

The demand modeling for pure endowment life insurance in definite and stochastic modes

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Abstract:
Pure Endowment life insurance is a type of life insurance in which the insurer makes a commitment to pay a sum of money to the insured in case he is alive in a predetermined date. It is distinctively designed for the people whose consumption in elderly years of their lives is more important than leaving a legacy for their heirs. In this theoretical paper, firstly, expected utility functions are defined and a wealth accumulation process constraint in deterministic as well as stochastic modes is implemented. Consequently, the utility functions have been optimized using definite optimal control techniques and ITO stochastic calculus. Our results exhibit that in definite mode interest rate affects demand for insurance positively, while factors like time preferences rate, degree of risk aversion, premium, propensity to consumption have a negative impact. However, although these results are similar in stochastic mode (that is, when the costumer has a risky asset to invest in), in this new setting the average returns of risky asset contributes positively to insurance demand.

Keywords: Life insurance, pure endowment life insurance, risk, Ito stochastic calculus, CRRA utility functions

JEL Classifications: C_{61} ،D_{14} ،D_{81} ،D_{91}

1- Introduction:

Pure endowment insurance is a contract between the insurance company and an insured person under which a specified amount of money is paid if the insured person survives by the end of the contract. As the costumer can devote his savings to a risk-free asset (definite mode) as well as a risky asset (stochastic mode), he is facing a decision problem through which he should find an optimal pattern of current consumption and the life insurance which will be paid when he is retired. Having said that, and imposing the assumption of no legacy in the end of lifetime, we have obtained the optimal pattern of insurance demand. We will firstly review the current literature on the issue and then derive the optimal pattern of demand for life insurance in definite and stochastic modes. Finally, the results of this paper will be discussed and summarized.

2- Review of literature:

The modern literature about the pure endowment life insurance is mostly referred to pioneering paper of Yaari (1965). He has discussed the demand function of insurance under the title of expected utility. He assumes that people maximize the $\int_0^T U(C_{\ t})\,dt$ Integral in which T is the finite lifetime as a random variable. In his paper it is shown as the integral bellow:

$$\int_0^T F_t\, U(C_t)dt$$

In this Integral, F is the probability of the representative agent being alive at time T. This simple approach has been a guideline for the analysis with stochastic lifetime.

Fischer (1973) and Louis (1989) and Iwaki and Komoribayashi(2004) have also obtained the demand function using different approaches.

Fischer (1973) derived the consumption and investment patterns under risky conditions within discrete time dynamic programming model. Louis (1989) within a discrete time model and using local analysis (Taylor Series), derives the expected utility function. Moreover, he defines the time preference rate as an exogenous variable.

Iwaki and Komoribayashi(2004) using Martingle’s method derived the optimal demand for insurance. Households in this method however, are permitted to purchase life insurance in time (t) and cannot alter their demand thereafter.

Merton (1969) introduced the prominent continuous model of consumption and investment choice, Richard (1975) combined the life insurance literature with Merton's model. His basic assumption is that the individual has a fixed planning horizon. On the other hand, Pliska and Ye (2006) expanded Richard's model and relaxed his assumption to let individual plan even after he is retired and hence, they presume the planning horizon as a random variable. They consequently have taken advantage of dynamic programming.

3. Optimal pattern of life insurance in definite mode:

Here the individual maximizes his lifetime expected utility subject to wealth accumulation process constraint. This lifetime utility is a function of his consumption (c_t) and the money he earns through the life insurance payment (I_t).

Therefore, the utility function is comprised of two sections; the first one refers to the expected utility earned through consumption and the insurance payment which is paid in his retirement. The second is referred to consumption in his active years. This objective function can be shown as:

$$Max \int_0^T [\varphi\, U(C_t, I_t) + (1-\varphi)B(C_t)]^{-\rho t} dt \qquad (1)$$

In which T is the lifetime of the individual, P time preference rate and U and B are utility for first and second parts respectively. (I_t) is the payment of life insurance which is interpreted as present value using the term $e^{-\rho t}$. It is assumed that no legacy is left. Now if we assume that the insured person is alive in the due date of contract with the probability of φ, the wealth accumulation process constraint can be written as follows:

$$W_t = W_0 + \int_0^T Y_t\, dt + \int_0^T r\, W_t\, dt - \int_0^T C_t\, dt - \int_0^T P_t dt \qquad (2)$$

In which r is the return of the risk-free asset, Y income, P is the premium paid to insurance company. If we differentiate equation (2) with respect to time and divide it by d_t, equation (3) is obtained.

$$W_t = Y_t + rW_t - C_t - \varphi(1+l)I_t \qquad (3)$$

Now we define the P_t as follows:

$$P_t = \varphi(1+l)I_t \qquad (4)$$

L is the excess cost of the insurance company which is a portion of total premium and is used to cover costs other than those related to compensation of insurers.

We should now maximize the objective function (1) subject to the constraint (3) described above. Hence, using Hamiltonian method we will have:

$$H = [\varphi U(C_t, I_t) + (1-\varphi)B(C_t)]e^{-\rho t} + \lambda_t[Y_t + rW_t - C_t - \varphi(1+l)I_t] \quad (5)$$

First order conditions of optimization are:

$$H = [\varphi U(C_t, I_t) + (1-\varphi)B(C_t)]e^{-\rho t} + \lambda_t[Y_t + rW_t - C_t - \varphi(1+l)I_t] \quad (5)$$

A constant relative risk aversion function is considered for the utility obtained through I_t. Risk aversion degree is equal to α which is set 0< α < 1 to show the individual is risk averse. To address the consumption in the utility function, we have simply put C^β in which β is Propensity to consumption and as 0< β <1 marginal utility will be positive and diminishing.

Placing the above relations into explicit optimal function, we obtain the pure endowment life insurance demand.

$$\dot{I} = -\frac{U'_i}{U''_{ii}}(r-\rho) \qquad \Rightarrow \qquad \frac{\dot{I}}{\mathrm{I}} = \frac{\mathrm{I}}{\alpha}(\mathrm{r}-\rho) \qquad (15)$$

Putting equations (7) and (9) as equal and replacing derivatives of utility function (14) we get (16):

$$I_t = \left[\frac{1}{(1+l)\beta}.C^{1-\beta}\right]^{\frac{1}{\alpha}} \Rightarrow I_0 = \left[\frac{1}{(1+l)^{\beta}}.C_0^{1-\beta}\right]^{\frac{1}{\alpha}} \qquad (16)$$

Solving the differential equation (15) we attain equation (17).

$$I_t = \mathrm{e}^{\frac{1}{\alpha}(r-p)t} + I_0 \qquad (17)$$

Now by putting (16) into (17), we get explicit function for optimized pattern of pure endowment life insurance demand.

$$I_t = e^{\frac{1}{\alpha}(r-p)t} + \left[\frac{1}{(1+l)\beta}C_0^{1-\beta}\right]^{\frac{1}{\alpha}} \qquad (18)$$

In order to evaluate the effects of different factor on demand, we take derivatives with respect to them:

$$\frac{\partial I_t}{\partial r} = \frac{1}{\alpha}te^{\frac{1}{\alpha}(r-\rho)t}$$

Since "ALPHA AND T>0 and thus $\frac{\partial I_t}{\partial r} > 0$, and thus, interest rate affects demand function positively.

$$\frac{\partial I_t}{\partial \rho} = -\frac{1}{\alpha}te^{\frac{1}{\alpha}(r-\rho)t}$$

As expected, time preference rate contributes to demand function negatively.

$$\frac{\partial I_t}{\partial \alpha} = \frac{-(r-\rho)t}{\alpha^2} \; e^{\frac{1}{\alpha}(r-\rho)t} - \frac{1}{\alpha^2} Ln[\frac{1}{(1+l)\beta} C_0^{\,1-\beta}] \left[\frac{1}{(1+l)\beta} C_0^{\,1-\beta}\right]^{\frac{1}{\alpha}}$$

Nonetheless, it is to some extent complicated to analyze the impact of risk aversion on demand function. If $r > \rho$ risk aversion contributes negatively and since C_0 is usually a big number $Ln[\frac{1}{(1+l)\beta} C_0^{\,1-\beta}]$ is usually is bigger than 1. Now if $r < \rho$ It is impossible to determine sign for the term $\frac{\partial I_t}{\partial \alpha}$. That is, if time preference rate is bigger than interest rate, one cannot definitely gauge the effects of the degree of risk aversion on demand function of pure endowment life insurance.

$$\frac{\partial I_t}{\partial l} = \frac{-1}{\alpha\beta(1+l)^2} C_0^{\,1-\beta} \quad \left[\frac{1}{(1+l)\beta} C_0^{\,1-\beta}\right]^{\frac{1}{\alpha}-1}$$

The excess cost of insurance company diminishes demand for life insurance too.

$$\frac{\partial I_t}{\partial \beta} = \frac{1}{\alpha}\left[\frac{1}{(1+l)\beta^2} C_0^{\,1-\beta} - lnC_0 . C_0^{\,1-\beta}\right]\left[\frac{1}{(1+l)\beta} C_0^{\,1-\beta}\right]^{\frac{1}{\alpha}}$$

As mentioned above, Ln C_0 is usually a big number and hence $\frac{1}{(1+l)\beta^2} C_0^{\,1-\beta} - lnC_0 . C_0^{\,1-\beta} < 0$ and it implies that the more propensity to current consumption increases (i.e. consumption taste changes) the less demand for life insurance is.

4. Optimal pattern of life insurance in stochastic mode:

Let's assume that there are two assets; one is riskless and with a return rate of r and the other is a risky asset with expected return b and standard deviation σ. S_1 and S_2 are prices of risky and riskless asset respectively. Also we define the share of investment on riskless asset as W and the share of risky asset as (1-W). The price of risky asset follows a geometric Brownian motion which gives:

$$\frac{ds_{2t}}{s_{2t}} = rdt \tag{19}$$

$$\frac{ds_{1t}}{s_{1t}} = b\,dt + \sigma dz_t \tag{20}$$

In which dz_t represents the variations in stochastic term which follows a VINER process with zero mean and unit variance. The wealth of investor in time t considering the assumption that he is alive will be:

$$W_t = W_0 \int_0^t y_t dt - \int_0^t C_t\, dt - \int_0^t \phi(1+l) I_t dt + \int_0^t \frac{wW_t}{S_1} dS_1 + \int_0^t \frac{(1-w)W_t}{S_2} ds_2 \tag{21}$$

Differentiating the (20) gives us:

$$dW_t = [y_t - c_t - \phi(1+l)I_t + rW_t + (b-r)wW_t]\,dt + w\sigma wW_t dz_t \tag{22}$$

$$\Pi_t = \int_0^T [\varphi\, U(C_t, I_t) + (1-\varphi)B(C_t)]\, e^{-\rho t} dt \tag{23}$$

What we are going to do here is to optimize the objective function (23) with respect to (22). What one needs to consider here is the fact that wealth accumulation process is comprised of a deterministic as well as a stochastic part. Thus, we use Ito stochastic calculus:

We assume that J=J(W,t) Represents optimal value function which is differentiable of order two. Based on the Ito lemma, we will have:

$$dJ = J_w dW_t + J_t dt + \frac{1}{2} J_{ww} dW_t^{\,2} + \frac{1}{2} J_{tt} dt^2 + J_{wt} dW_t dt \tag{24}$$

Now replacing dw_t from (22) into (24) and take expected derivative and since we have $E[dW_t]^2 = \sigma^2 w^2 W_t^{\,2} dt$ it results in:

$$\frac{1}{dt} \mathrm{EdJ_w} = [Y_t + rW_t - C_t - \varphi(1+l)I_t + (b-r)wW_t]\mathrm{J_{ww}} + \mathrm{J_{wt}} + \frac{1}{2}\mathrm{J_{www}} \sigma^2 w^2 W_t^{\,2} \tag{25}$$

Now we form the Stochastic Bellman Equation

$$max\Pi_t + [y_t - C_t - \phi(1+l)I_t + rW_t + (b-r)wW_t]J_w + J_t + \frac{1}{2}\,\sigma^2 w^2 W_t J_{ww} \tag{26}$$

Partial derivatives of the first order condition are as described in following equations:

$$\frac{\partial \Pi_t}{\partial C_t} - J_w = 0\ , \qquad \frac{\partial \Pi_t}{\partial C_t} = \mathrm{U'}_c . e^{-\rho t} \tag{27}$$

$$\frac{\partial \Pi_t}{\partial I_t} - \varphi(1+l)J_w = 0\ , \qquad \frac{\partial \Pi_t}{\partial I_t} = \mathrm{U'}_i . e^{-\rho t} \tag{28}$$

$$\frac{\partial \Pi_t}{\partial W_t} + [r + (\alpha - r)w]J_w + [y_t - C_t - \phi(1+l)I_t + rW_t + (b-r)wW_t]J_{ww} + J_{tw} + J_{ww}\sigma^2 w^2 W_t + \frac{1}{2} J_{www}\sigma^2 w^2 W_t^{\,2} = 0 \tag{29}$$

Now replacing (25) into (29) we have:

$$\frac{\partial \Pi_t}{\partial W_{\ t}} + [r-(b-r)w]J_w + \mathrm{J_{ww}}\sigma^2 w^2 W_t = -\frac{1}{dt}\mathrm{EdJ_w} \qquad (30)$$

Taking stochastic expectation and time derivative from (28), we have:

$$-\frac{1}{dt}\mathrm{Ed}\left(\frac{1}{\varphi(1+l)}\frac{\partial \Pi_t}{\partial I_{\ t}}\right) = -\frac{1}{dt}\mathrm{EdJ_w} \qquad (31)$$

Equating (30) and (31) and using (27) and (28) and considering that .... we will have:

$$[r+(b-r)w]J_{\ w} + J_{\ ww}\,W^{\ 2}\sigma^2 W_{\ t} = -\frac{1}{dt}\mathrm{E}d\left(\frac{1}{1+l}U'_{\ i}.e^{\ -\rho t}\right) \qquad (32)$$

Having done arithmetic calculations on the right hand side of equation (32) and rewriting it, we have:

$$-\frac{1}{dt}\frac{1}{(1+l)}.\mathrm{E}\big(U''_{\ i\ i}.\dot{\mathrm{I}}.\mathrm{dt.e}^{\ -\rho t} - \rho \mathrm{e}^{\ -\rho t}.U'_{\ i}.\mathrm{dt}\big) = \frac{-e^{\ -\rho t}}{1+l}E\big(U''_{\ i\ i}.\dot{I} - \rho.U'_{\ i}\big)$$

$$\Rightarrow [r+(\propto - r)W]J_{\ w} + J_{\ ww}\,W^{\ 2}\delta^{\ 2}W_{\ t} = \frac{-e^{\ -\rho t}}{1+l}.E\big(U''_{\ i\ i}.\dot{I} - \rho.U'_{\ i}\big)$$

$$J_{\ ww} = 0 \qquad \Rightarrow \qquad \frac{(1-w)r+wb}{1+l}.U'_{\ i}.e^{\ -\rho t} = \frac{-e^{\ -\rho t}}{1+l}.E(U''_{\ i\ i}.\dot{I} - \rho.U'_{\ i})$$

After a simple series of arithmetic calculations, we will get the explicit demand function for pure endowment life insurance in stochastic mode:

$$[(1-w)r+wb]U'_{\ i} = -U''_{\ i\ i}.\dot{I} + \rho.U'_{\ i} \quad \Rightarrow \dot{I} = -\frac{U'_{\ i}}{U''_{\ i\ i}}[(1-w)r+wb-\rho]$$

$$\dot{I}/_{I} = 1/_{\propto}[(1-w)r+wb-\rho]$$

$$\Rightarrow \quad I_{\ t} = e^{\ 1/_{\propto}[(1-w)r+wb-\rho]\mathrm{t}} + [\frac{1}{(1+l)\beta}.C_{\ 0}^{\ 1-\beta}]^{\ 1/_{\propto}}$$

The demand function for life insurance in stochastic mode is similar to the one in definite mode with this difference that in the former we have $(1-w)r+wb$ which is weighted average of both risky and risk-free asset returns instead of r. Since 0<w<1, all factors are of the same contribution to the demand function as the definite mode. The only difference is “b” which is average return of risky asset which affects demand function positively beside all other factors.

$$\frac{\partial I_{\ t}}{\partial b} = \frac{1}{\alpha}w.t.e^{\ 1/_{\propto}[(1-w)r+wb-\rho]\mathrm{t}}$$

It is worth noting that as is evident in the final equation of the demand function, standard deviation of the risky asset return doesn’t affect life insurance demand.

5- Conclusion:

Having controlled for definite as well as stochastic modes, this paper models demand function for pure endowment life insurance using dynamic optimization methods. The numerical results of ours corroborate that factors such as interest rate, time preference rate, risk aversion, excess cost of insurance company, and propensity to consumption impact the demand function. It has also been elucidated that these effects are similar in both definite and stochastic modes. As regards the direction of these factors, we have exhibited that interest rate is the only factor which has a positive effect on demand function, whereas all other elements contribute negatively to the demand function.

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