Results with Quadruple Random Fixed Point

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Abstract
In this paper, we introduced concept of ICS mapping for quadruple random fixed point in partially ordered metric space. The present results generalized the result of Karapinar E. [23]. Results are motivated by Choudhary [11] and Gupta [24]

2. Introduction and Preliminaries
Probabilistic functional analysis has emerged as one of the important mathematical disciplines in view of its role in analyzing probabilistic models in the applied sciences. The study of fixed points of random operators forms a central topic in this area. The Prague school of probabilistic initiated its study in the 1950. However, the research in this area flourished after the publication of the survey article of Bharucha-Reid [10]. Since then many interesting random fixed point results and several applications have appeared in the literature. The Banach contraction principle, which is the most famous metrical fixed point theorem, play a very important role in nonlinear analysis and its applications are well known. Many authors have extended this theorem, including more general contractive conditions, which imply the existence of a fixed point. Existence of fixed points in ordered metric spaces was investigated in 2004 by Ran and Reurings [17] and then by Nieto and Lopez [16]. After this various results in have been obtained in this direction, see e.g. [1,15,18].

Bhaskar and Lakshmikantham [8] introduced the concept of a coupled fixed point of mapping $F: X\times X \to X$ and investigated some coupled fixed point theorems in partially ordered metric spaces. Later, various results in coupled fixed point have been obtained, see e.g. [2, 3, 4,5,6,12,13,14,15].

On the other hand, Berinde and Borcut [9] introduced the concept of triple fixed point and proof some related fixed point theorem. After this various results on tripled fixed point have been obtaind.

Further studied by Nieto and Rodriguez - Lopez [16], Samet and Vetro [19] introduced the notion of fixed point of N order in case of single-valued mappings. In particular for N = 4 (Quadruple case), i.e., Let $(X,\leq)$ be partially ordered set and (X,d) be a complete metric space. We consider the following partial order on the product space $X^4 = X\times X\times X\times X$:

$$(u,v,r,t) \leq (x,y,z,w) \text{ iff } x \geq u,\ y \leq v,\ z \geq r,\ t \leq w,$$

where $(u,v,r,t), (x,y,z,w) \in X^4$.

Regarding this partial order, Karapinar E. [23] introduced the concept of Quadruple fixed point and proves some new fixed point theorems. We define the following concept of quadruple random fixed point motivated by [23] Karapinar E.

3. Privious definition

Definition 3.1.1:- let $(X,\leq)$ be a partially ordered set, $F: X^4 \to X$ mapping. The mapping F is said to have the mixed monotone property for random operator if for any $x(\xi), y(\xi), z(\xi), w(\xi) \in X$.

i. $x_1(\xi), x_{2(\xi)} \in X,\ x_1(\xi) \leq x_2(\xi) \to F\left(\xi, x_{1(\xi)}, y(\xi), z(\xi), w(\xi)\right) \leq F(\xi, x_2(\xi), y(\xi), z(\xi), w(\xi))$,

ii. $y_1(\xi), y_2(\xi) \in X,\ y_1(\xi) \geq y_2(\xi) \to F(\xi, x(\xi), y(\xi), z_1(\xi), w(\xi)) \leq F(\xi, x(\xi), y(\xi), z_2(\xi), w(\xi))$

iii. $w_1(\xi), w_2(\xi) \in X,\ w_1(\xi) \geq w_2(\xi) \to F(\xi, x(\xi), y(\xi), z(\xi), w_1(\xi)) \geq F(\xi, x(\xi), y(\xi), z(\xi), w_2(\xi))$.

Definition 4:- An element $(x(\xi), y(\xi), z(\xi), w(\xi)) \in X^4$ is called a quadruple fixed point of $F: X^4 \to X$ if

$$F(x(\xi), y(\xi), z(\xi), w(\xi)) = x(\xi),$$

$$F(y(\xi), z(\xi), w(\xi), x(\xi)) = y(\xi), \quad F(z(\xi), w(\xi), x(\xi), y(\xi)) = z(\xi)$$

$$\text{and } F(w(\xi), x(\xi), y(\xi), z(\xi)) = w(\xi).$$

5. Throughout this paper (Ω, Σ) denotes a measurable space, X be a partially ordered metric space.and C is non empty subset of X.

Definition 5.1. (a): A function $f : \Omega \to C$ is said to be measurable if $f'(B \cap C) \in \Sigma$ for every Borel subset B of X.

Definition 5.1. (b): A function $f: \Omega \times C \to C$ is said to be random operator, if $f(., X): \Omega \to C$ is measurable for every$X \in C$.

Definition 5.1 (c): A random operator $f: \Omega \times C \to C$ is said to be continuous if for fixed $t \in \Omega, f(t, .): C \times C$ is continuous.

6.1. Random Fixed Point: A measurable function $g : \Omega \to C$ is said to be random fixed point of the random operator $f: \Omega \times C \to C$, if $f(t, g(t)) = g(t), \forall t \in \Omega$.

In this paper we give some quadruple random fixed point theorems for mapping having the mixed monotone property in partially ordered metric spaces depended on another function.

6. 2. Main Results

Definition 6.2 .1:- Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be ICS if T is injuctive, continuous and has the property: for every sequence $\{x_n\}$ in X, if $\{Tx_n\}$ is convergent then $\{x_n(\xi)\}$ is also convergent.

Let ϕ be the set of all functions $\phi : [0, \infty) \to [0, \infty)$ such that

i. ϕ is non- decreasing,
ii. $\phi(t) < t$ for all $t > 0$,
iii. $\lim_{r \to t^+} \phi(r) < t$ for all $t > 0$

From now on, we denote$X^4 = X \times X \times X \times X$. Our first result is given by the following:

Theorem 6.2.2:- Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d in X such that (X, d) is a complete metric space. Suppose $T: X \to X$ is a ICS mapping and $F: X^4 \to X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \Phi$ and $\xi \varepsilon \Omega$, be a measerable sellector such that

$$D\left(TF(\xi, x(\xi), y(\xi), z(\xi), w(\xi)), TF(\xi, u(\xi), v(\xi), p, q(\xi))(\xi)\right)$$

$$\leq \phi\left(\max\left\{\begin{matrix} d(T(\xi, x(\xi)), T(\xi, u(\xi)), d(T(\xi, y(\xi), T(\xi, v(\xi)), \\ d(T(\xi, z(\xi), T(\xi, p(\xi)), d(w, T(\xi, q(\xi)) \end{matrix}\right\}\right) \quad (6.2.\ 2.1)$$

for any $x(\xi), y(\xi), z(\xi), w(\xi) \in X$ for which $x(\xi) \leq u(\xi), v(\xi) \leq y(\xi)$,

$z(\xi) \leq p(\xi),\ q(\xi) \leq w(\xi)$. Suppose either

i. F is continuous, or
ii. X has the following property:
(a) if non decreasing sequence $x_n(\xi) \to x(\xi)$ (respectively,
(b) $z_n(\xi) \to z(\xi))$, then $x_n(\xi) \leq x(\xi)$, (respectively, $z_n(\xi) \leq z(\xi))$ for all n,
(c) if non increasing sequence $y_n(\xi) \to x(\xi)$(respectively, $w_n(\xi) \to z(\xi))$, then $y_n(\xi) \geq y(\xi)$, (respectively, $w_n(\xi) \geq w(\xi))$ for all n.

If there exists $x_0(\xi), y_0(\xi), z_0(\xi), w(\xi)_0 \in X$ such that

$$X_0(\xi) \geq F\left(\xi, x_0(\xi), y_{0(\xi)}, z_0(\xi), w_0(\xi)\right),$$

$$y_0 \leq F(\xi, y_0(\xi), z_0(\xi), w_0(\xi), x_0(\xi)),$$

$$z_0(\xi) \leq F(\xi, z_0(\xi), w_0(\xi), x_0, (\xi), y_0(\xi))$$

and $w_0(\xi) \geq F(\xi, w_0(\xi), x_0(\xi), y_0(\xi), z_0(\xi)),$

then there exist $x(\xi), y(\xi), z(\xi), w(\xi) \in X$ such that

$$x(\xi) = F(\xi, x(\xi), y(\xi), z(\xi), w(\xi)),$$

$$y(\xi) = F(\xi, y(\xi), z(\xi), w(\xi), x(\xi)),$$

$$z(\xi) = F(\xi, z(\xi), w(\xi), x(\xi), y(\xi)),$$

$$w(\xi) = F(\xi, w(\xi), x(\xi), y(\xi), z(\xi))$$

that is, F has a quadrupled fixed point.

Proof: Let $x_0(\xi), y_0(\xi), z_0(\xi), w_0(\xi) \in X$ such that $x_0(\xi) \geq F((\xi, x_0(\xi), y_0(\xi), z_0(\xi), w_0(\xi)),$

$y_0(\xi) \leq F(\xi, y_0(\xi), z_0(\xi), w_0(\xi), x_0(\xi)),$ and

$$z_0(\xi) \leq F(\xi, z_0(\xi), w_0(\xi), x_0(\xi), y_0(\xi))$$

$w_0(\xi) \geq F(\xi, w_0(\xi), S(\xi, x_0(\xi)), y_0(\xi), z_0(\xi))$ set

$$X_1(\xi) = F(\xi, x_0(\xi), y_0(\xi), z_0, w_0(\xi)),$$

$$y_1(\xi) = F(\xi, y_0(\xi), z_0(\xi), w_0(\xi), x_0(\xi))$$

$$z_1(\xi) = F(\xi, z_0(\xi), w_0(\xi), x_0(\xi), y_0(\xi)),$$

$$w_1(\xi) = F(\xi, w_0(\xi), S(\xi, x_0(\xi), y_0(\xi), z_0(\xi)) \qquad (6.2.\ 2.2)$$

Continuing this process, we can construct sequences $\{x_n(\xi)\}, \{y_n(\xi)\}, \{z_n(\xi)\}$ and $\{w_n(\xi)\}$ in X such that

$$x_{n+1}(\xi) = F(\xi, x_n(\xi), y_n(\xi), z_n(\xi), w_n(\xi))$$

$$y_{n+1} = F(\xi, y_n(\xi), z_n(\xi), w_n(\xi), x_n(\xi))$$

$$z_{n+1}(\xi) = F(\xi, z_n(\xi), w_n(\xi), x_n(\xi), y_n(\xi)) \quad w_{n+1} = F(\xi, w_n(\xi), S(\xi, x_n(\xi), y_n(\xi), z_n(\xi)) \qquad (6.2.\ 2.3)$$

Since F has the mixed monotone property, then using the mathematical induction it is easy that

$$x_n(\xi) \leq x_{n+1}(\xi), \quad y(\xi)_n \geq y_{n+1}(\xi),$$

$$z_n(\xi) \leq z_{n+1}(\xi), \quad w_n(\xi) \geq w_{n+1}(\xi) \qquad (6.2.\ 2.4)$$

for $n = 0,1,2,3, \ldots$.

Assume for some $n \in N$

$$x_n(\xi) = x_{n+1}(\xi), \quad y_n(\xi) = y_{n+1}(\xi), \quad z_n(\xi) = z_{n+1}(\xi),$$

$$w_n(\xi) = w_{n+1} \qquad (6.2.\ 2.5)$$

then, $(x_n(\xi), y_n(\xi), z_n(\xi), w_n(\xi))$ is the quadrupled fixed point of F. From now on, assume for any $n \in N$ that atleast,

$$x_n(\xi) \neq x_{n+1}(\xi),\ y_n(\xi) \neq y_{n+1}(\xi),\ z_n(\xi) \neq z_{n+1}(\xi),$$

$$w_n(\xi) \neq w_{n+1}(\xi) \qquad (6.2.\ 2.6)$$

Since T is injective, then by (2.6), for any $n \in N$

$$0 \leq \phi\left(\max\left\{\begin{array}{c} d\left(T(\xi, x_{n+1}(\xi)), T(\xi, x_n(\xi))\right) \\ , d(T(\xi, y_{n+1}(\xi)), T(\xi, y_n(\xi))), \\ d(T(\xi, z_{n+1}(\xi)), T(\xi, z_n(\xi))), \\ d(w_{n+1}(\xi), T(\xi, w_n(\xi))) \end{array}\right\}\right) \qquad (6.2.\ 2.7)$$

Now we have

$$d\left(T(\xi, x_n(\xi)), T(\xi, x_{n+1}(\xi))\right)$$
$$= d\left(TF(\xi, (x_{n-1}(\xi), y_{n-1}(\xi), z_{n-1}(\xi), w_{n-1}(\xi))), TF(\xi, (x_n(\xi), y_n(\xi), z_n(\xi), w_n(\xi))\right)$$

$$\leq \phi\left(\max\left\{\begin{array}{c} d(T(\xi, x_{n-1}(\xi)), T(\xi, x_n(\xi))) \\ , d(T(\xi, y_{n-1}(\xi)), T(\xi, y_n(\xi))), \\ d\left(T(\xi, z_{n-1}(\xi)), T(\xi, z_n(\xi))\right), \\ d(w_{n-1}(\xi), T(\xi, w_n(\xi))) \end{array}\right\}\right) \qquad (6.2.\ 2.8)$$

$$d(T(\xi, y_n(\xi)), T(\xi, y_{n+1}(\xi))) =$$
$$d\left(TF(\xi, (y_{n-1}(\xi)), z_{n-1}(\xi), w_{n-1}(\xi), x_{n-1}(\xi)), TF(\xi, (y_n(\xi)), z_n(\xi), w_n(\xi), x_n(\xi))\right)$$

$$\leq \phi\left(\max\left\{\begin{array}{c} d\left(T(\xi, x_{n-1}(\xi)), T(\xi, x_n(\xi))\right) \\ , d(T(\xi, y_{n-1}(\xi)), T(\xi, y_n(\xi))), \\ d(T(\xi, z_{n-1}(\xi)), T(\xi, z_n(\xi))), \\ d(w_{n-1}(\xi), T(\xi, w_n(\xi))) \end{array}\right\}\right) \qquad (6.2.\ 2.9)$$

$$d(T(\xi, z_n(\xi)), T(\xi, z_{n+1}(\xi))) =$$
$$d(TF(\xi, (z_{n-1}(\xi), w_{n-1}(\xi), x_{n-1}(\xi), y_{n-1}(\xi))), TF(\xi, (z_n(\xi), w_n(\xi), x_n(\xi), y_n(\xi))))$$
$$\leq \phi\left(\max\left\{\begin{array}{c} d\left(T(\xi, x_{n-1}(\xi)), T(\xi, x_n(\xi))\right) \\ , d(T(\xi, y_{n-1}(\xi)), T(\xi, y_n(\xi))), \\ d\left(T(\xi, z_{n-1}(\xi)), T(\xi, z_n(\xi))\right), \\ d(w_{n-1}(\xi), T(\xi, w_n(\xi))) \end{array}\right\}\right) \qquad (6.2.\ 2.10)$$

and

$$d\left(T(\xi, w_n(\xi)), T(\xi, w_{n+1}(\xi))\right) =$$

$$d\big(TF(\xi,(w_{n-1}(\xi),x_{n-1}(\xi),y_{n-1}(\xi),z_{n-1}(\xi))),TF(\xi,(w_n,x_n(\xi),y_n(\xi),z_n(\xi)))\big)$$

$$\leq \phi\left(\max\left\{\begin{array}{c} d(T(\xi,x_{n-1}(\xi)),T(\xi,x_n(\xi))),\\ d\Big(T(\xi,y_{n-1}(\xi)),T(\xi,y_n(\xi))\Big),\\ d\Big(T(\xi,z_{n-1}(\xi)),T(\xi,z_n(\xi))\Big),\\ d(w_{n-1}(\xi),T(\xi,w_n(\xi)))\end{array}\right\}\right) \qquad (6.2.\ 2.11)$$

Since we have $\phi(t) < t$ for all $t > 0$, so from (6.2. 2.8) to

(6.2. 2.11)

We obtain that

$$0 < \max\left\{\begin{array}{c} d\Big(T(\xi,x_n(\xi)),T(\xi,x_{n+1}(\xi))\Big)\\ ,d(T(\xi,y_n(\xi)),T(\xi,y_{n+1}(\xi))),\\ d\Big(T(\xi,z_n(\xi)),T(\xi,z_{n+1}(\xi))\Big),\\ d(T(\xi,w_n(\xi)),T(\xi,w_{n+1}(\xi)))\end{array}\right\} \leq \phi\left(\max\left\{\begin{array}{c} d\Big(T(\xi,x_{n-1}(\xi)),T(\xi,x_n(\xi))\Big)\\ ,d\Big(T(\xi,y_{n-1}(\xi)),T(\xi,y_n(\xi))\Big),\\ d(T(\xi,z_{n-1}(\xi)),T(\xi,z_n(\xi))),\\ d(w_{n-1}(\xi),T(\xi,w_n))\end{array}\right\}\right) \quad (6.2.\ 2.12)$$

$$\leq \max\left\{\begin{array}{c} d\Big(T(\xi,x_{n-1}(\xi)),T(\xi,x_n(\xi))\Big)\\ ,d\Big(T(\xi,y_{n-1}(\xi)),T(\xi,y_n(\xi))\Big),\\ d(T(\xi,z_{n-1}(\xi)),T(\xi,z_n(\xi))),\\ d(w_{n-1}(\xi),T(\xi,w_n(\xi)))\end{array}\right\}$$

It follows that

$$\max\left\{\begin{array}{c} d\Big(T(\xi,x_n(\xi)),T(\xi,x_{n+1}(\xi))\Big)\\ ,d\Big(T(\xi,y_n(\xi)),T(\xi,y_{n+1}(\xi))\Big),\\ d\Big(T(\xi,z_n(\xi)),T(\xi,z_{n+1}(\xi))\Big),\\ d(T(\xi,w_n(\xi)),T(\xi,w_{n+1}(\xi)))\end{array}\right\}$$

$$< \max\left\{\begin{array}{c} d\Big(T(\xi,x_{n-1}(\xi)),T(\xi,x_n(\xi))\Big)\\ ,d(T(\xi,y_{n-1}(\xi)),T(\xi,y_n(\xi))),\\ d(T(\xi,z_{n-1}(\xi)),T(\xi,z_n(\xi))),\\ d(w_{n-1}(\xi),T(\xi,w_n(\xi)))\end{array}\right\}$$

Thus, $\left\{\max\left\{\begin{array}{c} d(T(\xi,x_n(\xi)),T(\xi,x_{n+1}(\xi)))\\ ,d(T(\xi,y_n(\xi)),T(\xi,y_{n+1}(\xi))),\\ d\Big(T(\xi,z_n(\xi)),T(\xi,z_{n+1}(\xi))\Big),\\ d(T(\xi,w_n(\xi)),T(\xi,w_{n+1}(\xi)))\end{array}\right\}\right\}$ is positive decreasing sequence. Hence, there exists $r \geq 0$ such that

$$\lim_{n\to+\infty} \max \left\{ \begin{array}{c} d\left(T(\xi, x_n(\xi)), T(\xi, x_{n+1}(\xi))\right), \\ d(T(\xi, y_n(\xi)), T(\xi, y_{n+1}(\xi))), \\ d\left(T(\xi, z_n(\xi)), T(\xi, z_{n+1}(\xi))\right), \\ d(T(\xi, w_n(\xi)), T(\xi, w_{n+1}(\xi))) \end{array} \right\} = r$$

Suppose that $r > 0$. letting $n \to +\infty$ in (2.12), we obtain that

$$0 < r$$

$$< \lim_{n\to+\infty} \phi \left(\max \left\{ \begin{array}{c} d\left(T(\xi, x_{n-1}(\xi)), T(\xi, x_n(\xi))\right), \\ d\left(T(\xi, y_{n-1}(\xi)), T(\xi, y_n(\xi))\right), \\ d(T(\xi, z_{n-1}(\xi)), T(\xi, z_n(\xi))), \\ d(w_{n-1}, T(\xi, w_n(\xi))) \end{array} \right\} \right) \quad (6.2.\ 2.13)$$

it is a contradiction. We deduce that

$$\lim_{n\to+\infty} \max \left\{ \begin{array}{c} d(T(\xi, x_n(\xi)), T(\xi, x_{n+1}(\xi))), \\ d(T(\xi, y_n(\xi)), T(\xi, y_{n+1}(\xi))), \\ d(T(\xi, z_n(\xi)), T(\xi, z_{n+1}(\xi))), \\ d(T(\xi, w_n(\xi)), T(\xi, w_{n+1}(\xi))) \end{array} \right\} = 0 \quad (6.2.\ 2.14)$$

We shall show that $\{T(\xi, x_n(\xi)\}, \{T(\xi, y_n(\xi))\}, \{T(\xi, z_n(\xi))\}$

and $\{T(\xi, w_n(\xi))\}$ are Chauchy sequences. Assume the contrary, that is $\{T(\xi, x_n(\xi)\}, \{T(\xi, y_n(\xi))\}, \{T(\xi, z_n(\xi))\}$ and $\{T(\xi, w_n(\xi))\}$ are not a Cauchy sequence.

$$\lim_{n,m\to+\infty} d(T(\xi, x_m(\xi), T(\xi, x_n(\xi)) \neq 0 \quad , \quad \lim_{n,m\to+\infty} d(T(\xi, y_m(\xi)), T(\xi, y_n(\xi))) \neq 0$$

$$\lim_{n,m\to+\infty} d(T(\xi, z_m(\xi)), T(\xi, z_n(\xi))) \neq 0 \quad , \quad \lim_{n,m\to+\infty} d(T(\xi, w_m(\xi)), T(\xi, w_n(\xi))) \neq 0$$

This means that there exists $\epsilon > 0$ for which we can find subsequences of integers (m_k) and (n_k) with $n_k > m_k > k$ such that

$$\max \left\{ \begin{array}{c} d(T(\xi, x_{nk}(\xi)), T(\xi, x_{mk}(\xi))), \\ d\left(T(\xi, y_{nk}(\xi)), T(\xi, y_m(\xi)\right), \\ d\left(T(\xi, z_{nk}(\xi)), T(\xi, z_{mk}(\xi))\right), \\ d(T(\xi, w_{nk}(\xi)), T(\xi, w_{mk}(\xi))) \end{array} \right\} \geq \epsilon \quad (6.2.\ 2.15)$$

Further corresponding to m_k we can choose n_k in such a way that it is the smallest integer with $n_k > m_k$ and satisfying (6.2. 2.15). Then

$$\max \left\{ \begin{array}{c} d\left(T(\xi, x_{nk-1}(\xi)), T(\xi, x_{mk}(\xi))\right) \\ , d(T(\xi, y_{nk-1}(\xi)), T(\xi, y_{mk}(\xi))), \\ d\left(T(\xi, z_{nk-1}(\xi)), T(\xi, z_{mk}(\xi))\right), \\ d(T(\xi, w_{nk-1}(\xi)), T(\xi, w_{mk}(\xi))) \end{array} \right\} < \epsilon \quad (6.2.\ 2.16)$$

By triangular inequality and(6.2. 2.16), we have

$$d\left(T(\xi, x_{mk}(\xi)), T(\xi, x_{nk}(\xi))\right) \le$$

$$d\left(T(\xi, x_{mk}(\xi)), T(\xi, x_{nk-1}(\xi))\right) + d\left(T(\xi, x_{nk-1}(\xi)), T(\xi, x_{nk}(\xi))\right)$$

$$< \epsilon + d\left(T(\xi, x_{nk-1}(\xi)), T(\xi, x_{nk}(\xi))\right) \qquad (6.2.\ 2.17)$$

Thus, by (6.2. 2.14)we obtain

$$\lim_{k\to+\infty} d\left(T(\xi, x_{mk}(\xi)), T(\xi, x_{nk}(\xi))\right)$$

$$\le \lim_{k\to+\infty} d\left(T(\xi, x_{mk}(\xi)), T(\xi, x_{nk-1}(\xi))\right) \le \epsilon \qquad (6.2.\ 2.18)$$

Similarly, we have

$$\lim_{k\to+\infty} d\left(T(\xi, y_{mk}(\xi)), T(\xi, y_{nk}(\xi))\right)$$

$$\le \lim_{k\to+\infty} d\left(T(\xi, y_{mk}(\xi)), T(\xi, y_{nk-1}(\xi))\right) \le \epsilon \qquad (6.2.\ 2.19)$$

$$\lim_{k\to+\infty} d\left(T(\xi, z_{mk}(\xi)), T(\xi, z_{nk}(\xi))\right) \qquad (6.2.\ 2.20)$$

$$\le \lim_{k\to+\infty} d\left(T(\xi, z_{mk}(\xi)), T(\xi, z_{nk-1}(\xi))\right) \le \epsilon$$

$$\lim_{k\to+\infty} d(T(\xi, w_{mk}(\xi)), T(\xi, w_{nk}(\xi)))$$

$$\le \lim_{k\to+\infty} d(T(\xi, w_{mk}(\xi)),\ T(\xi, w_{nk-1}(\xi))) \le \epsilon \qquad 2.21$$

Again by (6.2. 2.16), we have

$$d\left(T(\xi, x_{mk}(\xi)), T(\xi, x_{nk}(\xi))\right)$$

$$\le d\left(T(\xi, x_{mk}(\xi)), T(\xi, x_{mk-1}(\xi))\right)$$

$$+ d\left(T(\xi, x_{mk-1}(\xi)), T(\xi, x_{nk-1}(\xi))\right) + d\left(T(\xi, x_{nk-1}(\xi)), T(\xi, x_{nk}(\xi))\right)$$

$$< d\left(T(\xi, x_{mk}(\xi)), T(\xi, x_{mk-1}(\xi))\right) + d\left(T(\xi, x_{mk-1}(\xi)), T(\xi, x_{mk}(\xi))\right)$$

$$+ d\left(T(\xi, x_{mk}(\xi)), T(\xi, x_{nk-1}(\xi))\right) + d(T(\xi, x_{nk-1}(\xi))), T(\xi, x_{nk}(\xi)))$$

$$< d\left(T(\xi, x_{m_k(\xi)}, T(\xi, x_{m_k-1}(\xi)\right) + d(T(\xi, xmk-1\ (\xi)), T(\xi, xmk(\xi))))$$

$$+ \epsilon + d(T(\xi, x_{n_k-1}(\xi)), T(\xi, x_{n_k}(\xi)))$$

Letting $k \to +\infty$ and using (6.2. 2.14)we get

$$\lim_{k\to+\infty} d(T(\xi, x_{m_k}(\xi)), T(\xi, x_{n_k}(\xi))) \le \quad \lim_{k\to+\infty} d(T(\xi, x_{m_{k-1}}(\xi)), T(\xi, x_{nk-1}(\xi))) \le \epsilon \qquad (6.2.\ 2.22)$$

Similarly, we have

$$\lim_{k\to+\infty} d(T(\xi, y_{m_k}(\xi)), T(\xi, y_{mn_k}(\xi)))$$

$\leq \ lim_{k\to+\infty} d\big(T(\xi, y_{m_{k-1}}(\xi)), T(\xi, y_{n_k-1}(\xi))\big) \leq \epsilon$ (6.2. 2.23)

$$lim_{k\to+\infty} d\big(T(\xi, z_{m_k}(\xi)), T(\xi, z_{n_k}(\xi))\big)$$

$$\leq \ lim_{k\to+\infty} d\big(T(\xi, z_{m_{k-1}}(\xi)), T(\xi, z_{n_{k-1}}(\xi))\big)$$

$$\leq \epsilon \qquad (6.2.\ 2.24)$$

$$lim_{k\to+\infty}\ d\big(Tw_{m_k}, Tw_{n_k}\big)$$

$$\leq \ lim_{k\to+\infty}\ d\big(T(\xi, w_{m_{k-1}}(\xi)), T(\xi, w_{n_{k-1}}(\xi))\big)$$

$$\leq \epsilon \qquad (6.2.\ 2.25)$$

Using (6.2. 2.15)and (6.2. 2.22)- (6.2. 2.25)we have

$$lim_{k\to+\infty}\ max \begin{Bmatrix} d\big(Tx_{m_k}, Tx_{n_k}\big), d\big(Ty_{m_k}, Ty_{n_k}\big), \\ d\big(Tz_{m_k}, Tz_{n_k}\big), d\big(Tw_{m_k}, Tw_{n_k}\big) \end{Bmatrix}$$

$$= lim_{k\to+\infty}\ max \begin{Bmatrix} d\big(T(\xi, x_{m_{k-1}}(\xi)), T(\xi, x_{n_{k-1}}(\xi))\big), \\ d\big(T(\xi, y_{m_{k-1}}(\xi)), T(\xi, y_{n_{k-1}}(\xi))\big), \\ d\big(T(\xi, z_{m_{k-1}}(\xi)), T(\xi, z_{n_{k-1}}(\xi))\big), \\ d\big(T(\xi, w_{m\ k-1}(\xi)), T(\xi, w_{n_{k-1}}(\xi))\big) \end{Bmatrix}$$

$$= \epsilon \qquad 2.26$$

Now, using inequality (2.1) we obtain

$$d\left(T(\xi, x_{m_k}(\xi), T(\xi, x_{n_k}(\xi)\right)$$

$$= \ d(TF\big(\xi, x_{m_k-1}(\xi), y_{m_k-1}(\xi), z_{m_k-1(\xi)}, w_{m_k-1(\xi)}\big), TF\big(\xi, x_{n_k-1(\xi)}, y_{n_k-1(\xi)}, z_{n_k-1(\xi)}, w_{n_k-1}(\xi)\big)$$

$$\leq \ \phi \left(max \begin{Bmatrix} d\left(T(\xi, x_{m_k-1}(\xi), Tx_{n_k-1}(\xi)\right), \\ d\big(T(\xi, y_{m_k-1}(\xi), Ty_{n_k-1}(\xi)\big), \\ d\left(T(\xi, z_{m_k-1}(\xi), Tz_{n_k-1}(\xi)\right), \\ d\big((\xi, w_{m_k-1}(\xi), Tw_{n_k-1}(\xi)\big) \end{Bmatrix} \right) \quad (6.2.\ 2.27)$$

$$d\big(T(\xi, y_{m_k}(\xi), T(\xi, y_{n_k}(\xi)\big) \ =$$
$$d(TF\big(\xi, y_{(m_k)-1}(\xi), z_{(m_k)-1}(\xi), w_{(m_k)-1}(\xi), x_{n-1}(\xi)\big), TF\big(\xi, y_{n_k-1}(\xi), z_{n_k-1}(\xi), w_{n_k-1}(\xi), x_{n_k-1}(\xi)\big)$$

$$\leq \ \phi \left(max \begin{Bmatrix} d\left(T(\xi, x_{m_k}(\xi), Tx_{n_k-1}(\xi)\right) \\ , d\big(T(\xi, y_{m_k-1}(\xi), Ty_{n_k-1}(\xi)\big), \\ d\big(T(\xi, z_{m_k-1}(\xi), T(\xi, z_{n_k-1}(\xi)\big), \\ d\big(w_{m_k-1}(\xi), Tw_{n_k-1}(\xi)\big) \end{Bmatrix} \right) \quad (6.2.\ 2.28)$$

$$d\big(T(\xi, z_{m_k}(\xi), T(\xi, z_{n_k}(\xi)\big) \ =$$
$$\boldsymbol{d\left(TF\big(\xi, z_{m_k-1}(\xi), w_{m_k-1}(\xi), x_{m_k-1}(\xi), y_{m_k-1}(\xi)\big), TF\big(\xi, z_{n_k-1}(\xi)\big), w_{n_k-1}(\xi), x_{n_k-1}(\xi), y_{n_k-1}(\xi)\big)\right)}$$

$$\leq \phi\left(max \begin{Bmatrix} d\left(T\left(\xi, x_{m_k-1}(\xi)\right), T(\xi, x_{n_k-1}(\xi)\right) \\ , d(T(\xi, y_{m_k-1}(\xi)), T(\xi, y_{n_k-1}(\xi)), \\ d\left(T\left(\xi, z_{m_k-1}(\xi)\right), T\left(\xi, z_{n_k-1}(\xi)\right)\right), \\ d(w_{m_k-1}(\xi), Tw_{n_k-1}(\xi))) \end{Bmatrix} \right) \qquad (6.2.\ 2.29)$$

and

$$d(T(\xi, w_{m_k}(\xi)), T(\xi, w_{n_k}(\xi)) =$$
$$d\left(TF((\xi, w_{m_k-1}(\xi), x_{m_k-1}(\xi), y_{m_k-1}(\xi), z_{m_k-1}(\xi)), TF((\xi, w_{n_k-1}(\xi), x_{n_k-1}(\xi), y_{n_k-1}(\xi), z_{n_k-1}(\xi))\right)$$

$$\leq \phi\left(max \begin{Bmatrix} d(T(\xi, x_{m_k-1}(\xi)), T(\xi, x_{n_k-1})) \\ , d(T(\xi, y_{m_k-1}(\xi)), T(\xi, y_{n_k-1}(\xi)), \\ d\left(T\left(\xi, z_{m_k-1}(\xi)\right), T\left(\xi, z_{n_k-1}(\xi)\right)\right), \\ d(w_{m_k-1}(\xi), Tw_{n_k-1}(\xi)) \end{Bmatrix} \right) \qquad (6.2.\ 2.30)$$

We deduce from (6.2. 2.27)- (6.2. 2.30) that

$$max \begin{Bmatrix} d(T(\xi, x_{m_k}(\xi)), T(\xi, x_{n_k}(\xi))) \\ , d(T(\xi, y_{m_k}(\xi)), T(\xi, y_{n_k}(\xi))), \\ d\left(T\left(\xi, z_{m_k}(\xi)\right), T\left(\xi, z_{n_k}(\xi)\right)\right), \\ d(T(\xi, w_{m_k}(\xi)), T(\xi, w_{n_k}(\xi))) \end{Bmatrix}$$

$$\leq \phi\left(max \begin{Bmatrix} d(T(\xi, x_{m_k-1}(\xi)), T(\xi, x_{n_k-1}(\xi))) \\ , d(T(\xi, y_{m_k-1}(\xi)), T(\xi, y_{n_k-1}(\xi))), \\ d(T(\xi, z_{m_k-1}(\xi)), T(\xi, z_{n_k-1}(\xi))), \\ d(w_{m_k-1}(\xi), T(\xi, w_{n_k-1}(\xi))) \end{Bmatrix} \right) \qquad (6.2.\ 2.31)$$

Letting $k \to +\infty$ in (2.31) and having in mind (2.16), we get that

$0 < \epsilon \leq lim_{t \to \epsilon^+} \phi(t) < \epsilon$ .it is a contradiction. Thus $\{T(\xi, x_n(\xi))\}, \{T(\xi, y_n(\xi))\}, \{T(\xi, z_n(\xi))\}$ and $\{T(\xi, w_n(\xi))\}$ are Chauchy sequences in (X, d). Since X is complete metric space, $\{T(\xi, x_n(\xi))\}, \{T(\xi, y_n(\xi))\}, \{T(\xi, z_n(\xi))\}$ and $\{T(\xi, w_n(\xi))\}$

are convergent sequences.

Since T is an ICS mapping, there exist $x(\xi), y(\xi), z(\xi), w(\xi) \in X$ such that

$$lim_{n \to +\infty} x_n(\xi) = x(\xi), lim_{n \to +\infty} y_n(\xi) = y(\xi),$$

$$lim_{n \to +\infty} z_n(\xi) = z(\xi), lim_{n \to +\infty} w_{n(\xi)} = w(\xi). \qquad (6.2.\ 2.32)$$

Since T is continuous, we have

$$lim_{n \to +\infty} T(\xi, x_n(\xi)) = T(\xi, x(\xi)), lim_{n \to +\infty} T(\xi, y_n(\xi)) = T(\xi, y(\xi)),$$

$$lim_{n \to +\infty} T(\xi, z_n(\xi)) = T(\xi, z(\xi)),$$

$$lim_{n \to +\infty} T(\xi, w_n(\xi)) = T(\xi, w(\xi)). \qquad (6.2.\ 2.33)$$

Suppose now the assumption (a) holds, that is, F is continuous. we obtain $x(\xi) = lim_{n \to +\infty} x_{n+1}(\xi) = lim_{n \to +\infty} F((\xi, x_n(\xi), y_n(\xi), z_n(\xi), w_n(\xi)))$

$$= F\left((\xi, lim_{n\to+\infty}x_n(\xi), lim_{n\to+\infty}y_n(\xi), lim_{n\to+\infty}z_n(\xi), lim_{n\to+\infty}w_n(\xi)\right))$$

$$= F((\xi, x(\xi), y(\xi), z(\xi), w(\xi)))$$

$$y(\xi) = lim_{n\to+\infty}\, y_{n+1}(\xi) = lim_{n\to+\infty}F\left((\xi, y_n(\xi), z_n(\xi), w_n(\xi), x_n(\xi)\right))$$

$$= F\left((\xi, lim_{n\to+\infty}y_n(\xi), lim_{n\to+\infty}z_n(\xi), lim_{n\to+\infty}w_n(\xi), lim_{n\to+\infty}x_n(\xi)\right))$$

$$= F\left((\xi, y(\xi), z(\xi), w(\xi), x(\xi)\right))$$

$$z\,(\xi) = lim_{n\to+\infty}\, z_{n+1}(\xi)$$

$$= lim_{n\to+\infty}F\left(z_n(\xi), w_n(\xi), x_n(\xi), y_n(\xi)\right)$$

$$= F\left((\xi, lim_{n\to+\infty}z_n(\xi), lim_{n\to+\infty}w_n(\xi), lim_{n\to+\infty}x_n(\xi), lim_{n\to+\infty}y_n(\xi)\right)$$

$$= F((\xi, z(\xi), w(\xi), x(\xi), y(\xi))$$

and $$w(\xi) = lim_{n\to+\infty}w_{n+1}(\xi) = lim_{(n\to+\infty)}F\left((\xi, w_n(\xi), x_n(\xi), y_n(\xi), z_n(\xi)\right)$$

$$= F((\xi, lim_{n\to+\infty}w_n(\xi), lim_{n\to+\infty}x_n(\xi), lim_{n\to+\infty}y\_(\xi), lim_{n\to+\infty}z_n\,(\xi)$$

$$= F((\xi, w(\xi), x(\xi), y(\xi), z(\xi))$$

We have proved that F has a quadrupled fixed point.

Suppose now the assumption (b) holds. Since $\{x_n(\xi)\}$, $\{z_n(\xi)\}$ are non- decreasing with $x_n(\xi) \to x(\xi), z_n(\xi) \to z(\xi)$ and $\{y_n(\xi)\}$, $\{w_n(\xi)\}$ are non- increasing

with $y_n(\xi) \to y(\xi), w_n(\xi) \to w(\xi)$ then we have

$$x_n(\xi) \le x(\xi),\ y_n(\xi) \ge y(\xi)\ , z_n(\xi) \le z(\xi),\ w_n(\xi) \ge w(\xi)$$

for all n. Consider now

$$d\left(T(\xi, x(\xi)), TF\left((\xi, x(\xi), y(\xi), z(\xi), w(\xi)\right)\right)$$

$$\le d(T(\xi, x(\xi), Tx_{n+1}(\xi))) + d\left(T(\xi, x_{n+1}, TF((\xi, x(\xi), y(\xi), z(\xi), w(\xi)))\right)$$

$$= d(T(\xi, x(\xi)), T(\xi, x_{n+1}(\xi)))$$

$$+ d\left(TF((\xi, x_n(\xi), y_n(\xi), z_n(\xi), w_n(\xi))), TF((\xi, x(\xi), y(\xi), z(\xi), w(\xi)))\right)$$

$$\le d(Tx(\xi), Tx_{n+1}(\xi)))$$

$$+ \phi\left(max\begin{Bmatrix} d(T(\xi, x_n(\xi)), Tx(\xi))), d(T(\xi, y_n(\xi)), Ty(\xi))), \\ d\left(T(\xi, z_n, Tz(\xi)\right), d(T(\xi, w_n(\xi), T(\xi, w(\xi))) \end{Bmatrix}\right) \quad (6.2.\ 2.34)$$

Taking as $n \to \infty$ and using (6.2. 2.33)the right hand side of

(6.2. 2.34)tends to 0,

So we get that $d(T(\xi, x(\xi)), TF((\xi, x(\xi), y(\xi), z(\xi), w(\xi)))) = 0.$

Thus $Tx(\xi)) = TF((\xi, x(\xi), y(\xi), z(\xi), w(\xi)))$ and T is injective,

We get that x(ξ)) = F(ξ, (x(ξ),y(ξ),z(ξ),w(ξ))). Similarly we find that

$$y = F((\xi, y, z, w, x),\ z = F((\xi, z(\xi), w(\xi), x(\xi), y(\xi)))$$

$$And\ w(\xi) = F((\xi, w(\xi), x(\xi), y(\xi), z(\xi)))$$

Hence F has a quadruple fixed point.

Corollary 6.3:- Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d in X such that (X, d) is a complete random metric space. Suppose $T: X \rightarrow X$ is a ICS mapping and $F: X^4 \rightarrow X$ is such that F has the mixed monotone property. Assume that there exists $\phi \in \phi$ $and\ \xi \varepsilon\ \Omega, be\ a\ measerable\ sellector$ such that

$$d\left(TF\big((\xi, x(\xi), y(\xi), z(\xi), w(\xi)\big), TF\big((\xi, u(\xi), v(\xi), p(\xi), q(\xi)\big)\right)$$

$$\leq \phi\left(\frac{d(T(\xi,x(\xi)),Tu(\xi,(\xi)))+ d(T(\xi,y(\xi)),(\xi,Tv(\xi)))+ d(T(\xi,z(\xi)),T(\xi,p(\xi)))+ d(w(\xi),T(\xi,q(\xi)))}{4}\right)$$

for any $x(\xi), y(\xi), z(\xi), w(\xi) \in X$ for which $x(\xi) \leq u(\xi)$,

$v(\xi) \leq y(\xi), z(\xi) \leq p(\xi),\ q(\xi) \leq w(\xi)$. Suppose either

i. F is continuous, or
ii. X has the following property:
(a) if non decreasing sequence $x_n(\xi) \rightarrow x(\xi)$ $(respectively,\ z_n(\xi) \rightarrow z(\xi))$, then $x_n(\xi) \leq x(\xi)$, $(respectively,\ z_n(\xi) \leq z(\xi))$ for all n,
(b) if non increasing sequence $y_n(\xi) \rightarrow y(\xi)$ $(respectively,\ w_n(\xi) \rightarrow w(\xi))$, then $y_n(\xi) \leq y(\xi)$, $(respectively,\ w_n(\xi) \geq w(\xi))$ for all n.

If there exists $x_0(\xi), y_0(\xi), z_0(\xi), w_0(\xi) \in X$ such that

$$x_0(\xi) \geq F((\xi, x_0(\xi), y_0(\xi), z_0(\xi), w_0(\xi))), y_0(\xi) \leq F((\xi, y_0(\xi), z_0(\xi), w_0(\xi), x_0(\xi))),$$

$$z_0(\xi) \leq F\big((\xi, z_0(\xi), w_0(\xi), x_0(\xi), y_0(\xi)\big)\ and\ w_0(\xi) \geq F\big((\xi, w_0(\xi), Sx_0(\xi), y_0(\xi), z_0(\xi)\big),$$

Then there exist $x(\xi), y(\xi), z(\xi), w(\xi) \in X$ such that

$$x(\xi) = F\big((\xi, x(\xi), y(\xi), z(\xi), w(\xi)\big),\ y(\xi) = F\big((\xi, y(\xi), z(\xi), w(\xi), x(\xi)\big),$$

$$z(\xi) = F((\xi, z(\xi), w(\xi), x(\xi), y(\xi))\ and$$

$$w(\xi) = F((\xi, w(\xi), x(\xi), y(\xi), z(\xi))$$

Then F has a quadrupled fixed point.

Proof:- If we take

$$\frac{1}{4}\left[\begin{matrix} d\left(T\big(\xi, x(\xi)\big), T\big(\xi, u(\xi)\big)\right) + d\left(T\big(\xi, y(\xi)\big), T\big(\xi, v(\xi)\big)\right) \\ d(T(\xi, z(\xi)), T(\xi, p) + d(w(\xi), T(\xi, q(\xi))) \end{matrix}\right]$$

$$\leq max \begin{Bmatrix} d(T(\xi, x(\xi)), T(\xi, u(\xi))), \\ d(T(\xi, y(\xi)), T(\xi, v(\xi))), \\ d(T(\xi, z(\xi)), T(\xi, p(\xi))), \\ d(w(\xi)), T(\xi, q(\xi))) \end{Bmatrix} \qquad (6.3.2)$$

Then, applying Theorem 6.2.2 result can be obtained easily.

Corollary 6.3.2:- Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d in X such that (X, d) is a complete random metric space. Suppose $T: X \rightarrow X$ is a ICS mapping and F: $X^4 \rightarrow$ X is such that F has the mixed monotone property. $\xi \varepsilon\, \Omega, be\ a\ measerable\ sellector$. Assume that there exists $k \in [0,1)$ such that

$$d\big(TF((\xi, x(\xi), y(\xi), z(\xi), w(\xi))), TF\big((\xi, u(\xi), v(\xi), p(\xi), q(\xi))\big)\big)$$

$$\leq k\, max \left\{ \begin{array}{c} d(T(\xi, x(\xi)), T(\xi, u(\xi))), \\ d(T(\xi, y(\xi)), T(\xi, v(\xi))), \\ d(T(\xi, z(\xi)), T(\xi, p(\xi))), \\ d(w(\xi), T(\xi, q(\xi))) \end{array} \right\} \qquad (6.3.3)$$

For any $x(\xi), y(\xi), z(\xi), w(\xi) \in X$ for which $x(\xi) \leq u(\xi), v(\xi) \leq y(\xi)$,

$z(\xi) \leq p(\xi),\ q(\xi) \leq w(\xi)$. Suppose either

i. F is continuous, or
ii. X has the following property:
(a) if non decreasing sequence $x_n(\xi) \rightarrow x(\xi)\, (respectively,\ z_n(\xi) \rightarrow z(\xi))$, then $x_n(\xi) \leq x(\xi), (respectively,\ z_n(\xi) \leq z(\xi))$ for all n,
(b) if non increasing sequence $y_n(\xi) \rightarrow y(\xi)\, (respectively,\ w_n(\xi) \rightarrow w(\xi))$, then $y_n(\xi) \leq y(\xi), (respectively,\ w_n(\xi) \geq w(\xi))$ for all n.

If there exists $x_0(\xi), y_0(\xi), z_0(\xi), w_0(\xi) \in X$ such that

$$x_0(\xi) \geq F\big((\xi, x_0(\xi), y_0(\xi), z_0(\xi), w_0(\xi)\big), y_0(\xi) \leq F\big((\xi, y_0(\xi), z_0(\xi), w_0(\xi), x_0(\xi)\big),$$

$$z_0(\xi) \leq F\big((\xi, z_0(\xi), w_0(\xi), x_0(\xi), y_0(\xi)\big)\ w_0(\xi) \geq F\big(w_0(\xi), S(\xi, x_0(\xi)), y_0(\xi), z_0(\xi)\big),$$

then there exist $x, y, z, w \in X$ such that

$$x(\xi) = F\big((\xi, x(\xi), y(\xi), z(\xi), w(\xi)\big), \qquad y(\xi) = F\big((\xi, y(\xi), z(\xi), w(\xi), x(\xi)\big),$$

$$z(\xi) = F((\xi, z(\xi), w(\xi), x(\xi), y(\xi))\ and\ w = F((\xi, w(\xi), x(\xi), y(\xi), z(\xi))$$

Then F has a quadrupled fixed point.

Proof:- It can be proved easily by Theorem 6.2.2.

Corollary 6.3.3:- Let $(X, \leq)$ be a partially ordered set and suppose there is a metric d in X such that (X, d) is a completerandom metric space. Suppose $T: X \rightarrow X$ is a ICS mapping

F: $X^4 \rightarrow$ X is such that F has the mixed monotone property. Assume that there exists $k \in [0,1)$ such that

$$d\left(TF\big((\xi, x(\xi), y(\xi), z(\xi), w(\xi)\big), TF\big((\xi, u(\xi), v(\xi), p(\xi), q(\xi)\big)\right)$$

$$\leq \frac{k}{4} \left(\begin{array}{c} d(T(\xi, x(\xi)), T(\xi, u(\xi))) + d(T(\xi, y(\xi)), T(\xi, v(\xi))) \\ + d(T(\xi, z(\xi)), T(\xi, p(\xi))) + d(w(\xi), T(\xi, q(\xi))) \end{array} \right) \qquad (6.3.4)$$

for any $x(\xi), y(\xi), z(\xi), w(\xi) \in X$ for which $x(\xi) \leq u(\xi), v(\xi) \leq y(\xi), z(\xi) \leq p(\xi),\ q(\xi) \leq w(\xi)$. Suppose either

i. F is continuous, or
ii. X has the following property:
(a) if non decreasing sequence $x_n(\xi) \rightarrow x(\xi)\, (respectively,\ z_n(\xi) \rightarrow z(\xi))$, then $x_n(\xi) \leq x(\xi), (respectively,\ z_n(\xi) \leq z(\xi))$ for all n,

(b) if non increasing sequence $y_n(\xi) \to y(\xi)(respectively,\ w_n(\xi) \to w(\xi))$, then $y_n(\xi) \leq y(\xi),(respectively,\ w_n(\xi) \geq w(\xi))$ for all n.

If there exists $x_0(\xi), y_0(\xi), z_0(\xi), w_0(\xi) \in X$

such that $x_0(\xi) \geq F((\xi, x_0(\xi), y_0(\xi), z_0(\xi), w_0(\xi))),$

$$y_0(\xi) \leq F((\xi, y_0(\xi), z_0(\xi), w_0(\xi), x_0(\xi))),$$

$$z_0(\xi) \leq F((\xi, z_0(\xi), w_0(\xi), x_0(\xi), y_0(\xi)))\ and\ w_0(\xi) \geq F((\xi, w_0(\xi), Sx_0(\xi), y_0(\xi), z_0(\xi))),$$

then there exist $x(\xi), y(\xi), z(\xi), w(\xi) \in X$ such that

$$x(\xi) = F((\xi, x(\xi), y(\xi), z(\xi), w(\xi))),\ y(\xi) = F((\xi, y(\xi), z(\xi), w(\xi), x(\xi))),$$

$$z(\xi) = F\big((\xi, z(\xi), w(\xi), x(\xi), y(\xi))\big)\ and\ w(\xi) = F\big((\xi, w(\xi), x(\xi), y(\xi), z(\xi))\big)$$

Then, F has a quadrupled fixed point.

Proof:- It can be proved easily by Corollary 6.3.1

Now, we shall prove the existence and uniqueness of a quadruple fixed point, for a product X^4 of a partially ordered set $(X, \leq)$, we define a partial ordering in the following way: for all $(x(\xi), y(\xi), z(\xi), w(\xi)), (u(\xi), v(\xi), p(\xi), q(\xi)) \in X^4$

$$\big(x(\xi), y(\xi), z(\xi), w(\xi)\big) \leq \big(u(\xi), v(\xi), p(\xi), q(\xi)\big)$$

$$\to x(\xi) \leq u(\xi),\ y(\xi) \geq v(\xi),\ z(\xi) \leq p(\xi)\ and$$

$$w(\xi) \geq q(\xi)$$

We say that $(x(\xi), y(\xi), z(\xi), w(\xi)), (u(\xi), v(\xi), p(\xi), q(\xi)) \in X^4$ are comparable if

$$\big(x(\xi), y(\xi), z(\xi), w(\xi)\big) \leq \big(u(\xi), v(\xi), p(\xi), q(\xi)\big)$$

$$or\ (x(\xi), y(\xi), z(\xi), w(\xi)) \geq (u(\xi), v(\xi), p(\xi), q(\xi)) \qquad 2.40$$

Also we say that $\big(x(\xi), y(\xi), z(\xi), w(\xi)\big) = \big(u(\xi), v(\xi), p(\xi), q(\xi)\big)$

if and only if $x(\xi) = u(\xi), y(\xi) = v(\xi), z(\xi) = p(\xi), w(\xi) = q(\xi)$.

Theorem 2.6:- In addition to hypothesis of theorem 6.2.2., suppose that for all $(x(\xi), y(\xi), z(\xi), w(\xi)), (u(\xi), v(\xi), p(\xi), q(\xi)) \in X^4$,

There exists $(a(\xi), b(\xi), c(\xi), e(\xi)) \in X^4$, $\xi \varepsilon\ \Omega, be\ a\ measerable\ sellector$

such that

$$F((\xi, a(\xi), b(\xi), c(\xi), e(\xi))), F((\xi, b(\xi), c(\xi), e(\xi), a(\xi))),$$

$$F\big((\xi, c(\xi), e(\xi), a(\xi), b(\xi))\big), F((\xi, e(\xi), a(\xi), b(\xi), c(\xi))))$$

is comparable to

$$(F((\xi, x(\xi)), y(\xi), z(\xi), w(\xi)), F\big((\xi, y(\xi), z(\xi), w(\xi), x(\xi))\big),$$

$$F\big((\xi, z(\xi), w(\xi), x(\xi), y(\xi))\big), F\big((\xi, w(\xi), x(\xi), y(\xi), z(\xi))\big))$$

And

$$\begin{pmatrix} F((\xi, u(\xi), v(\xi), p(\xi), q(\xi))), F((\xi, v(\xi), p(\xi), q(\xi), u(\xi))), \\ F((\xi, p(\xi), q(\xi), u(\xi), v(\xi))), F((\xi, q(\xi), u(\xi), v(\xi), p(\xi))) \end{pmatrix}.$$

Then, F has a unique quadruple random fixed point $(x(\xi), y(\xi), z(\xi), w(\xi))$.

Proof: - The set of quadruple fixed points of F is non empty due to Theorem. Assume, now,

$(x(\xi), y(\xi), z(\xi), w(\xi)), (u(\xi), v(\xi), p(\xi), q(\xi)) \in X^4$ are two quadruple fixed points of F, that is,

$$F((\xi, x(\xi), y(\xi), z(\xi), w(\xi))) = x(\xi), \ \ F((\xi, u(\xi), v(\xi), p(\xi), q(\xi))) = u(\xi),$$

$$\big((\xi, y(\xi), z(\xi), w(\xi), x(\xi))\big) = y(\xi),$$

$$F((\xi, v(\xi), p(\xi), q(\xi), u(\xi))) = v$$

$$F((\xi, z(\xi), w(\xi), x(\xi), y(\xi)) = z(\xi), \ \ F((\xi, p(\xi), q(\xi), u(\xi), v(\xi))) = p,$$

$$F((\xi, w, x, y, z) = w,$$

$$F((\xi, q(\xi), u(\xi), v(\xi), p(\xi))) = q(\xi)$$

We shall show that $\big(x(\xi), y(\xi), z(\xi), w(\xi)\big) and \ \big(u(\xi), v(\xi), p(\xi), q(\xi)\big)$

are equal.

By assumption, there exists $(a(\xi), b(\xi), c(\xi), d(\xi)) \in X^4$ such that

$F((\xi, a(\xi), b(\xi), c(\xi), e(\xi))), F((\xi, b, c, e, a)$, And

$F((\xi, c(\xi), e(\xi), a(\xi), b(\xi)), F(\xi, e(\xi), a(\xi), b(\xi), c(\xi)))$

Are comparable to

$$F\big((\xi, x(\xi), y(\xi), z(\xi), w(\xi)\big), F\big((\xi, y(\xi), z(\xi), w(\xi), x(\xi)\big),$$

$$F((\xi, z, w, x, y), F((\xi, w(\xi), x(\xi), y(\xi), z(\xi)))$$

Also

$(F((\xi, u(\xi), v(\xi), p(\xi), q(\xi)), F((\xi, v(\xi), p(\xi), q(\xi), u(\xi)), F((\xi, p(\xi), q(\xi), u(\xi), v(\xi)), F((\xi, q(\xi), u(\xi), v(\xi), p(\xi))$.

Define sequences $\{a_n(\xi)\}, \{b_n(\xi)\}, \{c_n(\xi)\} \ and \ \{e_n(\xi)\}$ such that

$$a_0(\xi) = a(\xi), \quad b_0(\xi) = b(\xi), \quad c_0(\xi) = c(\xi) \ and \ e_0(\xi) = e(\xi)$$

and for any $n \geq 1$

$$a_n(\xi) = F\big((\xi, a_{n-1}(\xi), b_{n-1}(\xi), c_{n-1}(\xi), e_{n-1}(\xi)\big),$$

$$b_n(\xi) = F\big((\xi, b_{n-1}(\xi), c_{n-1}(\xi), e_{n-1}(\xi), a_{n-1}(\xi)\big)$$

$$c_n(\xi) = F\big((\xi, c_{n-1}(\xi), e_{n-1}(\xi), a_{n-1}(\xi), b_{n-1}(\xi)\big),$$

$$e_n = F\big((\xi, e_{n-1}(\xi), a_{n-1}(\xi), b_{n-1}(\xi), c_{n-1}(\xi)\big) \qquad 2.42$$

for all n. Further, set $x_0(\xi) = x(\xi), y_0(\xi) = y(\xi), z_0(\xi) = z(\xi),$

$w_0(\xi) = w(\xi)$ and $u_0(\xi) = u(\xi), v_0(\xi) = v(\xi), p_0(\xi) = p(\xi), q_0(\xi) = q(\xi)$, and on the same way define the sequences

$\{x_n(\xi)\}, \{y_n(\xi)\}, \{z_n(\xi)\}, \{w_n(\xi)\}$ and $\{u_n(\xi)\}, \{v_n(\xi)\}, \{p_n(\xi)\}, \{q_n(\xi)\}$. Then it is easy that

$$X_n(\xi) = F((\xi, x(\xi), y(\xi), z(\xi), w(\xi)),$$

$$u_n(\xi) = F((\xi, u(\xi), v(\xi), p(\xi), q(\xi)),$$

$$y_n(\xi) = F((\xi, y(\xi), z(\xi), w(\xi), x(\xi)), \; v_n(\xi) = F((\xi, v(\xi), p(\xi), q(\xi), u(\xi)),$$

$$z_n = F((\xi, z(\xi), w(\xi), x(\xi), y(\xi)),$$

$$p_n(\xi) = F((\xi, p(\xi), q(\xi), (\xi)u, v(\xi)),$$

$$w_n(\xi) = F((\xi, w(\xi), x(\xi), y(\xi), z(\xi)),$$

$$q_n(\xi) = F((\xi, q(\xi), u(\xi), v(\xi), p(\xi))$$

for all $n \geq 1$. Since

$$\{x_n(\xi)\}, \{y_n(\xi)\}, \{z_n(\xi)\}, \{w_n(\xi)\} = (x_1(\xi), y_1(\xi), z_1(\xi), w_1(\xi))$$

$= (x(\xi), y(\xi), z(\xi), w(\xi))$ is comparable to

$$\begin{Bmatrix} (F((\xi, a(\xi), b(\xi), c(\xi), e(\xi)\,), \\ (F((\xi, b(\xi), c(\xi), e(\xi), a(\xi)\,), \\ (F((\xi, c(\xi), e(\xi), a(\xi), b(\xi)), \\ (F((\xi, e(\xi), a(\xi), b, c)(\xi))) \end{Bmatrix}$$

$= (a_1(\xi), b_1(\xi), c_1(\xi), e_1(\xi))$, Then it is easy to show

$(x(\xi), y(\xi), z(\xi), w(\xi)) \leq (a_1(\xi), b_1(\xi), c_1(\xi), e_1(\xi))$. Recursively, we get that

$$(x(\xi), y(\xi), z(\xi), w(\xi))$$
$$\leq (a_n(\xi), b_n(\xi), c_n(\xi), e_n(\xi)) \quad for \ all \, n \geq 1$$

Now we have

$$d\left(T(\xi, x(\xi)), T(\xi, a_{n+1}(\xi)\right) =$$

$$d\left(TF((\xi, x(\xi), y(\xi), z(\xi), w(\xi)), TF(\xi, (a_n(\xi), b_n(\xi), c_n(\xi), e_n(\xi))\right)$$

$$\leq \phi\left(max \begin{Bmatrix} d(T(\xi, x(\xi)), T(\xi, a_n(\xi))), \\ d(T(\xi, y(\xi)), T(\xi, b_n(\xi))), \\ d(T(\xi, z(\xi), T(\xi, c_n(\xi))), \\ d(T(\xi, w(\xi)), T(\xi, e_n(\xi))) \end{Bmatrix} \right)$$

$$d(T(\xi, y(\xi), T(\xi, b_{n+1}(\xi)))$$

$$= d\big(TF(\xi,(y(\xi),z(\xi),w(\xi),x(\xi))),TF(\xi,\big(b_n(\xi),c_n(\xi),e_n(\xi),a_n(\xi)\big))\big)$$

$$\le \phi\left(max\begin{Bmatrix} d(T(\xi,x(\xi)),T(\xi,a_n(\xi))), \\ d(T(\xi,y(\xi)),T(\xi,b_n(\xi))), \\ d(T(\xi,z(\xi)),T(\xi,c_n(\xi))), \\ d(T(\xi,w(\xi)),T(\xi,e_n(\xi))) \end{Bmatrix}\right)$$

$$d\left(T\big(\xi,z(\xi)\big),T\big(\xi,c_{n+1}(\xi)\big)\right)$$

$$= d\big(TF(\xi,(z,w(\xi),x(\xi),y(\xi))),TF(\xi,\big(c_n(\xi),e_n(\xi),a_n(\xi),b_n(\xi)\big))\big)$$

$$\le \phi\left(max\begin{Bmatrix} d(T(\xi,x(\xi)),T(\xi,a_n(\xi))), \\ d(T(\xi,y(\xi)),T(\xi,b_n(\xi))), \\ d(T(\xi,z(\xi)),T(\xi,c_n(\xi))), \\ d(T(\xi,w(\xi)),T(\xi,e_n(\xi))) \end{Bmatrix}\right)$$

And

$$d(T(\xi,w(\xi)),T(\xi,e_{n+1}(\xi)))$$

$$= d\big(TF(\xi,(w(\xi),x(\xi),y(\xi),z(\xi))),TF(\xi,\big(e_n(\xi),a_n(\xi),b_n(\xi),c_n(\xi)\big))\big)$$

$$\le \phi\left(max\begin{Bmatrix} d(T(\xi,x(\xi)),T(\xi,a_n(\xi))), \\ d(T(\xi,y(\xi),T(\xi,b_n(\xi))), \\ d(T(\xi,z(\xi),T(\xi,c_n(\xi))), \\ d(T(\xi,w(\xi)),T(\xi,e_n(\xi))) \end{Bmatrix}\right)$$

It follows from (2.45) - (2.48)

$$max\begin{Bmatrix} d(T(\xi,x(\xi)),T(\xi,a_{n+1}(\xi))), \\ d(T(\xi,y(\xi)),T(\xi,b_{n+1}(\xi))), \\ d(T(\xi,z(\xi)),T(\xi,c_{n+1}(\xi))), \\ d\big(T(\xi,w(\xi)),T(\xi,e_{n+1}(\xi))\big) \end{Bmatrix}$$

$$\le \phi\left(max\begin{Bmatrix} d(T(\xi,x(\xi)),T(\xi,a_n(\xi))), \\ d(T(\xi,y(\xi)),T(\xi,b_n(\xi))), \\ d(T(\xi,z(\xi)),T(\xi,c_n(\xi))), \\ d(T(\xi,w(\xi)),T(\xi,e_n(\xi))) \end{Bmatrix}\right)$$

Therefore, for each $n \ge 1$,

$$max\begin{Bmatrix} d(T(\xi,x(\xi)),T(\xi,a_n(\xi))), \\ d(T(\xi,y(\xi)),T(\xi,b_n(\xi))), \\ d(T(\xi,z(\xi)),T(\xi,c_n(\xi))), \\ d(T(\xi,w(\xi)),T(\xi,e_n(\xi))) \end{Bmatrix}$$

$$\le \phi^n\left(max\begin{Bmatrix} d(T(\xi,x(\xi)),T(\xi,a_0(\xi))), \\ d(T(\xi,y(\xi)),T(\xi,b_0(\xi))), \\ d(T(\xi,z(\xi)),T(\xi,c_0(\xi))), \\ d(T(\xi,w(\xi)),T(\xi,e_0(\xi))) \end{Bmatrix}\right)$$

It is known that $\phi(t) < t$ and $lim_{r\to t^+}\phi(r) < t$

imply $lim_{n\to\infty}\phi^{n(t)} = 0$ for each $t > 0$.

Thus

$$lim_{n\to\infty}\, max \begin{Bmatrix} d(T(\xi,x(\xi)),T(\xi,a_n(\xi))), \\ d(T(\xi,y(\xi)),T(\xi,b_n(\xi))), \\ d(T(\xi,z(\xi)),T(\xi,c_n(\xi))), \\ d(T(\xi,w(\xi)),T(\xi,e_n(\xi))) \end{Bmatrix} = 0$$

This yield that

$lim_{n\to\infty} d(T(\xi,x(\xi)),T(\xi,a_n(\xi))) = 0$, $lim_{n\to\infty} d(T(\xi,y(\xi)),T(\xi,b_n(\xi))) = 0$

$lim_{n\to\infty} d(T(\xi,z(\xi)),T(\xi,c_n(\xi))) = 0$, $lim_{n\to\infty} d(T(\xi,w(\xi)),T(\xi,e_n(\xi))) = 0$

Analogously, we show that

$$lim_{n\to\infty}\, d(T(\xi,u(\xi)),T(\xi,a_n(\xi))) = 0\,,\ lim_{n\to\infty} \mathrm{d}(\mathrm{T}(\xi,\mathrm{v}(\xi)),\mathrm{T}(\xi,\mathrm{b_n}(\xi))) = 0$$

$$\mathrm{lim}_{n\to\infty}\mathrm{d}(\mathrm{T}(\xi,\mathrm{p}(\xi)),\mathrm{T}(\xi,\mathrm{c_n}(\xi))) = 0\ ,\mathrm{lim}_{n\to\infty}\mathrm{d}(\mathrm{T}(\xi,\mathrm{q}(\xi)),\mathrm{T}(\xi,\mathrm{e_n}(\xi))) = 0$$

Combining above yields that (T(ξ,x(ξ)), T(ξ, (ξ))y,T(ξ,z(ξ)),T(ξ,w(ξ))

and (T(ξ,u,T(ξ,v(ξ))),T(ξ,p(ξ))).T(ξ,q(ξ)))) are equal.

The fact that T is injective gives us x(ξ) = u(ξ) , y(ξ) = v(ξ) ,

z (ξ) = p(ξ) and w(ξ) = q(ξ))).

Hence the theorem is proved

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7. References

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