A new lexicographical approach for ranking fuzzy numbers

Uday Sharma
Department of Mathematics, University of Delhi, Delhi-110007, India
E-mail: udaysharma88@yahoo.com

Abstract
In the literature many ranking methods have been proposed for comparing the fuzzy numbers, most of them suffer from plenty of shortcomings such as complex calculations, inconsistency with human intuition. To overcome such shortcomings, a new ranking method is proposed for *L-R* flat fuzzy numbers which is based on the lexicographical ordering approach. It is shown that proposed ranking method satisfies all the reasonable properties of the ordering fuzzy quantities proposed by Wang & Kerre (Fuzzy Sets and Systems 118(2001) 375-385). Finally a comprehensive comparison is done between the existing ranking methods with the proposed one to demonstrate the effectiveness of the proposed ranking method.
Keywords: Ranking method, *L-R* flat fuzzy numbers

1. Introduction
In fuzzy set theory comparison of fuzzy numbers plays a vital role. Due to its application in fuzzy optimization problems ranking method has been extensively researched in the past few years. The ranking method of fuzzy numbers was first introduced by Jain (1976). In the literature a significant number of ranking methods have been suggested on the basis of Hamming distance (Yager 1980), the possibility and probability measure of fuzzy events (Lee & Li 1988), minimizing and maximizing sets (Chen1985), the total integral values (Lious & Wang 1992), the artificial neural networks (Requena et al. 1995), the area measure (Fortemps & Roubens 1996), the coefficient of variance (Cheng 1998), the centroid point of fuzzy number (Chu & Tsao 2002), the radius of gyration (Deng et al. 2006) and so on. Most of them exhibit several shortcomings associated with non-discrimination, counter-intuitive problems and complex calculations. The most commonly approach to compare fuzzy numbers is to map fuzzy numbers onto a real line by an appropriate mapping and then subsequently realize a comparison of them. Still no single existing ranking method in the literature has been superior to the other existing methods since almost each method have different drawback such as the lack of discrimination, difficulty in implementation, inconsistency with human intuition and producing counterintuitive ordering. To overcome all these problems, a new ranking method is proposed for comparing *L-R* flat fuzzy numbers based on the lexicographical ordering approach.
The paper is organized as follows: Some basic definitions and the idea of area compensation are presented in Section 2. A new ranking method for *L-R* flat fuzzy numbers is introduced in Section 3. In Section 4, a comparative study between the proposed method and the existing methods is done. The paper is concluded in Section 5.

2. Preliminaries
In this section, some basic definitions and idea of area compensation are presented which will be used in the rest of the paper.

2.1 Basic Definitions
Definition 2.1.1. (Farhadinia 2009). A fuzzy set $\tilde{A}$ in X is characterized by a membership function $\mu_{\tilde{A}}$: X $\rightarrow$ [0, 1] and denoted by

$$\tilde{A} = \left\{ \left(x, \mu_{\tilde{A}}\left(x \right) \right) : x \in X \right\}.$$

Definition 2.1.2. (Farhadinia 2009). Let $\tilde{A}$ is a fuzzy set then it is called fuzzy number if it satisfied the following conditions:

1. $\tilde{A}$ is normal
2. $\mu_{\tilde{A}}$ is upper semi-continuous.
3. $\mu_{\tilde{A}}$ is quasi-convex.
4. $Supp\left(\tilde{A}\right)$ is bounded in X.

Definition 2.1.3. (Asady 2010) A fuzzy number $\tilde{A}=(m,n,\alpha_1,\alpha_2)_{LR}$ is said to be an *L-R* flat fuzzy number if its membership function is given by

$$\mu_{\tilde{A}}(x)=\begin{cases} L\left(\dfrac{m-x}{\alpha_1}\right), & if \;\; x\le m, \alpha_1>0, \\ 1, & if \;\; m\le x\le n, \\ R\left(\dfrac{x-n}{\alpha_2}\right), & if \;\; x\ge n,\; \alpha_2>0, \end{cases}$$

where *L* and *R* are called reference function. $L, R:[0,\infty]\to[0,1]$ be two upper semi-continuous, non-increasing functions satisfying *L (0) = R (0) = 1and L (1) = R (1) = 0* that define the left and right shape of $\mu_{\tilde{A}}(x)$ respectively. If m = n then $\tilde{A}=(m,\alpha_1,\alpha_2)_{LR}$ is called *L-R* fuzzy number.
F(R) denotes the set of all the *L-R* flat fuzzy numbers.

Definition 2.1.4. (Farhadinia 2009) Let $\tilde{A}$ be a fuzzy set in X and $\alpha \in$ (0, 1]. The α - cut of the fuzzy set $\tilde{A}$ is the crisp set given by

$$\tilde{A}_\alpha=\{x\in X:\ \mu_{\tilde{A}}(x)\ge\alpha\}.$$

If $\tilde{A}=(m,n,\alpha_1,\alpha_2)_{LR}$ is a *L-R* flat fuzzy number then $\tilde{A}_\alpha=[\underline{a}(\alpha),\ \overline{a}(\alpha)]$ for all 0 < α ≤ 1, where $\underline{a}(\alpha)=m-\alpha_1L^{-1}(\alpha)$ and $\overline{a}(\alpha)=n+\alpha_2R^{-1}(\alpha)$.

Definition 2.1.5. (Farhadinia 2009) Let $\tilde{A}\in$ F(R). Then $C(\tilde{A})=\inf\{x\in \text{supp}(\tilde{A}):\ \mu_{\tilde{A}}(x)=1\}$.

2.2. Idea of Area Compensation

Roubens (1990) proposed the area comparison method based on the area compensation using α - cut of two fuzzy numbers.

Let us defined

$$S_L(\tilde{A}\ge\tilde{B})=\int_{U(\tilde{A},\tilde{B})}[\underline{a}(\alpha)-\underline{b}(\alpha)]d\alpha,$$

$$S_R(\tilde{A}\ge\tilde{B})=\int_{V(\tilde{A},\tilde{B})}[\overline{a}(\alpha)-\overline{b}(\alpha)]d\alpha,$$

where $U(\tilde{A},\tilde{B})=\{\alpha:\ 0\le\alpha\le1,\ \underline{a}(\alpha)\ge\underline{b}(\alpha)\}$,

$V(\tilde{A},\tilde{B})=\{\alpha:\ 0\le\alpha\le1,\ \overline{a}(\alpha)\ge\overline{b}(\alpha)\}$,

$S_L(\tilde{A}\ge\tilde{B})$ is the area which claims that the left slope of $\tilde{A}$ is greater to the corresponding part of $\tilde{B}$,

$S_R(\tilde{A}\ge\tilde{B})$ is the area which claims that the right slope of $\tilde{A}$ is greater to the corresponding part of $\tilde{B}$,

$C^*(\tilde{A}\ge\tilde{B})$ is the degree to which $\tilde{A}$ is larger than $\tilde{B}$ shown in Fig. 1.

$C^*: F(R)\times F(R)\to\Re$ is defined as

$$C^*(\tilde{A}\ge\tilde{B})=\frac{1}{2}\{S_L(\tilde{A}\ge\tilde{B})+S_R(\tilde{A}\ge\tilde{B})-S_L(\tilde{B}\ge\tilde{A})-S_R(\tilde{B}\ge\tilde{A})\}.$$

In this paper, we are taking $\Re$ (set of real numbers) to be the range of C^* where as in (Fortemps & Roubens 1996) the range of C^* was taken to be $\Re^+$ (set of non- negative real numbers).

3. New ranking method

In this section, a new lexicographical approach for ranking *L-R* flat fuzzy numbers is introduced. The proposed method calculates three parameters on the basis of which it compares the two *L-R* flat fuzzy numbers. Farhadinia (2009) was the first one to used the lexicographical ordering concept for ranking fuzzy numbers in four steps where as the proposed method will be using only three steps. Before presenting the new ranking method an important notation is defined as follows:

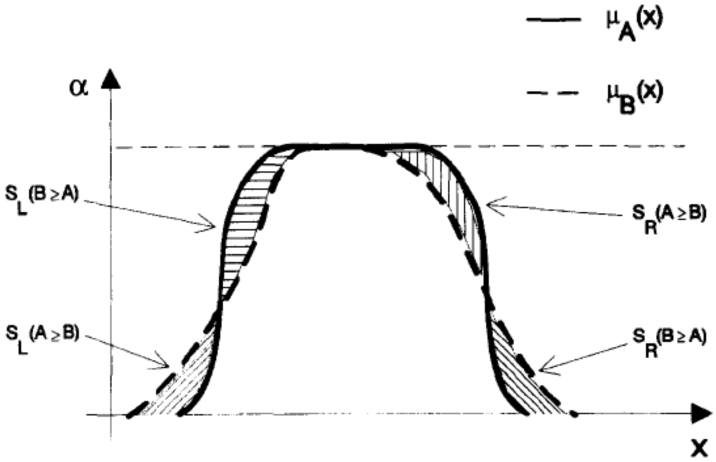

Fig. 1. Comparing $\tilde{A}$ and $\tilde{B}$.

Definition 3.1 Let $\tilde{A} \in F(R)$ be a fuzzy number. Define

$$W'(\tilde{A}) = \sup\{x \in supp(\tilde{A}) : \mu_{\tilde{A}}(x) = 1\} - \inf\{x \in supp(\tilde{A}) : \mu_{\tilde{A}}(x) = 1\}.$$

The proposed ranking method for comparing any two *L-R* flat fuzzy numbers is defined as follows:

Let $\tilde{A}$ = (m, n, α_1, α_2)$_{LR}$ and $\tilde{B}$ = (p, q, β_1, β_2)$_{LR}$ be two *L-R* flat fuzzy number. Then we go through the following steps to compare $\tilde{A}$ and $\tilde{B}$.

Step 1 Find $C(\tilde{A})$ = m and $C(\tilde{B})$ = p. If m < p then $\tilde{A} \prec \tilde{B}$ and stop. If m = p go to Step 2.

Step 2 Find $W'(\tilde{A})$ = n - m and $W'(\tilde{B})$ = q - p. If (n – m) < (q – p) then $\tilde{A} \prec \tilde{B}$ and stop. If (n – m) = (q - p) then go to Step 3.

Step 3 Find $C^*(\tilde{A} \geq \tilde{B})$.

If $C^*(\tilde{A} \geq \tilde{B}) > 0$ then $\tilde{A} \succ \tilde{B}$.

If $C^*(\tilde{A} \geq \tilde{B}) < 0$ then $\tilde{A} \prec \tilde{B}$.

If $C^*(\tilde{A} \geq \tilde{B}) = 0$ then $\tilde{A} \cong \tilde{B}$.

It can be easily shown that proposed ranking method satisfies all the reasonable properties of the fuzzy quantities proposed by Wang and Kerre (2001) for the validation of the ranking method.

$\tilde{A}_1$: For an arbitrary finite subset Γ of F(R) and $\tilde{A} \in \Gamma, \tilde{A} \succeq \tilde{A}$.

$\tilde{A}_2$: For an arbitrary finite subset Γ of F(R) and $(\tilde{A}, \tilde{B}) \in \Gamma^2, \tilde{A} \succeq \tilde{B}$ *and* $\tilde{B} \succeq \tilde{A}$, we should have $\tilde{A} \cong \tilde{B}$.

$\tilde{A}_3$: For an arbitrary finite subset Γ of F(R) and $(\tilde{A}, \tilde{B}, \tilde{C}) \in \Gamma^3, \tilde{A} \succeq \tilde{B}$ *and* $\tilde{B} \succeq \tilde{C}$, we should have $\tilde{A} \succeq \tilde{C}$.

$\tilde{A}_4$: For an arbitrary finite subset Γ of F(R) and $(\tilde{A}, \tilde{B}) \in \Gamma^2$, inf $supp(\tilde{A})$ > sup $supp(\tilde{B})$, we should have $\tilde{A} \succeq \tilde{B}$

$\tilde{A}'_4$: For an arbitrary finite subset Γ of F(R) and $(\tilde{A}, \tilde{B}) \in \Gamma^2$, inf $supp(\tilde{A})$ > sup $supp(\tilde{B})$, we should have $\tilde{A} \succ \tilde{B}$.

$\tilde{A}_5$: Let Γ and Γ' be two arbitrary finite subsets of F(R) in which $\tilde{A}$ *and* $\tilde{B}$ are in $\Gamma \cap \Gamma'$. We obtain $\tilde{A} \succ \tilde{B}$

on Γ if and only if $\tilde{A} \succ \tilde{B}$ on Γ'.

$\tilde{A}_6$: Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{A} \oplus \tilde{B}$ *and* $\tilde{B} \oplus \tilde{C}$ be elements of F(R). If $\tilde{A} \succeq \tilde{B}$, then $\tilde{A} \oplus \tilde{C} \succeq \tilde{B} \oplus \tilde{C}$.

$\tilde{A}_7$: Let $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{A} \otimes \tilde{B}$ *and* $\tilde{B} \otimes \tilde{C}$ be elements of F(R). If $\tilde{A} \succeq \tilde{B}$ and $\tilde{C} \succeq \tilde{0}$ then $\tilde{A} \otimes \tilde{C} \succeq \tilde{B} \otimes \tilde{C}$.

Remark: If $\tilde{A}, \tilde{B} \in F(R)$ *and* $\tilde{A} \prec \tilde{B}$ *then* $-\tilde{A} \succ -\tilde{B}$.

4. Comparative study

In this section, the proposed method is compared with the some of the existing methods which have been appeared in the literature and their shortcomings have been discussed. In all the examples the reference functions are defined as $L(x) = R(x) = \max(0, 1-x)$.

Example 4.1. Consider the following sets: (see Shureshjani & Hmiraki 2013).

Set 1	$\tilde{A}$ = (0.3, 0.3, 0.2, 0.2)$_{LR}$	$\tilde{B}$ = (0.5, 0.5, 0.2, 0.2)$_{LR}$	
Set 2	$\tilde{A}$ = (0.2, 0.4, 0.1, 0.1)$_{LR}$	$\tilde{B}$ = (0.3, 0.3, 0.2, 0.2)$_{LR}$	
Set 3	$\tilde{A}$ = (-0.3, -0.3, 0.2, 0.1)$_{LR}$	$\tilde{B}$ = (0.3, 0.3, 0.2, 0.2)$_{LR}$	
Set 4	$\tilde{A}$ = (0.5, 0.5, 0.2, 0.5)$_{LR}$	$\tilde{B}$ = (0.6, 0.6, 0.5, 0.2)$_{LR}$	
Set 5	$\tilde{A}$ = (0.4, 0.6, 0.4, 0.2)$_{LR}$	$\tilde{B}$ = (0.5, 0.5, 0.3, 0.4)$_{LR}$	$\tilde{C}$ = (0.6, 0.7, 0.5, 0.2)$_{LR}$

Figs. 2 - 6 present the five sets in Example 4.1. The ranking results of set 1, set 2, set 3, set 4 and set 5 are given in Table 1. The shortcoming of Yager's method (1978), Murakami et al.'s method (1983), Cheng's method (1998), Chu & Tsao's method (2002), Chen & chen's method (2009), Chen & Sanguansar's method (2011) and Shureshjani & Hmiraki's method (2013) are discussed below.

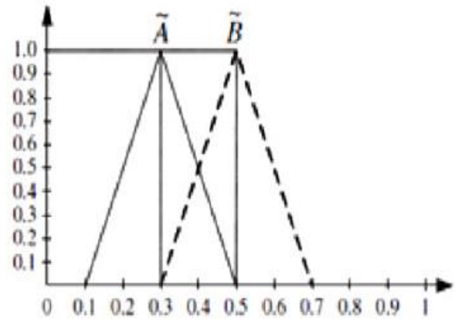

Fig. 2. Set 1 of Example 4.1.

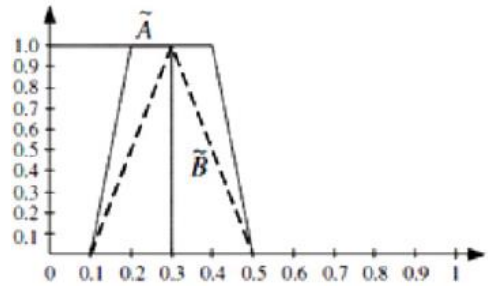

Fig. 3. Set 2 of Example 4.1.

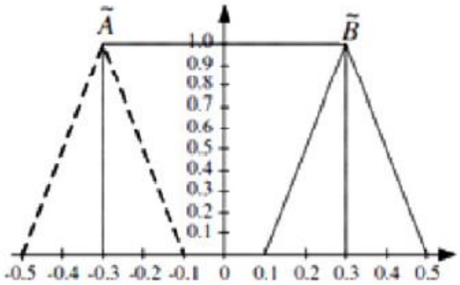

Fig. 4. Set 3 of Example 4.1.

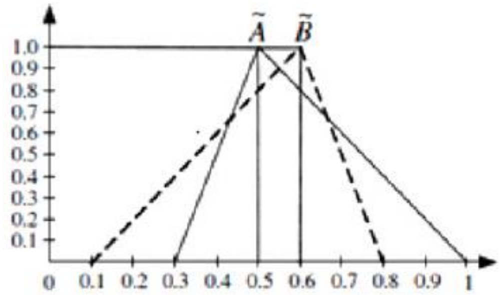

Fig. 5. Set 4 of Example 4.1.

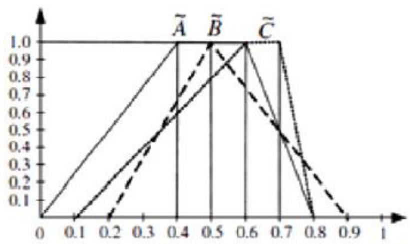

Fig. 6. Set 5 of Example 4.1.

From Table 1, we can observe that

Set 1: The proposed method and all the existing methods yield the same ranking order.

Set 2 : $C(\tilde{A}) = 0.2 < C(\tilde{B}) = 0.3$, therefore $\tilde{A} \prec \tilde{B}$ by proposed method, Murakami's method and Chen & Chen's method where as by Cheng's method, Chu & Tsao's method, Yager's method, Chen & Sanguansar's method and Shureshjani & Hmiraki's method gives $\tilde{A} = \tilde{B}$, which is not intuitive as one is trapezoidal fuzzy number and other is triangular fuzzy number.

Set 3 : Comparing positive and negative of the fuzzy numbers Cheng's method gets unreasonable result while the other existing methods and proposed method provide the same result i.e. $\tilde{A} \prec \tilde{B}$.

Set 4: Two triangular fuzzy numbers are compared which are intersecting each other and both having different left and right spread. The proposed method, Shureshjani & Hmiraki's method at α = 0.8 yield $\tilde{A} \prec \tilde{B}$ where as other existing methods gets unreasonable result (see Shureshjani & Hmiraki 2013).

Set 5: The proposed method and all the methods produce $\tilde{A} \prec \tilde{B} \prec \tilde{C}$ except Yager's method and Murakami et al.'s method gives $\tilde{A} \prec \tilde{C} \prec \tilde{B}$ which is counter-intuitive with human intuition.

Table 1 Comparative results of Example 4.1.

Method	Set 1	Set 2	Set 3	Set 4	Set 5
Yager's method (1978)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} = \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$
Murakami et al.'s method (1983)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$
Cheng's method (1998)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} = \tilde{B}$	$\tilde{A} = \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chu & Tsao's method (2002)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} = \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chen & chen's method (2009)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chen & Sanguansar's method (2011)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} = \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Shureshjani & Hmiraki's method (2013)	$\tilde{A} \prec \tilde{B}$	$\tilde{A} = \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$ (α = 0.8)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Proposed method	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$

Example 4.2. Consider the following sets used in (Nasseri et al.'s method 2013), as shown in Figs. 7 – 10, in order to compare the proposed method with Yager's method (1981), Choobineh and li's method (1985), Chen's method (1985), Cheng CV method (1998), Cheng distance method (1998), Yao and Wu's method (2000), Chu and Tsao's method (2002), Abbasbandy and Asady's method (2006), Nejad and Mashinchi's method (2011), Nasseri et al.'s method (2013). The results are shown in Table 2.

Set 1 $\tilde{A} = (0.5, 0.5, 0.1, 0.5)_{LR}$ $\tilde{B} = (0.7, 0.7, 0.3, 0.3)_{LR}$ $\tilde{C} = (0.9, 0.9, 0.5, 0.1)_{LR}$

Set 2 $\tilde{A} = (0.4, 0.7, 0.1, 0.2)_{LR}$ $\tilde{B} = (0.7, 0.7, 0.4, 0.2)_{LR}$ $\tilde{C} = (0.7, 0.7, 0.2, 0.2)_{LR}$

Set 3 $\tilde{A} = (0.5, 0.5, 0.2, 0.2)_{LR}$ $\tilde{B} = (0.5, 0.8, 0.2, 0.1)_{LR}$ $\tilde{C} = (0.5, 0.5, 0.2, 0.4)_{LR}$

Set 4 $\tilde{A} = (0.4, 0.7, 0.4, 0.1)_{LR}$ $\tilde{B} = (0.5, 0.5, 0.3, 0.4)_{LR}$ $\tilde{C} = (0.6, 0.6, 0.5, 0.2)_{LR}$

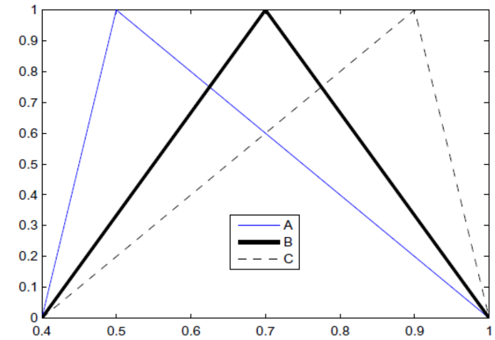

Fig. 7. Set 1 of Example 4.2.

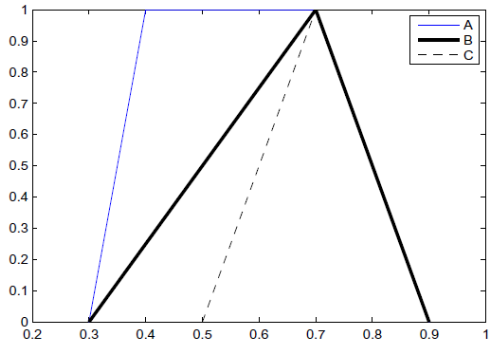

Fig. 8. Set 2 of Example 4.2.

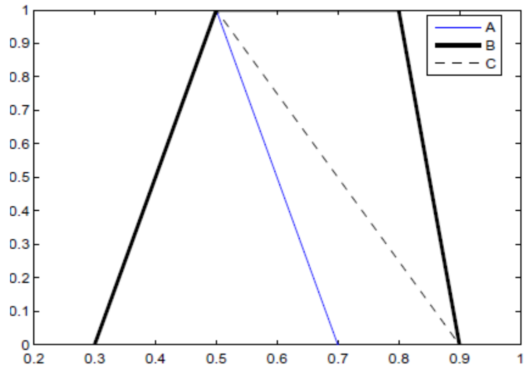

Fig. 9. Set 3 of Example 4.2.

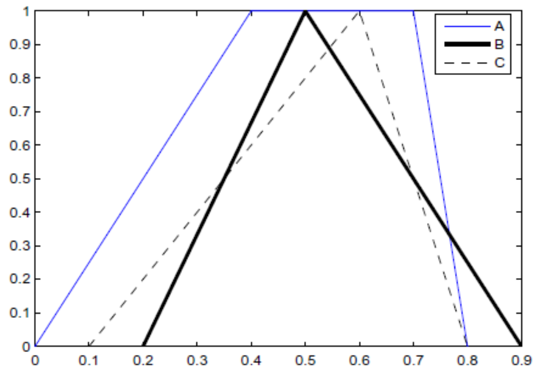

Fig. 10. Set 4 of Example 4.2.

From Table 2, we can see that

Set 1: By proposed method, we have $C(\tilde{A})=0.5<C(\tilde{B})=0.7<C(\tilde{C})=0.9$. Hence the ranking order is $\tilde{A}\prec\tilde{B}\prec\tilde{C}$ which is same as all the existing ranking methods except for Cheng CV method we have $\tilde{B}\prec\tilde{C}\prec\tilde{A}$.

Set 2: $C(\tilde{A})=0.4<C(\tilde{B})=0.7=C(\tilde{C})=0.7, W'(\tilde{B})=W'(\tilde{C})=0$ and $C^*(\tilde{C}\geq\tilde{B})\geq 0$. Hence by proposed method we get $\tilde{A}\prec\tilde{B}\prec\tilde{C}$. In Cheng CV index method, the ranking order is $\tilde{C}\prec\tilde{B}\prec\tilde{A}$.The result of our proposed method is similar to the other existing methods i.e. $\tilde{A}\prec\tilde{B}\prec\tilde{C}$.

Set 3 : $C(\tilde{A})=0.5=C(\tilde{B})=0.5=C(\tilde{C})=0$, $W'(\tilde{A})=W'(\tilde{C})=0<W'(\tilde{B})=0.5$ and $C^*(\tilde{C}\geq\tilde{A})\geq 0$, producing the ranking order $\tilde{A}\prec\tilde{C}\prec\tilde{B}$. The result of Choobineh et al.'s method, Yager's method and Chen's method are $\tilde{A}\prec\tilde{B}\prec\tilde{C}$ but the result of other existing methods is as same as the proposed method result i.e. $\tilde{A}\prec\tilde{C}\prec\tilde{B}$.

Set 4 : $C(\tilde{A})=0.4<C(\tilde{B})=0.5<C(\tilde{C})=0.6$, hence by proposed method the ranking order is $\tilde{A}\prec\tilde{B}\prec\tilde{C}$ but different methods yielding different results.

Example 4.3. Consider the two sets shown in Figs. 11 – 12 as follows (see Nasseri et al. 2013):

Set 1 $\tilde{A}=(6, 6, 1, 1)_{LR}$ $\tilde{B}=(6, 6, 0.1, 1)_{LR}$ $\tilde{C}=(6, 6, 0, 1)_{LR}$

Set 2 $\tilde{A}=(-0.3, -0.3, 0.2, 0.1)_{LR}$ $\tilde{B}=(-0.32, -0.32, 0.26, 0.35)_{LR}$ $\tilde{C}=(-0.4, -0.4, 0.3, 0.15)_{LR}$

The ranking result of Chen's method (1985), Cheng's method (1998), Chu and Tsao's method (2002), Nasseri's method (2010), Nasseri et al. method (2013) and proposed method is given in Table 3.

Table 2 Comparative results of Example 4.2.

Method	Set 1	Set 2	Set 3	Set 4
Yager's method (1981)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Choobineh and li's method (1985)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chen's method (1985)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Cheng CV method (1998)	$\tilde{B} \prec \tilde{C} \prec \tilde{A}$	$\tilde{C} \prec \tilde{B} \prec \tilde{A}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{B} \prec \tilde{C} \prec \tilde{A}$
Cheng distance method (1998)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$
Yao and Wu's method (2000)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \sim \tilde{C}$
Chu and Tsao's method (2002)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$
Abbasbandy and Asady's method (2006)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \sim \tilde{C}$
Nejad and Mashinchi's method (2011)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$
Nasseri et al.'s method (2013)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Proposed method	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$

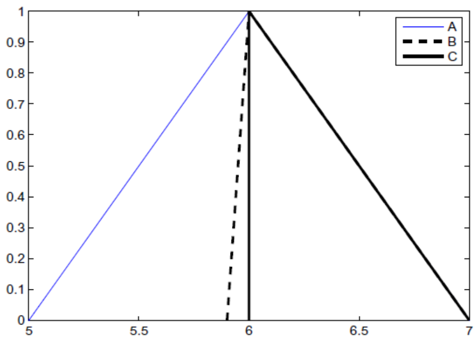

Fig. 11. Set 1 of Example 4.3.

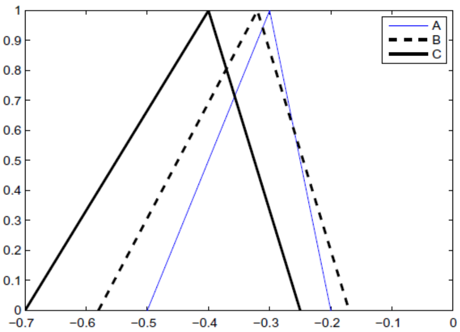

Fig. 12. Set 2 of Example 4.3.

From Table 3, we can conclude that

Set 1: $C(\tilde{A}) = C(\tilde{B}) = C(\tilde{C}) = 6$, $W'(\tilde{A}) = W'(\tilde{C}) = W'(\tilde{B}) = 0$, $C^*(\tilde{B} \geq \tilde{A}) \geq 0$ and $C^*(\tilde{C} \geq \tilde{B}) \geq 0$. Therefore the ranking order is $\tilde{A} \prec \tilde{B} \prec \tilde{C}$. Proposed method, Nasseri's method, Cheng's method and Nasseri et al.'s method yields the same result where as the result of Chu and Tsao's method and Chen's method are unreasonable (see Nasseri et al. 2013).

Set 2: Using the proposed method, $C(\tilde{A}) = -0.4 < C(\tilde{B}) = -0.32 < C(\tilde{C}) = -0.3$. The result is $\tilde{A} \succ \tilde{B} \succ \tilde{C}$ which is similar to Nasseri's method, Chu and Tsao's method, Chen's method and Nasseri et al.'s method. The result of chen's method is unreasonable (see Abbasbandy & Asady 2006).

Table 3 Comparative results of Example 4.4.

Method	Set 1	Set 2
Chen's method (1985)	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B} \succ \tilde{C}$
Cheng's method (1998)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$
Chu and Tsao's method (2002)	$\tilde{A} \prec \tilde{C} \prec \tilde{B}$	$\tilde{A} \succ \tilde{B} \succ \tilde{C}$
Nasseri's method (2010)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \succ \tilde{B} \succ \tilde{C}$
Nasseri et al. method (2013)	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \succ \tilde{B} \succ \tilde{C}$
Proposed method	$\tilde{A} \prec \tilde{B} \prec \tilde{C}$	$\tilde{A} \succ \tilde{B} \succ \tilde{C}$

Example 4.4 Consider the following sets shown in Figs. 13 – 15 (see Yu et al. 2013).

Set 1 $\tilde{A}_1 = (4, 4, 2, 2)_{LR}$ $\tilde{A}_2 = (5, 5, 4, 1)_{LR}$ $\tilde{A}_3 = (5, 5, 2, 1)_{LR}$

Set 2 $\tilde{A}_1 = (2, 2, 1, 4)_{LR}$ $\tilde{A}_2 = (2.75, 2.75, 0.25, 0.25)_{LR}$ $\tilde{A}_3 = (3, 3, 1, 1)_{LR}$

Set 3 $\tilde{A}_1 = (1, 1, 0, 14)_{LR}$ $\tilde{A}_2 = (2, 7, 2, 1)_{LR}$ $\tilde{A}_3 = (4, 4, 6, 6)_{LR}$

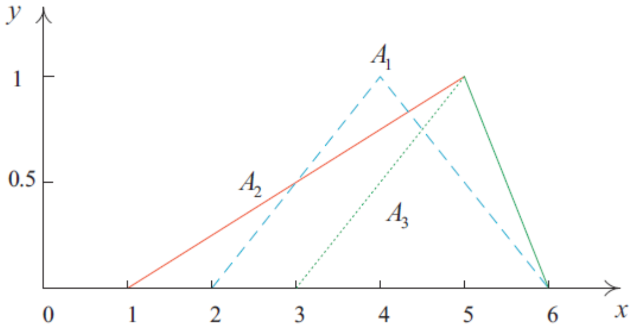

Fig. 13. Set 1 of Example 4.4.

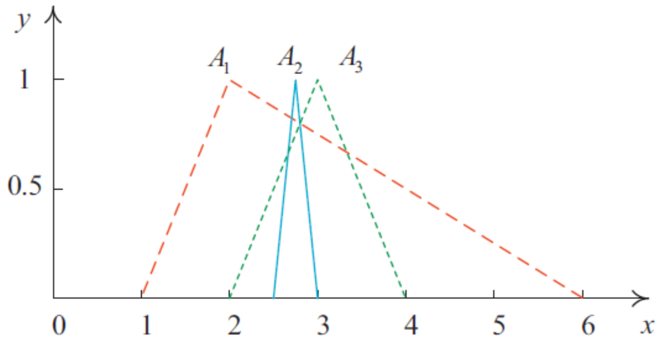

Fig. 14. Set 2 of Example 4.4.

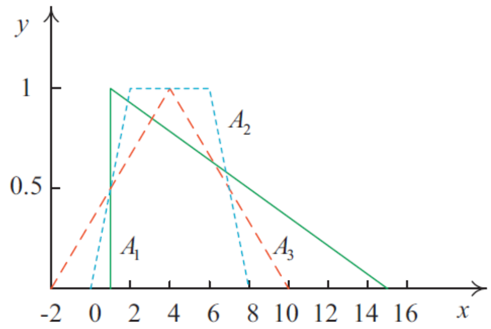

Fig. 15. Set 3 of Example 4.4.

The shortcoming of Abbasbandy & Hajari's method (2009), Wang et al.'s method (2009), Asady's method (2010) and Ezzati et al.'s (2012) methods are discussed below.

Set 1: In Wang et al.'s method whenever centroid point of fuzzy number is equal to a maximal or minimal of centroid point it gives inconsistent result. The ranking results are shown in Table 4.

Set 2: In Wang et al.'s method and Asady's method the order of their images is not logical as shown in Table 5.

Set 3: $C(\tilde{A}_1) = 1 < C(\tilde{A}_2) = 2 < C(\tilde{A}_3) = 4$. By proposed method we get $\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$. The ranking order of Abbasbandy & Hajari's method and Ezzati et al.'s method are $\tilde{A}_1 \prec \tilde{A}_2 \sim \tilde{A}_3$ and $\tilde{A}_1 \prec \tilde{A}_3 \prec \tilde{A}_2$ respectively which is counter-intuitive.

5. Conclusion

In this paper, a new lexicographical approach based on the position and area compensation of fuzzy number for ranking *L-R* Flat fuzzy numbers is proposed. The proposed method is illustrated by numerical examples and comparison is made with some of the existing methods. It is shown that our proposed method overcomes all the shortcoming of the existing methods. Application of ranking method in fuzzy risk analysis and in fuzzy optimization is left as future work.

Acknowledgments

The authors would like to express his sincerest thanks to Dr. Shashi Aggarwal, Associate Professor, Department of Mathematics, Miranda House, University of Delhi, Delhi 110007, India and Dr. Aparna Mehra, Associate Professor, Department of Mathematics, Indian Institute of Technology, Delhi, Hauz Khas, New Delhi 110016, India. Also, the author is thankful to the Council of Scientific and Industrial Research (CSIR), New Delhi, India, for the financial support (09/045(1153)/2012-EMR-I).

Table 4

Method	Set 1
Abbasbandy and Hajari's method (2009)	$\tilde{A}_1 \prec \tilde{A}_3 \prec \tilde{A}_2$
Wang et al.'s method (2009)	$\tilde{A}_1 \sim \tilde{A}_2 \prec \tilde{A}_3$
Asady's method (2010)	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$
Nejad and Mashinchi's method (2011)	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$
Asady's method (2011)	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$
Chen and Sanguansat's method (2011)	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$
Yu et al.'s method (2013)	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$
Proposed method	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$

Table 5

Method	Set 2	Image of Set 2	Result
Wang et al.'s method (2009)	$\tilde{A}_2 \prec \tilde{A}_3 \prec \tilde{A}_1$	$-\tilde{A}_3 \sim -\tilde{A}_1 \prec -\tilde{A}_2$	Inconsistent
Asady's method (2010)	$\tilde{A}_2 \prec \tilde{A}_3 \prec \tilde{A}_1$	$-\tilde{A}_3 \sim -\tilde{A}_1 \prec -\tilde{A}_2$	Inconsistent
Nejad and Mashinchi's method (2011)	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	$-\tilde{A}_3 \prec -\tilde{A}_1 \prec -\tilde{A}_2$	Inconsistent
Asady's method (2011)	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	$-\tilde{A}_3 \prec -\tilde{A}_2 \sim -\tilde{A}_1$	Consistent
Yu et al.'s method (2013)	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	$-\tilde{A}_3 \prec -\tilde{A}_2 \prec -\tilde{A}_1$	Consistent
Proposed method	$\tilde{A}_1 \prec \tilde{A}_2 \prec \tilde{A}_3$	$-\tilde{A}_3 \prec -\tilde{A}_2 \prec -\tilde{A}_1$	Consistent

References

Abbasbandy, S. & Asady, B. (2006), "Ranking of fuzzy numbers by sign distance", *Information Sciences* 176, 2405-2416.

Abbasbandy, S. & Hajjari, T. (2009), "A new approach for ranking trapezoidal fuzzy numbers", *Computer and Mathematics with Application* 57, 413-419.

Asady B. (2010), "The revised method of ranking LR fuzzy number based on deviation degree", *Expert Systems with Applications* 37, 5056-5060.

Asady B. (2011), "Revision of distance minimization method for ranking of fuzzy numbers", *Applied Mathematical Modelling* 35, 1306-1313.

Choobineh F. & Li. H. (1985), "An index for ordering fuzzy number", *Fuzzy Sets and Systems* 5, 1-19.

Chen, S. (1985), "Ranking fuzzy numbers with maximizing set and minimizing set", *Fuzzy sets and Systems* 17(2), 113-129.

Cheng C.H. (1998), "A new approach for ranking fuzzy numbers by distance method", *Fuzzy Sets and Systems* 95, 307-317.

Chu, T.C. & Tsao, C.T. (2002), "Ranking fuzzy numbers with an area between the centroid point and the original point", *Computers and Mathematics with Applications* 43, 111-117.

Chen, S. M. & Chen, J. H. (2009), "Fuzzy risk analysis based on ranking generalized fuzzy numbers with different heights and different spreads", *Expect Systems with Applications* 36, (3) 6833-6842.

Chen, S. M. & Sanguansat, K. (2011), "Analyzing fuzzy risk analysis based on a new fuzzy ranking method between generalized fuzzy numbers", *Expect Systems with Applications* 38, 2163-2171.

Deng, Y., Zhenfu, Z. & Qi, L. (2006), "Ranking fuzzy numbers with an area method using radius of gyration", *Computers and Mathematics with Applications* 51, 1127-1136.

Ezzati, R., Allahviranloo, T., Khezerloo, S. & Khezerloo, M. (2002), "An approach for ranking of fuzzy numbers", *Expect Systems with Applications* 39, 690-695.

Fortemps, P. & Roubens, M. (1996), "Ranking and defuzzification methods based on area compensation", *Fuzzy Sets and Systems* 82, 319-330.

Farhadinia B. (2009), "Ranking fuzzy numbers based on lexicographical ordering", *International Journal of Mathematical and Computer Sciences* 5(4) 220-223.

Jain, R. (1976), "Decision making in the presence of fuzzy variables", *IEEE Trans. System Man. Cybern*. 6, 698-703.

Lee E. S. & Li R. J. (1988), "Comparison of fuzzy numbers based on the probability measure of fuzzy events", *Computers and Mathematics with Application*s 15, 887-896.

Lious, T. S. & Wang, M. J. (1992), "Ranking fuzzy numbers with integral value", *Fuzzy Sets and Systems* 50, 247-255.

Murakami, S., Maeda, S. & Imamura, S. (1983), "Fuzzy decision analysis on the development of centralized regional energy control system", *Proceeding of the IFAC Symposium on Fuzzy Information, Knowledge Representation and Decision Analysis*, 363-368.

Nasseri, S. H. (2010), "Ranking trapezoidal fuzzy numbers by using Hadi method", *Austrian Journal of Basic Application and Science* 4(10), 4630-4637.

Nejad A. M. & Mashinchi M. (2011), "Ranking fuzzy numbers based on the area on the left and right sides of fuzzy number", *Computer Mathematics and Application* 61(2), 431-442.

Naseri, S.H., Zadeh, M.M., Kardoost, M. & Behmanesh, E. (2013), "Ranking fuzzy quantities based on the angle of the reference functions", *Applied Mathematical Modeling* 37, 9230-9241.

Roubens, M. (1990), Inequality constraints between fuzzy numbers and their use in mathematical programming, *in: R. Slowinski and J. Teghem, Eds., Stochastic versus fuzzy approach to multi-objective mathematical programming under uncertainly*, 321-330.

Requena, I, Delgado, M. and Verdegay, J. I. (1995), "Automatic ranking of fuzzy numbers with the criterion of decision-making learnt by an artificial neural network", *Fuzzy Sets and Systems* 73, 185-199.

Shureshjani, R. A. & Hmiraki, M. Dare, (2013), "A new parametric method for ranking fuzzy numbers", *Indagationes Mathematicae* 24, 518-529.

Wang, X. & Kere, E. E. (2001), "Reasonable properties for the ordering of fuzzy quantities (1)", *Fuzzy Sets and Systems* 118, 375 - 385.

Wang, Z. X., Liu, Y. J., Fan, Z.P. & Feng, B. (2009), "Ranking L-R fuzzy number based on deviation degree", *Expect Systems with Applications* 179, 2070-2077.

Yager, R.R. (1978), "Ranking fuzzy subsets over the unit interval", *Proceeding of the 17th IEEE International Conference on Decision and Control*, San Diego, CA, 1435-1437.

Yager, R.R. (1980), "On choosing between fuzzy subsets", *Kybernetcs* 9, 151-154.

Yager, R. R. (1981), "A procedure for ordering fuzzy subsets of the unit interval", *Information Science* 24, 143-161.

Yao, J.S. & Wu, K. (2000), "Ranking fuzzy numbers based on the decomposition principal and signed distance", *Fuzzy Sets and Systems* 116, 275-288.

Yu, V. F., Chi, H. T. X. & Shen, C-W. (2013), "Ranking fuzzy numbers based on epsilon-deviation degree", *Applied Soft Computing* 13, 3621-3627.