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A Fixed Point Theorems for a Contractive Condition of Integral Type by Using Altering Distance Functions

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Abstract

In this article, we prove some fixed point theorems in metric space by using altering distance function. Our result are generalization of many previously known results

***Key words*: -** Metric space, fixed point, Common fixed point, Altering Distance function.

Introduction

A new category of contractive fixed point problem was introduced by M.S. Khan , M. Swalech and S.Sessa [10]. In this work, they introduced the concept of altering distance function which is a control function that alters distance between two points in a metric space.

2 Preliminary

Definition 2.1: If T is a mapping of a complete metric space (X,d) into itself satisfying the condition:

$$d(Tx,Ty) \leq kd(x,y)$$

For all $x,y \in X$ and for some $k, 0 \leq k < 1$ then T has a unique fixed point. A mapping satisfying above condition is called contraction mapping.

Definition 2.2: The function $\psi: [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and non-decreasing.

(ii) $\psi(t) = 0$ if and only if $t = 0$.

Definition 2.3: If $\xi: [0,\infty) \to [0,\infty)$ is subadditive on each $[a,b] \subset [0,\infty)$ then

$$\int_0^{a+b} \xi(t)dt \leq \int_0^a \xi(t)dt + \int_0^b \xi(t)dt.$$

Lemma 2.4: Let (X,d) be a metric space. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} \psi[d(x_n,x_{n+1})] = 0$$

If $\{x_n\}$ is not Cauchy sequence in X, then there exist an $\epsilon_0 > 0$ and sequence of integer positive $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$d\big(x_{m(k)}, x_{n(k)}\big) \geq \epsilon_0,\ d\big(x_{m(k)-1}, x_{n(k)}\big) < \epsilon_0.$$

Main Results

Theorem 3.1: Let (X,d) be a complete metric space, let $S: X \to X$ be a mapping which satisfies the following condition:

$$\psi\Big(\int_0^{d(Sx,Sy)} \xi(t)dt\Big) \leq \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big) \quad (3.1.1)$$

For each $x,y \in X$ with $a,b,c > 0$ such that $a + b + 2c < 1$, where ψ, φ are altering distance functions, and

$$M(x,y) = a\int_0^{d(x,y)} \xi(t)dt + b\int_0^{\frac{d(y,Sy)[1+d(x,Sx)]}{1+d(x,y)}} \xi(t)dt \quad + c\int_0^{\frac{d(x,Sx).+d(y,Sy)}{1+d(x,y).d(y,Sx)}} \xi(t)dt \quad (3.1.2)$$

Where $\xi: R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable, sub-additive on each compact subset of R^+, non-negative and such that for each $\epsilon_0 > 0$,

$\int_0^{\epsilon_0} \xi(t)dt > 0.$

Then S has a unique fixed point $z_0 \in X$.

Proof: Let $x_0 \in X$ be an arbitrary point and let $\{x_n\}$ be a sequence defined as follow:

$x_{n+1} = Sx_n$ for each $n \geq 0$.

Now

$$M(x_{n-1}, x_n) = a\int_0^{d(x_{n-1},x_n)} \xi(t)dt \quad +b\int_0^{\frac{d(x_n,Sx_n)[1+d(x_{n-1},Sx_{n-1})]}{1+d(x_{n-1},x_n)}} \xi(t)dt$$
$$+c\int_0^{\frac{d(x_{n-1},Sx_{n-1}).+d(x_n,Sx_n)}{1+d(x_{n-1},x_n).d(x_n,Sx_{n-1})}} \xi(t)dt$$
$$= a\int_0^{d(x_{n-1},x_n)} \xi(t)dt + b\int_0^{d(x_n,x_{n+1})} \xi(t)dt \quad +c\int_0^{d(x_{n-1},x_n)+d(x_n,x_{n+1})} \xi(t)dt$$

By sub-additivity of ξ, we get

$$M(x_{n-1}, x_n) \leq (a+c)\int_0^{d(x_{n-1},x_n)} \xi(t)dt \qquad +(b+c)\int_0^{d(x_n,x_{n+1})} \xi(t)dt$$

From (3.1.1), we have

$$\psi\left(\int_0^{d(x_n,x_{n+1})} \xi(t)dt\right) = \psi\left(\int_0^{d(Sx_{n-1},Sx_n)} \xi(t)dt\right)$$
$$\leq \psi(M(x_{n-1},x_n)) - \varphi(M(x_{n-1},x_n))$$
$$\leq \psi\begin{pmatrix} (a+c)\int_0^{d(x_{n-1},x_n)} \xi(t)dt \\ +(b+c)\int_0^{d(x_n,x_{n+1})} \xi(t)dt \end{pmatrix} \qquad -\varphi(M(x_{n-1},x_n))$$

Since ψ is non-decreasing, we get

$$\int_0^{d(x_n,x_{n+1})} \xi(t)dt \leq \frac{a+c}{(1-b-c)}\int_0^{d(x_{n-1},x_n)} \xi(t)dt$$

Continuing this process, we get in general

$$\int_0^{d(x_n,x_{n+1})} \xi(t)dt \leq k^n \int_0^{d(x_0,x_1)} \xi(t)dt \qquad \text{Let } k = \frac{a+c}{(1-b-c)} < 1$$

Taking $n \to \infty$, we get

$\lim_{n\to\infty} \int_0^{d(x_n,x_{n+1})} \xi(t)dt = 0$ .Therefore $\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$ (3.1.3)

Now, we will show that $\{x_n\}$ is a Cauchy sequence in X. suppose that $\{x_n\}$ is not a Cauchy sequence, which means that there is a constant $\epsilon_0 > 0$ such that for each positive integer k, there are positive integer $m(k)$ and $n(k)$ with $m(k) > n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon_0, d(x_{m(k)-1}, x_{n(k)}) < \epsilon_0$$

By triangle inequality

$$\epsilon_0 \leq d(x_{m(k)}, x_{n(k)}) \leq d(x_{m(k)}, x_{m(k)-1}) \quad +d(x_{m(k)-1}, x_{n(k)}) \quad < d(x_{m(k)}, x_{m(k)-1}) + \epsilon_0$$

Letting $k \to \infty$ and using (3.1.3), we get

$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon_{0.}$ (3.1.4)

Similarly, we have

$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon_{0.}$ (3.1.5)

For $x = x_{m(k)}$ and $y = x_{n(k)}$ from (3.1.2), (3.1.3) and (3.1.4) we have,

$+\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}) = a\int_0^{\epsilon_0} \xi(t)dt$ (3.1.6)

From (3.1.1), we have

$$\psi\left(\int_0^{d(x_{m(k)+1},x_{n(k)+1})} \xi(t)dt\right) \quad = \psi\left(\int_0^{d(Sx_{m(k)},Sx_{n(k)})} \xi(t)dt\right)$$
$$\leq \psi\left(M(x_{m(k)}, x_{n(k)})\right) - \varphi\left(M(x_{m(k)}, x_{n(k)})\right)$$

Taking $k \to \infty$ and using (3.1.5), (3.1.6) and the continuity of ψ and φ, we get

$$\psi(\int_0^{\epsilon_0} \xi(t)dt) \leq \psi(a\int_0^{\epsilon_0} \xi(t)dt) \quad -\varphi(a\int_0^{\epsilon_0} \xi(t)dt) \quad \leq \psi(\int_0^{\epsilon_0} \xi(t)dt) - \varphi(a\int_0^{\epsilon_0} \xi(t)dt)$$

This leads to $\varphi(a\int_0^{\epsilon_0} \xi(t)dt) = 0$, and property of φ we get $\int_0^{\epsilon_0} \xi(t)dt = 0.$

This is contradiction. Thus $\{x_n\}$ is a Cauchy Sequence in (X, d), which is complete. Thus, there is $z_0 \in X$ such that

$$\lim_{n\to\infty} x_n = z_0 \quad ,$$

Setting $x = x_n$ and $y = z_0$ in (3.1.2) we have

$M(x_n,z_0) = a\int_0^{d(x_n,z_0)}\xi(t)dt \quad +b\int_0^{\frac{d(z_0,Sz_0)[1+d(x_n,Sx_n)]}{1+d(x_n,z_0)}}\xi(t)dt \quad +c\int_0^{\frac{d(x_n,Sx_n)+d(z_0,Sz_0)}{1+d(x_n,z_0).d(z_0,Sx_n)}}\xi(t)dt$

$= a\int_0^{d(x_n,z_0)}\xi(t)dt + b\int_0^{\frac{d(z_0,Sz_0)[1+d(x_n,x_{n+1})]}{1+d(x_n,z_0)}}\xi(t)dt$

$+c\int_0^{\frac{d(x_n,x_{n+1})+d(z_0,Sz_0)}{1+d(x_n,z_0).d((z_0,x_{n+1})}}\xi(t)dt$

Let $n \to \infty$ and using (3.1.3) and (3.1.7), we get

$\lim_{n\to\infty} M(x_n,z_0) = (b+c)\int_0^{d(z_0,Sz_0)}\xi(t)dt$ (3.1.8)

From (3.1.1) we have

$\psi\left(\int_0^{d(x_{n+1},Sz_0)}\xi(t)dt\right) = \psi\left(\int_0^{d(Sx_n,Sz_0)}\xi(t)dt\right)$

$\leq \psi\big(M(x_n,z_0)\big) - \varphi\big(M(x_n,z_0)\big)$

Using (3.1.7) and (3.1.8), we get

$\psi\left(\int_0^{d(z_0,Sz_0)}\xi(t)dt\right) \leq \psi\left((b+c)\int_0^{d(z_0,Sz_0)}\xi(t)dt\right) \quad -\varphi\left((b+c)\int_0^{d(z_0,Sz_0)}\xi(t)dt\right)$

$\leq \psi\left(\int_0^{d(z_0,Sz_0)}\xi(t)dt\right) \quad -\varphi\left((b+c)\int_0^{d(z_0,Sz_0)}\xi(t)dt\right)$

Which implies$\varphi\left((b+c)\int_0^{d(z_0,Sz_0)}\xi(t)dt\right) = 0$, so $d(z_0,Sz_0) = 0$, that is $Sz_0 = z_0$.

Uniqueness: Let y_0, z_0 be two fixed point of S such that $y_0 \neq z_0$.

Putting $x = y_0$ and $y = z_0$ in (3.1.2) we have

$M(y_0,z_0) = a\int_0^{d(y_0,z_0)}\xi(t)d \quad +b\int_0^{\frac{d(z_0,Sz_0)[1+d(y_0,Sy_0)]}{1+d(y_0,z_0)}}\xi(t)dt \quad +c\int_0^{\frac{d(y_0,Sy_0)+d(z_0,Sz_0)}{1+d(y_0,z_0).d(z_0,Sy_0)}}\xi(t)dt$

$M(y_0,z_0) = a\int_0^{d(y_0,z_0)}\xi(t)dt$ (3.1.9)

So $\varphi\left(a\int_0^{d(y_0,z_0)}\xi(t)dt\right) = 0$, so $d(y_0,z_0) = 0$, that is $y_0 = z_0$.

Theorem 3.2: Let (X,d) be a complete metric space, let $S: X \to X$ be a mapping which satisfies the following condition:

$\psi(\int_0^{d(Sx,Sy)}\xi(t)dt) \leq \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big)$ (3.2.1)

For each $x,y \in X, x \neq y, \alpha,\beta > 0, 2\alpha + 3\beta < 1$, where ψ,φ are altering distance functions, and

$M(x,y) = \alpha\int_0^{\left[\frac{d^2(x,Sx)+d^2(y,Sy)+d^2(y,Sx)}{d(x,Sx)+d(y,Sy)+d(y,Sx)}\right]}\xi(t)dt \quad +\beta\int_0^{\left[\frac{d^2(x,Sy)+d^2(y,Sx)+d^2(x,y)}{d(x,Sy)+d(y,Sx)+d(x,y)}\right]}\xi(t)dt$

(3.2.2)

Where $\xi: R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable, sub-additive on each compact subset of R^+, non-negative and such that for each $\epsilon_0 > 0$,

$\int_0^{\epsilon_0}\xi(t)dt > 0$

Then S has a unique fixed point $z_0 \in X$.

Proof: Can be proved easily as theorem 3.1

Theorem 3.3: Let (X,d) be a complete metric space, let $S,T: X \to X$ be a mapping which satisfies the following condition:

$\psi(\int_0^{d(Sx,Sy)}\xi(t)dt) \leq \psi\big(M(x,y)\big) - \varphi\big(M(x,y)\big)$ (3.3.1)

For each $x,y \in X, x \neq y, \alpha,\beta > 0, 2\alpha + 3\beta < 1$, where ψ,φ are altering distance functions, and

$M(x,y) = \alpha\int_0^{\left[\frac{d^2(x,Sx)+d^2(y,Ty)+d^2(y,Sx)}{d(x,Sx)+d(y,Ty)+d(y,Sx)}\right]}\xi(t)dt \quad +\beta\int_0^{\left[\frac{d^2(x,Ty)+d^2(y,Sx)+d^2(x,y)}{d(x,Ty)+d(y,Sx)+d(x,y)}\right]}\xi(t)dt$ (3.3.2)

Where $\xi: R^+ \to R^+$ is a lesbesgue- integrable mapping which is summable, sub-additive on each compact subset of R^+, non-negative and such that for each $\epsilon_0 > 0$,

$\int_0^{\epsilon_0}\xi(t)dt > 0$.Then S and T have a common fixed point $z_0 \in X$.

Proof: Can be proved easily as theorem 3.1 and 3.2

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References:

1. Branciari, A., “A fixed point theorem for mappings satisfying a general contractive condition of integral type”, Int. J. Math. Math. Sci. 29 (2002) 531–536.
2. Babu, G. U. R. and Sailaja, P.P., “ A fixed point theorem of generalized weakly contractive maps in orbitally complete metric space”, Thai Journal of Math., 9 1 (2011) 1–10.

3. Chugh, R., Kadian, T., Rani A. and Rhoades, B. E., " Property P in G- Metric Spaces", Fixed Point Theory and Applications, Vol. 2010, Article ID 401684, 12 pages, 2010. doi:10.1155/2010/401684
4. Das, B.K. and Gupta, S. , "An extension of Banach contractive principle through rational expression", Indian Jour. Pure and Applied Math., 6 (1975) 1455–1458.
5. Dulta, P.N. and Choudhury, B.S., " A Generalisation of Contraction Principle in Metric Spaces", Fixed Point Theory and Applications, Vol. 2008, Article ID 406368, 8 pages, 2008. doi:10.1155/2008/406368
6. Jaggi, D. S. "Some unique fixed point theorems," Indian Journal of Pure and Applied Mathematics, vol. 8, no. 2, pp. 223–230, 1977.
7. Jeong, G.S. and B.E. Rhoades, B.E. Maps for which F(T) = F(Tn), Fixed Point Theory and Appl., 6 (2005) 87–131.
8. Jeong, G.S. and B.E. Rhoades, B.E. More maps for which F(T) = F(Tn), Demostratio Math., 40 (2007) 671–680.
9. Aydi, H. "A fixed point theorems for a contractive condition of integral type involving altering distances", Int. J. Nonlinear Anal. Appl. 3(2012)42-53.
10. Khan, M. S., Swalech, M. and Sessa, S. "Fixed point theorems by altering distances between the points", Bull. Austral Math. Soc., 30 (1984) 1–9.
11. Sharma, M. and Saluja, A. S., "Some fixed point theorems by using altering distance functions, IOSR Journal of Engineering 2 6 (2012), 1462-1472.
12. Naidu, S.V.R., "Some fixed point theorems in metric spaces by altering distances", Czechoslovak Math. Jour. 53 1 (2003) 205–212.
13. Popa, V. and Mocanu, M., "Altering distance and common fixed points under implicit relations", Hacettepe Jour. Math. and Stat., 38 3 (2009) 329–337.