\right)}{[p(G^2x_{2n-1}(w), FGx_{2n-1}(w))+p(G^2x_{2n-1}(w), FHx_{2n}(w))].[p^2(G^2x_{2n-1}(w), FGx_{2n-1}(w))-p^2(H^2x_{2n}(w), FHx_{2n}(w))]}$$

$$+\gamma \frac{p^2\big(H^2x_{2n}(w),FGx_{2n-1}(w)\big)+p^2\big(G^2x_{2n-1}(w),FHx_{2n}(w)\big)}{p(G^2x_{2n-1}(w),FGx_{2n-1}(w))+p(G^2x_{2n-1}(w),FHx_{2n}(w))}+\ p\big(Fx_{2n}(w),Hx_{2n}(w)\big)$$

$$\leq p\big(Gx_{2n-1}(w),Fx_{2n-1}(w)\big)$$

$$+\alpha \frac{p^2\big(G^2x_{2n-1}(w),FGx_{2n-1}(w)\big)+p^2\big(H^2x_{2n}(w),FHx_{2n}(w)\big)+p\big(G^2x_{2n-1}(w),FGx_{2n-1}(w)\big).p\big(H^2x_{2n}(w),FHx_{2n}(w)\big)}{[p(G^2x_{2n-1}(w),FGx_{2n-1}(w))+p(G^2x_{2n-1}(w),FHx_{2n}(w))]}$$

$$+\beta \frac{p^2\big(G^2x_{2n-1}(w),FGx_{2n-1}(w)\big)+p^2\big(H^2x_{2n}(w),FHx_{2n}(w)\big)}{[p(G^2x_{2n-1}(w),FGx_{2n-1}(w))+p(G^2x_{2n-1}(w),FHx_{2n}(w))]}$$

$$+\gamma \frac{p^2\big(H^2x_{2n}(w),FGx_{2n-1}(w)\big)+p^2\big(G^2x_{2n-1}(w),FHx_{2n}(w)\big)}{p(G^2x_{2n-1}(w),FGx_{2n-1}(w))+p(G^2x_{2n-1}(w),FHx_{2n}(w))}+\ p\big(Fx_{2n}(w),Hx_{2n}(w)\big)$$

$$\leq p\big(Gx_{2n-1}(w),Fx_{2n-1}(w)\big)$$

$$+\alpha \frac{\begin{bmatrix}p\big(G^2x_{2n-1}(w),GFx_{2n-1}(w)\big)\\+p\big(Gx_{2n-1}(w),Fx_{2n-1}(w)\big)\end{bmatrix}^2+\begin{bmatrix}p\big(H^2x_{2n}(w),HFx_{2n}(w)\big)\\+p\big(Hx_{2n}(w),Fx_{2n}(w)\big)\end{bmatrix}^2+\begin{bmatrix}p\big(G^2x_{2n-1}(w),GFx_{2n-1}(w)\big)\\+p\big(Gx_{2n-1}(w),Fx_{2n-1}(w)\big)\end{bmatrix}.\begin{bmatrix}p\big(H^2x_{2n}(w),HFx_{2n}(w)\big)\\+p\big(Hx_{2n}(w),Fx_{2n}(w)\big)\end{bmatrix}}{[\{p(G^2x_{2n-1}(w),GFx_{2n-1}(w))+p(Gx_{2n-1}(w),Fx_{2n-1}(w))\}+\{p(G^2x_{2n-1}(w),HFx_{2n}(w))+p(Hx_{2n}(w),Fx_{2n}(w))\}]}$$

$$+\beta \frac{\big[p\big(G^2x_{2n-1}(w),GFx_{2n-1}(w)\big)+p\big(Gx_{2n-1}(w),Fx_{2n-1}(w)\big)\big]^2+\big[p\big(H^2x_{2n}(w),HFx_{2n}(w)\big)+p\big(Hx_{2n}(w),Fx_{2n}(w)\big)\big]^2}{[\{p(G^2x_{2n-1}(w),GFx_{2n-1}(w))+p(Gx_{2n-1}(w),Fx_{2n-1}(w))\}+\{p(G^2x_{2n-1}(w),HFx_{2n}(w))+p(Hx_{2n}(w),Fx_{2n}(w))\}]}$$

$$+\gamma \frac{\big[p\big(H^2x_{2n}(w),GFx_{2n-1}(w)\big)+p\big(Gx_{2n-1}(w),Fx_{2n-1}(w)\big)\big]^2+\big[p\big(G^2x_{2n-1}(w),HFx_{2n}(w)\big)+p\big(Hx_{2n-1}(w),Fx_{2n}(w)\big)\big]^2}{[p(G^2x_{2n-1}(w),GFx_{2n-1}(w))+p(Gx_{2n-1}(w),Fx_{2n-1}(w))]+[p(G^2x_{2n-1}(w),HFx_{2n}(w))+p(Hx_{2n}(w),Fx_{2n}(w))]}$$

$$+p\big(Fx_{2n}(w),Hx_{2n}(w)\big)$$

Letting $n \to \infty$, we get

$$p\big(Gz(w),Hz(w)\big) \leq p\big(z(w),z(w)\big)$$

$$+\alpha \frac{\begin{bmatrix}p\big(Gz(w),Gz(w)\big)\\+p\big(z(w),z(w)\big)\end{bmatrix}^2+\begin{bmatrix}p\big(Hz(w),Hz(w)\big)\\+p\big(z(w),z(w)\big)\end{bmatrix}^2+\begin{bmatrix}p\big(Gz(w),Gz(w)\big)\\+p\big(z(w),z(w)\big)\end{bmatrix}.\begin{bmatrix}p\big(Hz(w),Hz(w)\big)\\+p\big(z(w),z(w)\big)\end{bmatrix}}{[\{p(Gz(w),Gz(w))+p(z(w),z(w))\}+\{p(Gz(w),Hz(w))+p(z(w),z(w))\}]}$$

$$+\beta \frac{[p(Gz(w),Gz(w))+p(z(w),z(w))]^2+[p(Hz(w),Hz(w))+p(z(w),z(w))]^2}{[\{p(Gz(w),Gz(w))+p(z(w),z(w))\}+\{p(Gz(w),Hz(w))+p(z(w),z(w))\}]}$$

$$+\gamma \frac{[p(Hz(w),Gz(w))+p(z(w),z(w))]^2+[p(Gz(w),Hz(w))+p(z(w),z(w))]^2}{[p(Gz(w),Gz(w))+p(z(w),z(w))]+[p(Gz(w),Hz(w))+p(z(w),z(w))]}+p\big(z(w),z(w)\big)$$

$$p\big(Gz(w),Hz(w)\big) \leq 2\gamma p\big(Hz(w),Gz(w)\big)$$

This is contradiction. Therefore $Gz(w) = Hz(w)$.

Now, we shall prove that $Fz(w) = Gz(w)$.

Consider

$$p\big(GFx_{2n+1}(w),Fz(w)\big) \leq p\big(GFx_{2n+1}(w),FGx_{2n+1}(w)\big)+p\big(FGx_{2n+1},Fz(w)\big)$$

By the weak commutatively of $\{F,G\}$, we have

$$p\big(GFx_{2n+1}(w),Fz(w)\big) \leq p\big(Gx_{2n+1}(w),Fx_{2n+1}(w)\big)$$

$$+\alpha \frac{p^3\big(G^2x_{2n+1}(w),FGx_{2n+1}(w)\big)-p^3(Hz(w),Fz(w))}{[p(G^2x_{2n+1}(w),FGx_{2n+1}(w))+p(G^2x_{2n+1}(w),Fz(w))].[p(G^2x_{2n+1}(w),FGx_{2n+1}(w))-p(Hz(w),Fz(w))]}$$

$$+\beta\frac{p^4\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)-p^4\left(Hz(w),Fz(w)\right)}{\left[p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right].\left[p^2\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)-p^2\left(Hz(w),Fz(w)\right)\right]}$$

$$+\gamma\frac{p^2\left(Hz(w),FGx_{2n+1}(w)\right)+p^2\left(G^2x_{2n+1}(w),Fz(w)\right)}{p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)}$$

$$\leq p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)$$

$$+\alpha\frac{p^2\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p^2\left(Hz(w),Fz(w)\right)+p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right).p\left(Hz(w),Fz(w)\right)}{\left[p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right]}$$

$$+\beta\frac{p^2\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p^2\left(Hz(w),Fz(w)\right)}{\left[p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right]}$$

$$+\gamma\frac{p^2\left(Hz(w),FGx_{2n+1}(w)\right)+p^2\left(G^2x_{2n+1}(w),Fz(w)\right)}{p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)}$$

$$\leq p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)$$

$$+\alpha\frac{\begin{Bmatrix}p\left(G^2x_{2n+1}(w),GFx_{2n+1}(w)\right)\\+p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)\end{Bmatrix}^2+p^2\left(Hz(w),Fz(w)\right)+\begin{Bmatrix}p\left(G^2x_{2n+1}(w),GFx_{2n+1}(w)\right)\\+p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)\end{Bmatrix}.p\left(Hz(w),Fz(w)\right)}{\left[\left\{p\left(G^2x_{2n+1}(w),GFx_{2n+1}(w)\right)+p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)\right\}+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right]}$$

$$+\beta\frac{\left[p\left(G^2x_{2n+1}(w),GFx_{2n+1}(w)\right)+p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)\right]^2+p^2\left(Hz(w),Fz(w)\right)}{\left[\left\{p\left(G^2x_{2n+1}(w),GFx_{2n+1}(w)\right)+p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)\right\}+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right]}$$

$$+\gamma\frac{\left[p\left(Hz(w),GFx_{2n+1}(w)\right)+p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)\right]^2+p^2\left(G^2x_{2n+1}(w),Fz(w)\right)}{\left[p\left(G^2x_{2n+1}(w),GFx_{2n+1}(w)\right)+p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)\right]+p\left(G^2x_{2n+1}(w),Fz(w)\right)}$$

Letting $n\to\infty$, we get

$$p\left(Gz(w),Fz(w)\right)\leq p\left(z(w),z(w)\right)$$

$$+\alpha\frac{\begin{Bmatrix}p\left(Gz(w),Gz(w)\right)\\+p\left(z(w),z(w)\right)\end{Bmatrix}^2+p^2\left(Hz(w),Fz(w)\right)+\begin{Bmatrix}p\left(Gz(w),Gz(w)\right)\\+p\left(z(w),z(w)\right)\end{Bmatrix}.p\left(Hz(w),Fz(w)\right)}{\left[\left\{p\left(Gz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right\}+p\left(Gz(w),Fz(w)\right)\right]}$$

$$+\beta\frac{\left[p\left(Gz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right]^2+p^2\left(Hz(w),Fz(w)\right)}{\left[\left\{p\left(Gz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right\}+p\left(Gz(w),Fz(w)\right)\right]}$$

$$+\gamma\frac{\left[p\left(Hz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right]^2+p^2\left(Gz(w),Fz(w)\right)}{\left[p\left(Gz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right]+p\left(Gz(w),Fz(w)\right)}$$

$$p\left(Gz(w),Fz(w)\right)\leq(\alpha+\beta+\gamma)p\left(Gz(w),Fz(w)\right)$$

This is contradiction. Hence $Gz(w)=Fz(w)$.

Thus $Fz(w)=Gz(w)=Hz(w)$.

It now follows that

$$p\left(Fz(w),Fx_{2n}(w)\right)\leq\alpha\frac{p^3\left(Gz(w),Fz(w)\right)-p^3\left(Hx_{2n}(w),Fx_{2n}(w)\right)}{\left[p\left(Gz(w),Fz(w)\right)+p\left(Gz(w),Fx_{2n}(w)\right)\right].\left[p\left(Gz(w),Fz(w)\right)-p\left(Hx_{2n}(w),Fx_{2n}(w)\right)\right]}$$

$$+\beta\frac{p^4\left(Gz(w),Fz(w)\right)-p^4\left(Hx_{2n}(w),Fx_{2n}(w)\right)}{\left[p\left(Gz(w),Fz(w)\right)+p\left(Gz(w),Fx_{2n}(w)\right)\right].\left[p^2\left(Gz(w),Fz(w)\right)-p^2\left(Hx_{2n}(w),Fx_{2n}(w)\right)\right]}+\gamma\frac{p^2\left(Hx_{2n}(w),Fz(w)\right)+p^2\left(Gz(w),Fx_{2n}(w)\right)}{p\left(Gz(w),Fz(w)\right)+p\left(Gz(w),Fx_{2n}(w)\right)}$$

$$\leq\alpha\frac{p^2\left(Gz(w),Fz(w)\right)+p^2\left(Hx_{2n}(w),Fx_{2n}(w)\right)+p\left(Gz(w),Fz(w)\right).p\left(Hx_{2n}(w),Fx_{2n}(w)\right)}{\left[p\left(Gz(w),Fz(w)\right)+p\left(Gz(w),Fx_{2n}(w)\right)\right]}$$

$$+\beta \frac{p^2(Gz(w),Fz(w))+p^2(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fx_{2n}(w))]}+\gamma \frac{p^2(Hx_{2n}(w),Fz(w))+p^2(Gz(w),Fx_{2n}(w))}{p(Gz(w),Fz(w))+p(Gz(w),Fx_{2n}(w))}$$

Taking limit $n \to \infty$, we get

$$p(Fz(w),z(w)) \le \alpha \frac{p^2(Fz(w),Fz(w))+p^2(z(w),z(w))+p(Fz(w),Fz(w)).p(z(w),z(w))}{[p(Fz(w),Fz(w))+p(Fz(w),z(w))]}$$

$$+\beta \frac{p^2(Fz(w),Fz(w))+p^2(z(w),z(w))}{[p(Fz(w),Fz(w))+p(Fz(w),z(w))]}+\gamma \frac{p^2(z(w),Fz(w))+p^2(Fz(w),z(w))}{p(Fz(w),Fz(w))+p(Fz(w),z(w))}$$

$$\Rightarrow p(Fz(w),z(w)) \le 2\gamma p(z(w),Fz(w))$$

This is a contradiction and therefore $Fz(w)=z(w)=Gz(w)=Hz(w)$.

Thus $z(w)$ is a common fixed point of F,G and H.

Uniqueness: Let $u(w)$ be another point of F,G and H. Then

$$p(z(w),u(w)) = p(Fz(w),Fu(w))$$

$$\le \alpha \frac{p^3(Gz(w),Fz(w))-p^3(Hu(w),Fu(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fu(w))].[p(Gz(w),Fz(w))-p(Hu(w),Fu(w))]}$$

$$+\beta \frac{p^4(Gz(w),Fz(w))-p^4(Hu(w),Fu(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fu(w))].[p^2(Gz(w),Fz(w))-p^2(Hu(w),Fu(w))]}+\gamma \frac{p^2(Hu(w),Fz(w))+p^2(Gz(w),Fu(w))}{p(Gz(w),Fz(w))+p(Gz(w),Fu(w))}$$

$$\le \alpha \frac{p^2(Gz(w),Fz(w))+p^2(Hu(w),Fu(w))+p(Gz(w),Fz(w)).p(Hu(w),Fu(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fu(w))]}$$

$$+\beta \frac{p^2(Gz(w),Fz(w))+p^2(Hu(w),Fu(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fu(w))]}+\gamma \frac{p^2(Hu(w),Fz(w))+p^2(Gz(w),Fu(w))}{p(Gz(w),Fz(w))+p(Gz(w),Fu(w))}$$

$$\le \alpha \frac{p^2(z(w),z(w))+p^2(u(w),u(w))+p(z(w),z(w)).p(u(w),u(w))}{[p(z(w),z(w))+p(z(w),u(w))]}$$

$$+\beta \frac{p^2(z(w),z(w))+p^2(u(w),u(w))}{[p(z(w),z(w))+p(z(w),u(w))]}+\gamma \frac{p^2(u(w),z(w))+p^2(z(w),u(w))}{p(z(w),z(w))+p(z(w),u(w))}$$

$$p(z(w),u(w)) \le 2\gamma p(u(w),z(w))$$

This is a contradiction. Hence $z(w)=u(w)$.

This completes the proof of the theorem.

Theorem 2.2: Let F be a self mapping and G and H be continuous self mappings of Polish metric space (X,d) with a w-distance p satisfying the conditions:

(i) $FX \subset GX \cap HX$

(ii) $$p(Fx(w),Fy(w)) \le \alpha\, max \left\{ \begin{array}{c} \frac{p^2(Hy(w),Fx(w))+p^2(Gx(w),Fy(w))}{p(Gx(w),Fx(w))+p(Gx(w),Fy(w))}, \\ \frac{p^3(Gx(w),Fx(w))+p^3(Hy(w),Fy(w))}{[p(Gx(w),Fx(w))+p(Gx(w),Fy(w))].[p(Gx(w),Fx(w))+p(Hy(w),Fy(w))]}, \\ \frac{p^4(Gx(w),Fx(w))-p^4(Hy(w),Fy(w))}{[p(Gx(w),Fx(w))+p(Gx(w),Fy(w))].[p^2(Gx(w),Fx(w))-p^2(Hy(w),Fy(w))]} \end{array} \right\}$$

For all $x,y \in X$ where $0<\alpha<\frac{1}{4}$ and $w \in \Omega$ be a selector.

(iii) $\{F,G\}$ and $\{F,H\}$ are weakly commuting pair.

Then F,G and H have a unique common fixed point in X.

Proof: Let $x_0(w)$ be an arbitrary point in X. Then $Fx_0(w) \in X$. Since $FX \subset GX$ there exist a point $x_1(w) \in X$ such that $Fx_0(w) = Gx_1(w)$. Since $FX \subset HX$ there exist a point $x_2(w) \in X$ such that $Fx_1(w) = Hx_2(w)$. In general one can choose point $x_{2n+1}(w)$ and $x_{2n+2}(w)$ such that

$$Fx_{2n}(w) = Gx_{2n+1}(w) \text{ and } Fx_{2n+1}(w) = Hx_{2n+2}(w), \text{ for } n = 0,1,2,3, \dots..$$

$$p(Fx_{2n}(w), Fx_{2n+1}(w)) = p(Fx_{2n+1}(w), Fx_{2n}(w))$$

$$\leq \alpha\, max \left\{ \begin{array}{c} \frac{p^2(Hx_{2n}(w),Fx_{2n+1}(w))+p^2(Gx_{2n+1}(w),Fx_{2n}(w))}{p(Gx_{2n+1}(w),Fx_{2n+1}(w))+p(Gx_{2n+1}(w),Fx_{2n}(w))}, \\ \frac{p^3(Gx_{2n+1}(w),Fx_{2n+1}(w))+p^3(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gx_{2n+1}(w),Fx_{2n+1}(w))+p(Gx_{2n+1}(w),Fx_{2n}(w))].[p(Gx_{2n+1}(w),Fx_{2n+1}(w))+p(Hx_{2n}(w),Fx_{2n}(w))]}, \\ \frac{p^4(Gx_{2n+1}(w),Fx_{2n+1}(w))-p^4(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gx_{2n+1}(w),Fx_{2n+1}(w))+p(Gx_{2n+1}(w),Fx_{2n}(w))].[p^2(Gx_{2n+1}(w),Fx_{2n+1}(w))-p^2(Hx_{2n}(w),Fx_{2n}(w))]} \end{array} \right\}$$

$$\leq \alpha\, max \left\{ \begin{array}{c} \frac{p^2(Fx_{2n-1}(w),Fx_{2n+1}(w))+p^2(Fx_{2n}(w),Fx_{2n}(w))}{p(Fx_{2n}(w),Fx_{2n+1}(w))+p(Fx_{2n}(w),Fx_{2n}(w))}, \\ \frac{p^3(Fx_{2n}(w),Fx_{2n+1}(w))+p^3(Fx_{2n-1}(w),Fx_{2n}(w))}{[p(Fx_{2n}(w),Fx_{2n+1}(w))+p(Fx_{2n}(w),Fx_{2n}(w))].[p(Fx_{2n}(w),Fx_{2n+1}(w))+p(Fx_{2n-1}(w),Fx_{2n}(w))]}, \\ \frac{p^4(Fx_{2n}(w),Fx_{2n+1}(w))-p^4(Fx_{2n-1}(w),Fx_{2n}(w))}{[p(Fx_{2n}(w),Fx_{2n+1}(w))+p(Fx_{2n}(w),Fx_{2n}(w))].[p^2(Fx_{2n}(w),Fx_{2n+1}(w))-p^2(Fx_{2n-1}(w),Fx_{2n}(w))]} \end{array} \right\}$$

$$\leq \alpha\, max \left\{ \begin{array}{c} \frac{p^2(Fx_{2n-1}(w),Fx_{2n+1}(w))}{p(Fx_{2n}(w),Fx_{2n+1}(w))}, \\ \frac{p^2(Fx_{2n}(w),Fx_{2n+1}(w))+p^2(Fx_{2n-1}(w),Fx_{2n}(w))-p(Fx_{2n}(w),Fx_{2n+1}(w)).p(Fx_{2n-1}(w),Fx_{2n}(w))}{[p(Fx_{2n}(w),Fx_{2n+1}(w))]}, \\ \frac{p^2(Fx_{2n}(w),Fx_{2n+1}(w))+p^2(Fx_{2n-1}(w),Fx_{2n}(w))}{[p(Fx_{2n}(w),Fx_{2n+1}(w))]} \end{array} \right\}$$

$$p^2(Fx_{2n}(w), Fx_{2n+1}(w)) \leq \alpha p^2(Fx_{2n-1}(w), Fx_{2n+1}(w))$$

$$p(Fx_{2n}(w), Fx_{2n+1}(w)) \leq \left(\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}\right) p(Fx_{2n-1}(w), Fx_{2n}(w))$$

$$p(Fx_{2n}(w), Fx_{2n+1}(w)) \leq k\, p(Fx_{2n-1}(w), Fx_{2n}(w))$$

Where $k = \left(\frac{\sqrt{\alpha}}{1-\sqrt{\alpha}}\right) < 1$ since $\alpha < \frac{1}{4}$.

Similarly,

$$p(Fx_{2n+1}(w), Fx_{2n+2}(w)) < k\, p(Fx_{2n}(w), Fx_{2n+1}(w))$$

In general,

$$p(Fx_n(w), Fx_{n+1}(w)) < K^n\, p(Fx_0(w), Fx_1(w))$$

Now, we shall prove that $\{Fx_n(w)\}$ is a Cauchy Sequence. Since $\lim_{n\to\infty} p(Fx_n(w), Fx_{n+1}(w)) = 0$, it is sufficient to show that the sequence $\{Fx_{2n}(w)\}$ is a Cauchy Sequence. Suppose that $\{Fx_{2n}(w)\}$ is not a Cauchy Sequence. Then there is $\epsilon > 0$ such that for each integer $2k, k = 0,1,2,\dots$ there even integers $2n(k)$ and $2m(k)$ with $2k \leq 2n(k) \leq 2m(k)$ such that

$$p\left(Fx_{2n(k)}(w), F_{2m(k)}(w)\right) > \epsilon \tag{2.2.1}$$

Let for each even integer $2k$, $2m(k)$ be the least exceeding $2n(k)$ and satisfying (2.2.1). Therefore

$$p\left(Fx_{2n(k)}(w), F_{2m(k)-2}(w)\right) \leq \epsilon \tag{2.2.2}$$

Then for each even integer $2k$, we have from (2.2.1)

$$\epsilon < p\left(Fx_{2n(k)}(w), F_{2m(k)}(w)\right)$$

$$\leq p\left(Fx_{2n(k)}(w), F_{2m(k)-2}(w)\right) + p\left(Fx_{2m(k)-2}(w), F_{2m(k)-1}(w)\right) + p\left(Fx_{2m(k)-1}(w), F_{2m(k)}(w)\right)$$

Using (2.2.2) and $\lim_{n\to\infty} p\big(Fx_n(w), Fx_{n+1}(w)\big) = 0$, we have

$$\lim_{k\to\infty} p\left(Fx_{2n(k)}(w), F_{2m(k)}(w)\right) = \epsilon \tag{2.2.3}$$

If follows immediately from the triangular inequality that

$$\left|p\left(Fx_{2n(k)+1}(w), Fx_{2m(k)-1}(w)\right) - p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)\right| \leq p\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right) + p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)$$

Using (2.2.2), we get

$$p\left(Fx_{2n(k)+1}(w), Fx_{2m(k)-1}(w)\right) \to \epsilon \tag{2.2.4}$$

Now

$$p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right) \leq p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) + p\left(Fx_{2n(k)+1}(w), Fx_{2m(k)}(w)\right)$$

$$\leq p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)$$

$$+\alpha \max \left\{ \begin{array}{l} \frac{p^2\left(Hx_{2m(k)}(w), Fx_{2n(k)+1}(w)\right) + p^2\left(Gx_{2n(k)+1}(w), Fx_{2m(k)}(w)\right)}{p\left(Gx_{2n(k)+1}(w), Fx_{2n(k)+1}(w)\right) + p\left(Gx_{2n(k)+1}(w), Fx_{2m(k)}(w)\right)}, \\ \frac{p^3\left(Gx_{2n(k)+1}(w), Fx_{2n(k)+1}(w)\right) + p^3\left(Hx_{2m(k)}(w), Fx_{2m(k)}(w)\right)}{\left[\begin{array}{l} p\left(Gx_{2n(k)+1}(w), Fx_{2n(k)+1}(w)\right) \\ +p\left(Gx_{2n(k)+1}(w), Fx_{2m(k)}(w)\right) \end{array}\right] \cdot \left[\begin{array}{l} p\left(Gx_{2n(k)+1}(w), Fx_{2n(k)+1}(w)\right) \\ +p\left(Hx_{2m(k)}(w), Fx_{2m(k)}(w)\right) \end{array}\right]}, \\ \frac{p^4\left(Gx_{2n(k)+1}(w), Fx_{2n(k)+1}(w)\right) - p^4\left(Hx_{2m(k)}(w), Fx_{2m(k)}(w)\right)}{\left[\begin{array}{l} p\left(Gx_{2n(k)+1}(w), Fx_{2n(k)+1}(w)\right) \\ +p\left(Gx_{2n(k)+1}(w), Fx_{2m(k)}(w)\right) \end{array}\right] \cdot \left[\begin{array}{l} p^2\left(Gx_{2n(k)+1}(w), Fx_{2n(k)+1}(w)\right) \\ -p^2\left(Hx_{2m(k)}(w), Fx_{2m(k)}(w)\right) \end{array}\right]} \end{array} \right\}$$

$$\leq p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)$$

$$+\alpha \max \left\{ \begin{array}{l} \frac{p^2\left(Fx_{2m(k)-1}(w), Fx_{2n(k)+1}(w)\right) + p^2\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)}{p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) + p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right)}, \\ \frac{p^3\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) + p^3\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)}{\left[\begin{array}{l} p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) \\ +p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right) \end{array}\right] \cdot \left[\begin{array}{l} p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) \\ +p\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right) \end{array}\right]}, \\ \frac{p^4\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) - p^4\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right)}{\left[\begin{array}{l} p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) \\ +p\left(Fx_{2n(k)}(w), Fx_{2m(k)}(w)\right) \end{array}\right] \cdot \left[\begin{array}{l} p^2\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right) \\ -p^2\left(Fx_{2m(k)-1}(w), Fx_{2m(k)}(w)\right) \end{array}\right]} \end{array} \right\}$$

$$\leq p\left(Fx_{2n(k)}(w), Fx_{2n(k)+1}(w)\right)$$

$$+\alpha \max\left\{\begin{array}{c}\frac{p^2\left(Fx_{2m(k)-1}(w),Fx_{2n(k)+1}(w)\right)+p^2\left(Fx_{2n(k)}(w),Fx_{2m(k)}(w)\right)}{p\left(Fx_{2n(k)}(w),Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w),Fx_{2m(k)}(w)\right)},\\ \frac{\left[\begin{array}{c}p^2\left(Fx_{2n(k)}(w),Fx_{2n(k)+1}(w)\right)\\+p^2\left(Fx_{2m(k)-1}(w),Fx_{2m(k)}(w)\right)\end{array}\right]-p\left(Fx_{2n(k)}(w),Fx_{2n(k)+1}(w)\right).p\left(Fx_{2m(k)-1}(w),Fx_{2m(k)}(w)\right)}{\left[p\left(Fx_{2n(k)}(w),Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w),Fx_{2m(k)}(w)\right)\right]},\\ \frac{p^2\left(Fx_{2n(k)}(w),Fx_{2n(k)+1}(w)\right)+p^2\left(Fx_{2m(k)-1}(w),Fx_{2m(k)}(w)\right)}{\left[p\left(Fx_{2n(k)}(w),Fx_{2n(k)+1}(w)\right)+p\left(Fx_{2n(k)}(w),Fx_{2m(k)}(w)\right)\right]}\end{array}\right\}$$

Using (2.2.3), (2.2.4) and $\lim_{n\to\infty} p\left(Fx_n(w), Fx_{n+1}(w)\right) = 0$ we have

$$\Rightarrow \varepsilon \leq 2\alpha\epsilon$$

This is a contradiction. Hence $\{Fx_n(w)\}$ is a Cauchy Sequence and then by completeness of X, there is a point $z(w) \in X$ such that $Fx_n(w) \to z(w)$.

Since the sequence $\{Gx_{2n+1}(w)\}$ and $\{Hx_{2n}(w)\}$ are subsequences of $\{Fx_n(w)\}$, they have the same limit $z(w)$. Since G and H are continuous, we have $GHx_{2n}(w) \to Gz(w)$ and $HGx_{2n+1}(w) \to Hz(w)$.

Now,

$$p\left(GHx_{2n}(w), HGx_{2n+1}(w)\right) = p\left(GFx_{2n-1}(w), HFx_{2n}(w)\right)$$

$$\leq p\left(GFx_{2n-1}(w), FGx_{2n-1}(w)\right) + p\left(FGx_{2n-1}(w), FHx_{2n}(w)\right) + p\left(FHx_{2n}(w), HFx_{2n}(w)\right)$$

Using condition (2.2-(iii)), the weak commutative pair of $\{F, G\}$ and $\{F, H\}$, we get,

$$p\left(GHx_{2n}(w), HGx_{2n+1}(w)\right) \leq p\left(Gx_{2n-1}(w), Fx_{2n-1}(w)\right) + p\left(FGx_{2n-1}(w), FHx_{2n}(w)\right)$$

$$+p\left(Fx_{2n}(w), Hx_{2n}(w)\right)$$

$$\leq p\left(Gx_{2n-1}(w), Fx_{2n-1}(w)\right)$$

$$+\alpha \max\left\{\begin{array}{c}\frac{p^2\left(H^2x_{2n}(w),FGx_{2n-1}(w)\right)+p^2\left(G^2x_{2n-1}(w),FHx_{2n}(w)\right)}{p\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p\left(G^2x_{2n-1}(w),FHx_{2n}(w)\right)},\\ \frac{p^3\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p^3\left(H^2x_{2n}(w),FHx_{2n}(w)\right)}{\left[p\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p\left(G^2x_{2n-1}(w),FHx_{2n}(w)\right)\right].\left[p\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p\left(H^2x_{2n}(w),FHx_{2n}(w)\right)\right]},\\ \frac{p^4\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)-p^4\left(H^2x_{2n}(w),FHx_{2n}(w)\right)}{\left[p\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p\left(G^2x_{2n-1}(w),FHx_{2n}(w)\right)\right].\left[p^2\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)-p^2\left(H^2x_{2n}(w),FHx_{2n}(w)\right)\right]}\end{array}\right\}$$

$$+ p\left(Fx_{2n}(w), Hx_{2n}(w)\right)$$

$$\leq p\left(Gx_{2n-1}(w), Fx_{2n-1}(w)\right)$$

$$+\alpha \max\left\{\begin{array}{c}\frac{p^2\left(H^2x_{2n}(w),FGx_{2n-1}(w)\right)+p^2\left(G^2x_{2n-1}(w),FHx_{2n}(w)\right)}{p\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p\left(G^2x_{2n-1}(w),FHx_{2n}(w)\right)},\\ \frac{p^2\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p^2\left(H^2x_{2n}(w),FHx_{2n}(w)\right)-p\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right).p\left(H^2x_{2n}(w),FHx_{2n}(w)\right)}{\left[p\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p\left(G^2x_{2n-1}(w),FHx_{2n}(w)\right)\right]},\\ \frac{p^2\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p^2\left(H^2x_{2n}(w),FHx_{2n}(w)\right)}{\left[p\left(G^2x_{2n-1}(w),FGx_{2n-1}(w)\right)+p\left(G^2x_{2n-1}(w),FHx_{2n}(w)\right)\right]}\end{array}\right\}$$

$$+ p\left(Fx_{2n}(w), Hx_{2n}(w)\right)$$

$$\leq p\left(Gx_{2n-1}(w), Fx_{2n-1}(w)\right)$$

$$+\alpha \max\left\{\begin{array}{c}\frac{\left[p\left(H^2x_{2n}(w),GFx_{2n-1}(w)\right)+p\left(Gx_{2n-1}(w),Fx_{2n-1}(w)\right)\right]^2+\left[p\left(G^2x_{2n-1}(w),HFx_{2n}(w)\right)+p\left(Hx_{2n-1}(w),Fx_{2n}(w)\right)\right]^2}{\left[p\left(G^2x_{2n-1}(w),GFx_{2n-1}(w)\right)+p\left(Gx_{2n-1}(w),Fx_{2n-1}(w)\right)\right]+\left[p\left(G^2x_{2n}(w),HFx_{2n}(w)\right)+p\left(Hx_{2n}(w),Fx_{2n}(w)\right)\right]},\\ \frac{\left[\left\{\begin{array}{c}p\left(G^2x_{2n-1}(w),GFx_{2n-1}(w)\right)\\+p\left(Gx_{2n-1}(w),Fx_{2n-1}(w)\right)\end{array}\right\}^2+\left\{\begin{array}{c}p\left(H^2x_{2n}(w),HFx_{2n}(w)\right)\\+p\left(Hx_{2n}(w),Fx_{2n}(w)\right)\end{array}\right\}^2\right]-\left[\left\{\begin{array}{c}p\left(G^2x_{2n-1}(w),GFx_{2n-1}(w)\right)\\+p\left(Gx_{2n-1}(w),Fx_{2n-1}(w)\right)\end{array}\right\}\cdot\left\{\begin{array}{c}p\left(H^2x_{2n}(w),HFx_{2n}(w)\right)\\+p\left(Hx_{2n}(w),Fx_{2n}(w)\right)\end{array}\right\}\right]}{\left[\left\{p\left(G^2x_{2n-1}(w),GFx_{2n-1}(w)\right)+p\left(Gx_{2n-1}(w),Fx_{2n-1}(w)\right)\right\}+\left\{p\left(G^2x_{2n}(w),HFx_{2n}(w)\right)+p\left(Hx_{2n}(w),Fx_{2n}(w)\right)\right\}\right]},\\ \frac{\left[p\left(G^2x_{2n-1}(w),GFx_{2n-1}(w)\right)+p\left(Gx_{2n-1}(w),Fx_{2n-1}(w)\right)\right]^2+\left[p\left(H^2x_{2n}(w),HFx_{2n}(w)\right)+p\left(Hx_{2n}(w),Fx_{2n}(w)\right)\right]^2}{\left[\left\{p\left(G^2x_{2n-1}(w),GFx_{2n-1}(w)\right)+p\left(Gx_{2n-1}(w),Fx_{2n-1}(w)\right)\right\}+\left\{p\left(G^2x_{2n}(w),HFx_{2n}(w)\right)+p\left(Hx_{2n}(w),Fx_{2n}(w)\right)\right\}\right]}\end{array}\right\}$$

$$+p\left(Fx_{2n}(w),Hx_{2n}(w)\right)$$

Letting $n\to\infty$, we get

$$p\left(Gz(w),Hz(w)\right)\le p\left(z(w),z(w)\right)$$

$$+\alpha \max\left\{\begin{array}{c}\frac{\left[p\left(Hz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right]^2+\left[p\left(Gz(w),Hz(w)\right)+p\left(z(w),z(w)\right)\right]^2}{\left[p\left(Gz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right]+\left[p\left(Gz(w),Hz(w)\right)+p\left(z(w),z(w)\right)\right]},\\ \frac{\left[\left\{\begin{array}{c}p\left(Gz(w),Gz(w)\right)\\+p\left(z(w),z(w)\right)\end{array}\right\}^2+\left\{\begin{array}{c}p\left(Hz(w),Hz(w)\right)\\+p\left(z(w),z(w)\right)\end{array}\right\}^2\right]-\left[\left\{\begin{array}{c}p\left(Gz(w),Gz(w)\right)\\+p\left(z(w),z(w)\right)\end{array}\right\}\cdot\left\{\begin{array}{c}p\left(Hz(w),Hz(w)\right)\\+p\left(z(w),z(w)\right)\end{array}\right\}\right]}{\left[\left\{p\left(Gz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right\}+\left\{p\left(Gz(w),Hz(w)\right)+p\left(z(w),z(w)\right)\right\}\right]},\\ \frac{\left[p\left(Gz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right]^2+\left[p\left(Hz(w),Hz(w)\right)+p\left(z(w),z(w)\right)\right]^2}{\left[\left\{p\left(Gz(w),Gz(w)\right)+p\left(z(w),z(w)\right)\right\}+\left\{p\left(Gz(w),Hz(w)\right)+p\left(z(w),z(w)\right)\right\}\right]}\end{array}\right\}+$$

$$p\left(z(w),z(w)\right)$$

$$p\left(Gz(w),Hz(w)\right)\le 2\alpha p\left(Hz(w),Gz(w)\right)$$

This is contradiction. Therefore $Gz(w)=Hz(w)$.

Now, we shall prove that $Fz(w)=Gz(w)$.

Consider

$$p\left(GFx_{2n+1}(w),Fz(w)\right)\le p\left(GFx_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(FGx_{2n+1},Fz(w)\right)$$

By the weak commutatively of $\{F,G\}$, we have

$$p\left(GFx_{2n+1}(w),Fz(w)\right)\le p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)$$

$$+\alpha \max\left\{\begin{array}{c}\frac{p^2\left(Hz(w),FGx_{2n+1}(w)\right)+p^2\left(G^2x_{2n+1}(w),Fz(w)\right)}{p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)},\\ \frac{p^3\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p^3\left(Hz(w),Fz(w)\right)}{\left[p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right].\left[p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(Hz(w),Fz(w)\right)\right]}\\ \frac{p^4\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)-p^4\left(Hz(w),Fz(w)\right)}{\left[p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right].\left[p^2\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)-p^2\left(Hz(w),Fz(w)\right)\right]}\end{array}\right\}$$

$$\le p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)$$

$$+\alpha \max\left\{\begin{array}{c}\frac{p^2\left(Hz(w),FGx_{2n+1}(w)\right)+p^2\left(G^2x_{2n+1}(w),Fz(w)\right)}{p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)},\\ \frac{p^2\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p^2\left(Hz(w),Fz(w)\right)-p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right).p\left(Hz(w),Fz(w)\right)}{\left[p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right]}\\ \frac{p^2\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p^2\left(Hz(w),Fz(w)\right)}{\left[p\left(G^2x_{2n+1}(w),FGx_{2n+1}(w)\right)+p\left(G^2x_{2n+1}(w),Fz(w)\right)\right]}\end{array}\right\}$$

$$\le p\left(Gx_{2n+1}(w),Fx_{2n+1}(w)\right)$$

$$+\alpha \max\left\{\begin{array}{c}\frac{[p(Hz(w),GFx_{2n+1}(w))+p(Gx_{2n+1}(w),Fx_{2n+1}(w))]^2+p^2(G^2x_{2n+1}(w),Fz(w))}{[p(G^2x_{2n+1}(w),GFx_{2n+1}(w))+p(Gx_{2n+1}(w),Fx_{2n+1}(w))]+p(G^2x_{2n+1}(w),Fz(w))},\\ \frac{\begin{bmatrix}p(G^2x_{2n+1}(w),GFx_{2n+1}(w))\\+p(Gx_{2n+1}(w),Fx_{2n+1}(w))\end{bmatrix}^2+p^2(Hz(w),Fz(w))-\begin{Bmatrix}p(G^2x_{2n+1}(w),GFx_{2n+1}(w))\\+p(Gx_{2n+1}(w),Fx_{2n+1}(w))\end{Bmatrix}.p(Hz(w),Fz(w))}{[\{p(G^2x_{2n+1}(w),GFx_{2n+1}(w))+p(Gx_{2n+1}(w),Fx_{2n+1}(w))\}+p(G^2x_{2n+1}(w),Fz(w))]}\\ \frac{[p(G^2x_{2n+1}(w),GFx_{2n+1}(w))+p(Gx_{2n+1}(w),Fx_{2n+1}(w))]^2+p^2(Hz(w),Fz(w))}{[\{p(G^2x_{2n+1}(w),GFx_{2n+1}(w))+p(Gx_{2n+1}(w),Fx_{2n+1}(w))\}+p(G^2x_{2n+1}(w),Fz(w))]}\end{array}\right\}$$

Letting $n \to \infty$, we get

$$p(Gz(w), Fz(w)) \leq p(z(w), z(w))$$

$$+\alpha \max\left\{\begin{array}{c}\frac{[p(Hz(w),Gz(w))+p(z(w),z(w))]^2+p^2(Gz(w),Fz(w))}{[p(Gz(w),Gz(w))+p(z(w),z(w))]+p(Gz(w),Fz(w))},\\ \frac{\begin{bmatrix}p(Gz(w),Gz(w))\\+p(z(w),z(w))\end{bmatrix}^2+p^2(Hz(w),Fz(w))-\begin{Bmatrix}p(Gz(w),Gz(w))\\+p(z(w),z(w))\end{Bmatrix}.p(Hz(w),Fz(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fz(w))]}\\ \frac{[p(Gz(w),Gz(w))+p(z(w),z(w))]^2+p^2(Hz(w),Fz(w))}{[\{p(Gz(w),Gz(w))+p(z(w),z(w))\}+p(Gz(w),Fz(w))]}\end{array}\right\}$$

$$p(Gz(w), Fz(w)) \leq \alpha p(Gz(w), Fz(w))$$

This is contradiction. Hence $Gz(w) = Fz(w)$.

Thus $Fz(w) = Gz(w) = Hz(w)$.

It now follows that

$$p(Fz(w), Fx_{2n}(w)) \leq \alpha \max\left\{\begin{array}{c}\frac{p^2(Hx_{2n}(w),Fz(w))+p^2(Gz(w),Fx_{2n}(w))}{p(Gz(w),Fz(w))+p(Gz(w),Fx_{2n}(w))},\\ \frac{p^3(Gz(w),Fz(w))+p^3(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fx_{2n}(w))].[p(Gz(w),Fz(w))+p(Hx_{2n}(w),Fx_{2n}(w))]},\\ \frac{p^4(Gz(w),Fz(w))-p^4(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fx_{2n}(w))].[p^2(Gz(w),Fz(w))-p^2(Hx_{2n}(w),Fx_{2n}(w))]}\end{array}\right\}$$

$$\leq \alpha \max\left\{\begin{array}{c}\frac{p^2(Hx_{2n}(w),Fz(w))+p^2(Gz(w),Fx_{2n}(w))}{p(Gz(w),Fz(w))+p(Gz(w),Fx_{2n}(w))},\\ \frac{p^2(Gz(w),Fz(w))+p^2(Hx_{2n}(w),Fx_{2n}(w))-p(Gz(w),Fz(w)).p(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fx_{2n}(w))]},\\ \frac{p^2(Gz(w),Fz(w))+p^2(Hx_{2n}(w),Fx_{2n}(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fx_{2n}(w))]}\end{array}\right\}$$

Taking limit $n \to \infty$, we get

$$p(Fz(w), z(w)) \leq \alpha \max\left\{\begin{array}{c}\frac{p^2(z(w),Fz(w))+p^2(Fz(w),z(w))}{p(Fz(w),Fz(w))+p(Fz(w),z(w))},\\ \frac{p^2(Fz(w),Fz(w))+p^2(z(w),z(w))-p(Fz(w),Fz(w)).p(z(w),z(w))}{[p(Fz(w),Fz(w))+p(Fz(w),z(w))]},\\ \frac{p^2(Fz(w),Fz(w))+p^2(z(w),z(w))}{[p(Fz(w),Fz(w))+p(Fz(w),z(w))]}\end{array}\right\}$$

$$p(Fz(w), z(w)) \leq 2\alpha p(z(w), Fz(w))$$

This is a contradiction and therefore $Fz(w) = z(w) = Gz(w) = Hz(w)$.

Thus $z(w)$ is a common fixed point of F, G and H.

Uniqueness: Let $u(w)$ be another point of F, G and H. Then

$$p(z(w), u(w)) = p(Fz(w), Fu(w))$$

$$\leq \alpha \max \left\{ \begin{array}{c} \frac{p^2(Hu(w),Fz(w))+p^2(Gz(w),Fu(w))}{p(Gz(w),Fz(w))+p(Gz(w),Fu(w))}, \\ \frac{p^3(Gz(w),Fz(w))+p^3(Hu(w),Fu(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fu(w))].[p(Gz(w),Fz(w))+p(Hu(w),Fu(w))]}, \\ \frac{p^4(Gz(w),Fz(w))-p^4(Hu(w),Fu(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fu(w))].[p^2(Gz(w),Fz(w))-p^2(Hu(w),Fu(w))]} \end{array} \right\}$$

$$\leq \alpha \max \left\{ \begin{array}{c} \frac{p^2(Hu(w),Fz(w))+p^2(Gz(w),Fu(w))}{p(Gz(w),Fz(w))+p(Gz(w),Fu(w))}, \\ \frac{p^2(Gz(w),Fz(w))+p^2(Hu(w),Fu(w))-p(Gz(w),Fz(w)).p(Hu(w),Fu(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fu(w))]}, \\ \frac{p^2(Gz(w),Fz(w))+p^2(Hu(w),Fu(w))}{[p(Gz(w),Fz(w))+p(Gz(w),Fu(w))]} \end{array} \right\}$$

$$\leq \alpha \max \left\{ \begin{array}{c} \frac{p^2(u(w),z(w))+p^2(z(w),u(w))}{p(z(w),z(w))+p(z(w),u(w))}, \\ \frac{p^2(z(w),z(w))+p^2(u(w),u(w))-p(z(w),z(w)).p(u(w),u(w))}{[p(z(w),z(w))+p(z(w),u(w))]}, \\ \frac{p^2(z(w),z(w))+p^2(u(w),u(w))}{[p(z(w),z(w))+p(z(w),u(w))]} \end{array} \right\}$$

$$p(z(w),u(w)) \leq 2\alpha p(u(w),z(w))$$

This is a contradiction. Hence $z(w) = u(w)$.

This completes the proof of the theorem.

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