Spectra of Some Simple Graphs

Essam El Seidy[1] Salah ElDin Hussein[2] Atef AboElkher[3*]

1,2. Department of Mathematics, Faculty of Science, Ain Shams University, Abbassia, Cairo, Egypt

3. Department of Mathematics, Faculty of Science, Assiut University, Assiut, Egypt

* E-mail: atefmohamed55@yahoo.com

Abstract

We consider a finite undirected and connected simple graph $G(E, V)$ with vertex set $V(G)$ and edge set $E(G)$. The spectra of some special simple graphs and different types of their matrices are discussed to represent a graph. In this discussion we are interested in the adjacency matrix, Laplacian matrix, signless Laplacian matrix, normalized Laplacian matrix, and seidel adjacency matrix.

Keywords: Laplacian matrix, signless Laplacian matrix, normalized Laplacian matrix, seidel adjacency matrix, spectral.

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1. Introduction

Spectral graph theory has a long history. In the early days, matrix theory and linear algebra were used to analyze adjacency matrices of graphs. Algebraic methods are especially effective in treating graphs which are regular and symmetric. Sometimes, certain eigenvalues have been referred to as the "algebraic connectivity" of a graph [7]. There is a large literature on algebraic aspects of spectral graph theory, well documented in several surveys and books, such as Biggs [2], Cvetkovi ́c, Doob and Sachs [4, 5], and Seidel [15]. In the past ten years, many developments in spectral graph theory have often had a geometric flavor. For example, the explicit constructions of expander graphs, due to Lubotzky-Phillips-Sarnak [10], are based on eigenvalue sand isoperimetric properties of graphs. The discrete analogue of the Cheeger inequality has been heavily utilized in the study of random walks and rapidly mixing Markov chains [1]. New spectral techniques have emerged and they are powerful and well-suited for dealing with general graphs. In a way, spectral graph theory has entered a new era. Just as astronomers study stellar spectra to determine the make-up of distant stars, one of the main goals in graph theory is to deduce the principal properties and structure of a graph from its graph spectrum (or from a short list of easily computable invariants). The spectral approach for general graphs is a step in this direction. We will see that eigenvalues are closely related to almost all major invariants of a graph, linking one extremal property to another. There is no question that eigenvalues play a central role in our fundamental understanding of graphs. The study of graph eigenvalues realizes increasingly rich connections with many other areas of mathematics. A particularly important development is the interaction between spectral graph theory and differential geometry. There is an interesting analogy between spectral Riemannian geometry and spectral graph theory. The concepts and methods of spectral geometry bring useful tools and crucial insights to the study of graph eigenvalues, which in turn lead to new directions and results in spectral geometry. Algebraic spectral methods are also very useful, especially for extremely examples and constructions. In this book, we take a broad approach which emphasis on the geometric aspects of graph eigenvalues, while including the algebraic aspects as well. The reader is not required to have special background in geometry; spectral graph theory has had applications to chemistry [16]. Eigenvalues were associated with the stability of molecules. Also, graph spectra arise naturally in various problems of theoretical physics and quantum mechanics, for example, in minimizing energies of Hamiltonian systems. The recent progress on expander graphs and eigenvalues was initiated by problems in communication networks. The development of rapidly mixing Markov chains has intertwined with advances in randomized approximation algorithms. Applications of graph eigenvalues occur in numerous areas and in different guises. However, the underlying mathematics of spectral graph theory through all its connections to the pure and applied, the continuous and discrete can be viewed as a single unified subject.[see, 5, 6, 11, 12, 13, 17]

Throughout this paper it is assumed that all graphs are undirected and simple connected [without loops or multiple edges] and all matrices are real and symmetric.

The following are the five different types of matrices that will take care for a graph G with n vertices

v_i; $ii = 1,2,\dots n$ n:

The adjacency matrix, $A = A(G) = a_{ij}$ of G is an $n \times n$ symmetric matrix,

$$a_{ij} = \begin{cases} 1, & \text{if} \quad v_i v_j \in E, \\ 0, & \text{otherwise}. \end{cases}$$

The Laplacian matrix of G is the matrix $L = L(G) = l_{ij} = D - A$,

$$l_{ij} = \begin{cases} d_i, & \text{if} \quad i = j, \\ -1, & \text{if} \quad v_i v_j \in E, \\ 0, & \text{otherwise}. \end{cases}$$

Where D is a diagonal matrix $\left(D = (d_1, d_2, \dots, d_n)\right)$.

The signless Laplacian matrix of G is the matrix $Q = Q(G) = q_{ij} = D + A$,

$$q_{ij} = \begin{cases} d_i, & \text{if} \quad i = j, \\ 1, & \text{if} \quad v_i v_j \in E, \\ 0, & \text{otherwise}. \end{cases}$$

The normalized Laplacian matrix of G is the matrix $£ = £(G) = £_{ij}$,

$$£_{ij} = \begin{cases} 1, & \text{if} \quad i = j, \\ -\frac{1}{\sqrt{d_i d_j}}, & \text{if} \quad v_i v_j \in E, \\ 0 & \text{otherwise}. \end{cases}$$

The Seidel adjacency matrix of a graph G with adjacency matrix A is the matrix S defined by

$S = J - I - 2A = s_{ij}$ (where J is the square matrix with all entries are equal one), i.e.

$$s_{ij} = \begin{cases} 0, & \text{if } i = j, \\ -1, & \text{if } v_i v_j \in E, \\ 1, & \text{if } v_i v_j \notin E. \end{cases}$$

The results of the adjacency matrix and Laplacian matrix are calculated in [3, 4]. We compute the new results in matrices the signless Laplacian, the normalized Laplacian, and the Seidel adjacency.

Definition 1.1**.** The trace $tr(A)$ of a square matrix $A = [aij]$ is the sum of the entries along the main diagonal i.e. $tr(A) = \sum_{i=1}^{n} a_{ii.}$

The following are some simple facts about Trace

Suppose $k_1, k_2 \in \mathbb{R}$ and A, A_1, A_2 are $n \times n$ matrices.Then

$tr(k_1 A_1 + k_2 A_2) = k_1 . tr(A_1) + k_2 . tr(A_2)$.

$tr(AB) = tr(BA)$. This is known as the cyclic property of the trace.

If $A = \left(a_{ij}\right)$ is $m \times n$ matrix, then $tr(AA^t) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2$, where A^t is the transpose of matrix A. Thus, if A is square symmetric matrix ($A = A^t$), we have $tr(A^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$.

Some previously established results about the spectrum are summarized in this section. They will play an important role throughout this article.

Lemma 1.2[9]. Let A be a $n \times n$ symmetric matrix. Then,

$tr(A) = \sum_{i=1}^{n} \lambda_i$, where λ_i are the eigenvalues of the adjacency matrix. □

Theorem 1.3[12]. Let G be a simple graph, then $\sum_{i=1}^{n} d_i = 2|E|$, where d_i is the degree of the vertex i of graph G and |E| is the number of edges. □

Theorem 1.4 [12]. Let G be a simple graph of order n, then

$$\text{(i)} \sum_{i=1}^{n} \lambda_i = 0.$$

$$\text{(ii)} \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} d_i.$$ □

2. Graph spectrum

In this section we showed how graphs can not only be represented as a picture but also be represented in matrix form. We now can introduce some ideas from linear algebra, as we will be working with matrices. In particular, we will introduce ideas that still relate to graphs.

Theorem 2.1 Let G be a simple graph of order n, then

$$\text{(i)} \sum_{i=1}^{n} \rho_i = \sum_{i=1}^{n} d_i = 2|E|.$$

$$\text{(ii)} \sum_{i=1}^{n} \rho_i^2 = \sum_{i=1}^{n} d_i\,(d_i + 1),$$

where ρ_i arethe eigenvalues of the Laplacian matrix.

Proof. Since G does not have self-loops, all the diagonal elements of A are zero. Thus

$\sum_{i=1}^{n} \lambda_i = tr(A) = \sum_{i=1}^{n} a_{ii} = 0$. Then we have

$\sum_{i=1}^{n} \rho_i = tr(L) = tr(D - A) = tr(D) - tr(A) = \sum_{i=1}^{n} d_i - \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} d_i = 2|E|$, (by Theorem 1.3).

We have $\sum_{i=1}^{n} \lambda_i^2 = tr(A^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2$, where $\sum_{j=1}^{n} a_{ij}^2$ is the degree of vertex i. Since $a_{ij}^2 = (a_{i\,1}\, a_{i\,2} \,\ldots .\, a_{i\,n}) \begin{pmatrix} a_{i\,1} \\ a_{i\,2} \\ \cdot \\ \cdot \\ \cdot \\ a_{i\,n} \end{pmatrix}$, A is symmetric and G is simple graph. Thus $\sum_{j=1}^{n} a_{i\,j}^2 = d_i$, and $tr(A^2)$ equal the sum of every vertex's degree in G. If A is symmetric matrix and D is n diagonal matrix, then we have $tr(AD) = tr(DA) = 0$. Thus

$$\sum_{i=1}^{n} \rho_i^2 = tr(L^2) = tr((D - A)^2) = tr(D^2) - tr(DA) - tr(AD) + tr(A^2) = tr(D^2) + tr(A^2)$$

$$= \sum_{i=1}^{n} {d_i}^2 + \sum_{i=1}^{n} d_i.$$ □

Theorem 2.2. Let G be a simple graph of order n, then

$$\text{(i)} \sum_{i=1}^{n} \mu_i = \sum_{i=1}^{n} d_i.$$

$$\text{(ii)} \sum_{i=1}^{n} \mu_i^2 = \sum_{i=1}^{n} d_i (d_i + 1),$$

where μ_i are the eigenvalues of the signless Laplacian matrix.

Proof. (i) As in the proof of Theorem 2.1, we have

$\sum_{i=1}^{n} \mu_i = tr(Q) = tr(D + A) = tr(D) + tr(A) = \sum_{i=1}^{n} d_{ii} + \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} d_{ii}$.

$\sum_{i=1}^{n} \mu_i^2 = tr(Q^2) = tr((D + A)^2) = tr(D^2) + tr(DA) + tr(AD) + tr(A^2) = tr(D^2) + tr(A^2) = \sum_{i=1}^{n} {d_i}^2 + \sum_{i=1}^{n} d_i$. □

Theorem 2.3. Let G be a simple graph of order n, then

$\sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} 1 = n$, where ν_i are the eigenvalues of the normalized Laplacian matrix.

Proof. Since £ is normalized Laplacian matrix, we have

$\sum_{i=1}^{n} \nu_i = tr(£) = \sum_{i=1}^{n} £_{ii} = \sum_{i=1}^{n} 1 = n$. □

Theorem 2.4. Let G be a simple graph of order n, then $\sum_{i=1}^{n} \omega_i = 0$, where ω_i are the eigenvalues of the seidel adjacency matrix.

Proof. (i) Since S is seidel adjacency matrix, then we have $\sum_{i=1}^{n} \omega_i = tr(S) = \sum_{i=1}^{n} s_{ii} = 0$. □

Theorem 2.5. Suppose that c_1 be a coefficient of x^{n-1} in characteristic polynomial of

adjacency, Laplacain, signless Laplacain, normalized Laplacain, and seidel adjacency matrices, then

(i) For adjacency matrix $c_1 = 0$.
(ii) For Laplacian matrix, $-c_1 = \sum_{i=1}^{n} d_i$, where d_i are the degrees of vertices v_i respectively.
(iii) For signless Laplacian matrix, $-c_1 = \sum_{i=1}^{n} d_i$.
(iv) For normalized Laplacian matrix, $-c_1 = n$.
(v) For seidel adjacency matrix, $c_1 = 0$.

Proof. Since $-c_1$ is a trace of all matrices mention above, then the assertion is clear. □

3. Spectrum of some graphs

We applied the matrices above to find the eigenvalues and the respective multiplicities of five different graphs as follows.

3.1 Star S_n

Matrices	Eigenvalues	Respective multiplicities
Adjacency matrix	$-\sqrt{n-1}, 0, \sqrt{n-1}$	$1, n-2, 1$
Laplacian matrix	$0, 1, \mathrm{n}$	$1, n-2, 1$
Signless Laplacian Matrix	$0, 1, n$	$1, n-2, 1$
Normalized Laplacian Matrix	$0, 1, 2$	$1, n-2, 1$
Seidel adjacency matrix	$-1, \mathrm{n}-1$	$n-1, 1$

Table 1: The eigenvalues and respective multiplicities for S_n.

3.2 Complete graph K_n

Matrix	Eigenvalues	Respective multiplicities
Adjacency matrix	$-1, \mathrm{n}-1$	$n-1, 1$
Laplacian matrix	$0, n$	$1, n-1$
Signless Laplacian Matrix	$\mathrm{n}-2, 2(\mathrm{n}-1)$	$n-1, 1$
Normalized Laplacian Matrix	$0, \frac{\mathrm{n}}{\mathrm{n}-1}$	$1, n-1$
Seidel adjacency matrix	$1-\mathrm{n}, 1$	$1, n-1$

Table 2: The eigenvalues and respective multiplicities for K_n.

3.3 Path P_n

Matrix	Eigenvalues	Respective multiplicities
Adjacency matrix	$2\cos\left(\frac{\pi \mathrm{i}}{\mathrm{n}+1}\right), \mathrm{i} = \{1,2,\cdots,\mathrm{n}\}$	$1,1,\ldots,1$
Laplacian matrix	$2-2\cos(\frac{\pi i}{n})$, $i = \{0, 1, 2, \cdots, n-1\}, \forall n \geq 3$	$1,1,\ldots,1$
SignlessLaplacian Matrix	$2+2\cos(\frac{\pi i}{n})$, $i = \{1, 2, \cdots, n\}, \forall n \geq 3$	$1,1,\ldots,1$
Normalized Laplacian Matrix	$1-\cos(\frac{\pi i}{n-1})$, $i = \{0, 1, \cdots, n-1\}$ $\forall n \geq 3$	$1,1,\ldots,1$

Table 3: The eigenvalues and respective multiplicities for P_n.

3.4 Cycle C_n

Matrices	Eigenvalues	Respective multiplicities
Adjacency matrix	$2\cos\left(\frac{2\pi i}{n}\right), i = \{0,1,2,\cdots,n-1\}$	2,1, ... , 1,2 for even 1,…,1,2 for odd
Laplacian matrix	$2 - 2\cos\frac{2\pi i}{n},\ i = \{0, 1, \cdots, n-1\}\ \forall\, n \geq 3$	
Signless Laplacian Matrix	$2 + 2\cos\frac{2\pi i}{n},\ i = \{0, 1, \cdots, n-1\}\ \forall\, n \geq 3$	
Normalized Laplacian Matrix	$1 - \cos\frac{2\pi i}{n},\ i = \{0, 1, \cdots, n-1\}\ \forall\, n \geq 3$	

Table 4: The eigenvalues and respective multiplicities for C_n.

3.5 Complete Bipartite $K_{n,m}$

Recall that the general form for the adjacency matrix of a complete bipartite graph is:

$A_{K_{n,m}} = \begin{bmatrix} 0 & C \\ C^T & 0 \end{bmatrix}$, where C is n ×m matrix in which all entries are 1.

Matrices	Eigenvalues	Respective multiplicities
Adjacency matrix	$-\sqrt{nm}, 0, \sqrt{nm}$	$1, n+m-2, 1$
Laplacian matrix	$0, n, m, m+n$	$1, m-1, n-1, 1$
Signless Laplacian Matrix	$0, n, m, m+n$	$1, m-1, n-1, 1$
Normalized Laplacian Matrix	$0, 1, 2$	$1, m+n-2, 1$
Seidel adjacency matrix	$-1, m+n-1$	$m+n-1, 1$

Table 5: The eigenvalues and respective multiplicities for $K_{n,m}$.

3.6 Petersen graph Pet_{10}

Matrices	Eigenvalues	Respective multiplicities
Adjacency matrix	$-2, 1, 3$	4, 5, 1
Laplacian matrix	$0, 2, 5$	1, 5, 4
Signless Laplacian Matrix	$1, 4, 6$	4, 5, 1
Normalized Laplacian Matrix	$0, \frac{2}{3}, \frac{5}{3}$	1, 5, 4
Seidel adjacency matrix	$-3, 3$	5, 5

Table 6: The eigenvalues and respective multiplicities for Pet_{10}.

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