

Coupled Fixed Point Results In G-Metric Spaces For W*-Compatible Mappings

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Abstract: In this paper, we consider a new class of pairs of generalized contractive type mappings defined in $G-$metric spaces. Some coincidence and common fixed point results for these mapping are presented.
Keywords: Coincidence Point, Coupled Fixed Point, Common Coupled Fixed Point, Common Fixed Point, Generalized Metric Space, w^*-Compatible Mappings.

1. Introduction and Preliminaries

Mustafa and Sims [5] introduced the notion of complete $G-$metric spaces as a generalization of complete metric spaces. For details on $G-$metric spaces, we refer to [5, 6, 7, 8]. The notion of a coupled fixed point in partially ordered metric spaces has been introduced by Bhaskar and Lakshmikantham in (2006)[9]. In this paper ,we prove a common coupled fixed point theorem for two mappings in $G-$metric spaces.

Definition 1.1 *[5] Let* X *be a nonempty set and* $G: X\times X\times X \to \mathbb{R}^+$ *a function satisfying the following properties:*

(G_1) $G(x,y,z)=0$ if $x=y=z=0,$

(G_2) $0<G(x,x,y),$ for all $x,y\in X$ with $x\neq y,$

(G_3) $G(x,x,y)\leq G(x,y,z)$ for all $x,y,z\in X$ with $z\neq y,$

(G_4) $G(x,y,z)=G(x,z,y)=G(y,z,x)=...,$ symmetry in all three variables,

(G_5) $G(x,y,z)\leq G(x,a,a)+G(a,y,z)$ for all $x,y,z,a\in X$

Then the function G is called a generalized metric, or, more specifically, a $G-$metric on $X,$ and the pair (X,G) is called a $G-$metric space.

Definition 1.2 *[5] Let* (X,G) *be a* $G-$*metric space and* (x_n) *a sequence of points of* X. *A point* $x\in X$ *is said to be the limit of the sequence* $(x_n),$ *if* $\lim_{m,n\to\infty} G(x,x_n,x_m)=0,$ *and we say that the sequence* (x_n) *is* $G-$*convergent to* x *or that* (x_n) $G-$*converges to* x.

Thus, $x_n\to x$ in a $G-$metric space (X,G) if for any $\varepsilon>0,$ there exists $k\in\mathbb{N}$ such that

$G(x, x_n, x_m) < \varepsilon$ for all $m, n \geq k.$

Proposition 1.1. [5] Let (X, G) be a $G-$metric space. Then the following are equivalent:

(1) (x_n) is $G-$convergent to $x.$

(2) $G(x_n, x_n, x) \to 0$ as $n \to \infty.$

(3) $G(x_n, x, x) \to 0$ as $n \to \infty.$

(4) $G(x_n, x_m, x) \to 0$ as $n, m \to \infty.$

Definition 1.3 *[5] Let* (X, G) *be a* $G-$*metric space, a sequence* (x_n) *is called* $G-$*Cauchy if for every* $\varepsilon > 0,$ *there is* $k \in \mathbb{N}$ *such that* $G(x_n, x_m, x_l) < \varepsilon,$ *for all* $n, m, l \geq k,$ *that is* $G(x_n, x_m, x_l) \to 0$ *as* $n, m, l \to \infty.$

Proposition 1.2. [5] Let (X, G) be a $G-$metric space, then the following statements are equivalent:

(1) The sequence (x_n) is $G-$Cauchy.

(2) For every $\varepsilon > \circ$ there is $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon,$ for all $n, m \geq k.$

Definition 1.4 *[5] A* $G-$*metric space* (X, G) *is called* $G-$*complete if every* $G-$*Cauchy sequence in* (X, G) *is* $G-$*convergent in* (X, G).

Proposition 1.3. [5] Let (X, G) be a $G-$metric space. Then, the function $G(x, y, z)$ is jointly continuous in all three of its variables.

Example 1.1. [5] Let $(\mathbb{R}, d)$ be the usual metric space. Define G_s by

$$G_s(x, y, z) = d(x, y) + d(y, z) + d(x, z)$$

for all $x, y, z \in \mathbb{R}$. Then it is clear that $(\mathbb{R}, G_s)$ is a $G-$metric space.

Proposition 1.4. [5] Let (X, G) be a $G-$metric space. Then $T : X \to X$ is $G-$continuous at $x \in X$ if and only if it is $G-$sequentially continuous at x ,that is, whenever (x_n) is $G-$convergent to $x,$ $(f(x_n))$ is $G-$convergent to $f(x).$

Definition 1.5 *[4] Let* (X, G) *be a* $G-$*metric space. A mapping* $F : X \times X \to X$ *is said to be* *continuous if for any two $G-$convergent sequences (x_n) and (y_n) converging to x and y respectively, $(F(x_n, y_n))$ is $G-$convergent to $F(x,y)$.*

Definition 1.6 *[3] An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F: X\times X\to X$ if $F(x,y)=x, F(y,x)=y.$*

Definition 1.7 *[9] An element $(x,y)\in X\times X$ is called a coupled coincidence point of a mapping $F: X\times X\to X$ and a mapping $g: X\to X$ if $F(x,y)=gx, F(y,x)=gy.$*

Note that if g is the identity mapping, then Definition 1.7 reduces to Definition 1.6.

Definition 1.8 *[1] An element $x\in X$ is called a common fixed point of a mapping $F: X\times X\to X$ and $g: X\to X$ if $F(x,x)=gx=x.$*

Abbas et al. [1] introduced the concept of $w-$compatible and w^*-compatible mappings and utilized this concept to prove an interesting uniqueness theorem of a coupled fixed point for mappings F and g in cone metric spaces.

Definition 1.9 *[1] Mappings $F: X\times X\to X$ and $g: X\to X$ are called*

(w_1) $w-$compatible if $g(F(x,y))=F(gx,gy)$ whenever $gx=F(x,y)$ and $gy=F(y,x)$.

(w_2) w^*-compatible if $g(F(x,x))=F(gx,gx)$ whenever $gx=F(x,x)$.

Example 1.2. [2] Let $X=\mathbb{R}^+$, define $F: X\times X\to X$ and $g: X\to X$ by

$$F(x,y)=\begin{cases} 8, & x=1, y=0,\\ 10, & x=0, y=1,\\ 4 & \text{other wise}, \end{cases} \quad \text{and} \quad g(x)=\begin{cases} 8, & x=1,\\ 10, & x=0,\\ 5, & x=4,\\ 4, & \text{other wise}. \end{cases}$$

Then it is clear that F and g are $w-$compatible but not w^*-compatible.

Definition 1.10 *[9] Let X be a nonempty set and $F: X\times X\to X$ and $g: X\to X$. One says F and g are commutative if for all $x,y\in X$, $F(gx,gy)=g(F(x,y)).$*

2. Main results

Our first result is the following.

Theorem 2.1 *Let (X,G) be a $G-$metric space. Set $T: X\times X\to X$ and $g: X\to X$. Assume* *there exist* $a_1, a_2, a_3 \geq 0$ *with* $2a_1 + 3a_2 + 3a_3 < 2$ *such that*

$$G(T(x,y),T(u,v),T(w,z)) \leq \frac{a_1}{2}[G(gx,gu,gw)+G(gy,gv,gz)] \tag{2.1}$$

$$+\frac{a_2}{2}[G(gx,T(x,y),T(x,y))+G(gu,T(u,v),T(u,v))+G(gy,gv,gz)]$$

$$+\frac{a_3}{2}[G(gx,T(u,v),T(u,v))+G(gu,T(x,y),T(x,y))+G(gy,gv,gz)],$$

for all $x,y,u,v,w,z \in X$. If $T(X \times X) \subseteq g(X)$, $g(X)$ is a $G-$complete subset of X, then T and g have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with g, then T and g have a unique common coupled fixed point.

Proof. Let x_0 and y_0 be in X. Since $T(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = T(x_0, y_0)$ and $gy_1 = T(y_0, x_0)$. Analogously, there exist $x_2, y_2 \in X$ such that $gx_2 = T(x_1, y_1)$ and $gy_2 = T(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$gx_{n+1} = T(x_n, y_n) \text{ and } gy_{n+1} = T(y_n, x_n) \text{ for all } n \geq 0 \tag{2.2}$$

From by (2.1), we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) = G(T(x_{n-1}, y_{n-1}), T(x_n, y_n), T(x_n, y_n))$$

$$\leq \frac{a_1}{2}[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)]$$

$$+\frac{a_2}{2}[G(gx_{n-1}, T(x_{n-1}, y_{n-1}), T(x_{n-1}, y_{n-1})) + G(gx_n, T(x_n, y_n), T(x_n, y_n)) +$$

$$G(gy_{n-1}, gy_n, gy_n)] + \frac{a_3}{2}[G(gx_{n-1}, T(x_n, y_n), T(x_n, y_n))$$

$$+G(gx_n, T(x_{n-1}, y_{n-1}), T(x_{n-1}, y_{n-1})) + G(gy_{n-1}, gy_n, gy_n)]$$

$$= \frac{a_1}{2}[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)]$$

$$+\frac{a_2}{2}[G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_{n-1}, gy_n, gy_n)]$$

$$+\frac{a_3}{2}[G(gx_{n-1}, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)].$$

Thus, we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \frac{a_1 + a_2 + a_3}{2 - a_2 - a_3}[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})], \tag{2.3}$$

and

$$\begin{aligned}G(gy_n, gy_{n+1}, gy_{n+1}) &= G(T(y_{n-1}, x_{n-1}), T(y_n, x_n), T(y_n, x_n)) \\ &\leq \frac{a_1}{2}[G(gy_{n-1}, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)] \\ &+ \frac{a_2}{2}[G(gy_{n-1}, T(y_{n-1}, x_{n-1}), T(y_{n-1}, x_{n-1})) + G(gy_n, T(y_n, x_n), T(y_n, x_n)) \\ &+ G(gx_{n-1}, gx_n, gx_n)] + \frac{a_3}{2}[G(gy_{n-1}, T(y_n, x_n), T(y_n, x_n)) \\ &+ G(gy_n, T(y_{n-1}, x_{n-1}), T(y_{n-1}, x_{n-1})) + G(gx_{n-1}, gx_n, gx_n)] \\ &= \frac{a_1}{2}[G(gy_{n-1}, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)] \\ &+ \frac{a_2}{2}[G(gy_{n-1}, gy_n, gy_n) + G(gy_n, gy_{n+1}, gy_{n+1}) + G(gx_{n-1}, gx_n, gx_n)] \\ &+ \frac{a_3}{2}[G(gy_{n-1}, gy_{n+1}, gy_{n+1}) + G(gy_n, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)].\end{aligned}$$

Thus, we obtain

$$G(gy_n, gy_{n+1}, gy_{n+1}) \leq \frac{a_1 + a_2 + a_3}{2 - a_2 - a_3}[G(gy_{n-1}, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)]. \tag{2.4}$$

From (2.3) and (2.4), we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1}) \leq \frac{2(a_1 + a_2 + a_3)}{2 - a_2 - a_3}[G(gy_{n-1}, gy_n, gy_n) + G(gx_{n-1}, gx_n, gx_n)].$$

Set $a_n = G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})$ and $\lambda = \frac{2(a_1 + a_2 + a_3)}{2 - a_2 - a_3}$, then the

sequence $\{a_n\}$ is decreasing as

$$0 \leq a_n \leq \lambda a_{n-1} \leq \lambda^2 a_{n-2} \leq \ldots \leq \lambda^n a_0$$

which implies

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}[G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1})] = 0.$$

Thus,

$$\lim_{n\to\infty} G(gx_n, gx_{n+1}, gx_{n+1}) = 0, \quad and \quad \lim_{n\to\infty} G(gy_n, gy_{n+1}, gy_{n+1})] = 0. \tag{2.5}$$

Next, let us prove that $\{gx_n\}$ and $\{gy_n\}$ are $G-$Cauchy sequences. In fact, for $m > n$, we have

$$\begin{aligned}G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) &\leq G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_n, gy_{n+1}, gy_{n+1}) \\ &+ G(gx_{n+1}, gx_{n+2}, gx_{n+2}) + G(gy_{n+1}, gy_{n+2}, gy_{n+2})\end{aligned}$$

$$+\ldots+G(gx_{m-1},gx_m,gx_m)+G(gy_{m-1},gy_m,gy_m)$$

$$=a_n+a_{n+1}+\ldots+a_{m-1}$$

$$\leq \lambda^n a_0+\lambda^{n+1}a_0+\ldots+\lambda^{m-1}a_0=(\lambda^n+\lambda^{n+1}+\ldots+\lambda^{m-1})a_0$$

$$\leq \frac{\lambda^n}{1-\lambda}a_0.$$

Letting $n,m\to\infty$, we have

$$\lim_{n,m\to\infty} G(gx_n,gx_m,gx_m)+G(gy_n,gy_m,gy_m)=0.$$

This imply that $\{gx_n\}$ and $\{gy_n\}$ are $G-$Cauchy sequences in $g(X)$. By $G-$completeness of $g(X)$, there exists $gx,gy\in g(X)$ such that $\{gx_n\}$ and $\{gy_n\}$ converge to gx and gy, respectively. We claim that $g(x)=T(x,y)$ and $g(y)=T(y,x).$ Indeed, from (2.1), we have

$$G(gx_{n+1},T(x,y),T(x,y))=G(T(x_n,y_n),T(x,y),T(x,y))$$

$$\leq \frac{a_1}{2}[G(gx_n,g(x),g(x))+G(gy_n,g(y),g(y))]$$

$$+\frac{a_2}{2}[G(gx_n,T(x_n,y_n),T(x_n,x_n))+G(g(x),T(x,y),T(x,y))$$

$$+G(gy_n,g(y),g(y))]+\frac{a_3}{2}[G(gx_n,T(x,y),T(x,y))$$

$$+G(g(x),T(x_n,y_n),T(x_n,y_n)+G(gy_n,g(y),g(y)))]$$

Letting $n\to\infty$, and using the fact that G is continuous on its variables, we get that

$$G(g(x),T(x,y),T(x,y))\leq \frac{a_2+a_3}{2}G(g(x),T(x,y),T(x,y)).$$

Hence $g(x)=T(x,y).$ Similarly, we may show that $g(y)=T(y,x).$ Then, (gx,gy) is a coupled point of coincidence of mappings T and g. Now we prove that $gx=gy.$ By (2.1), we have

$$G(g(x),g(y),g(y))=G(T(x,y),T(y,x),T(y,x))$$

$$\leq \frac{a_1}{2}[G(g(x),g(y),g(y))+G(g(y),g(x),g(x))]$$

$$+\frac{a_2}{2}[G(g(x),T(x,y),T(x,y))+G(g(y),T(y,x),T(y,x))+G(g(y),g(x),g(x))]$$

$$+\frac{a_3}{2}[G(g(x),T(y,x),T(y,x))+G(g(y),T(x,y),T(x,y))+G(g(y),g(x),g(x))]$$

$$= \frac{a_1 + a_3}{2} G(g(x), g(y), g(y)) + \frac{a_1 + a_2 + 2a_3}{2} G(g(y), g(x), g(x)).$$

Similarly, we may show that

$$G(g(y), g(x), g(x)) \le \frac{a_1 + a_2 + 2a_3}{2} G(g(x), g(y), g(y)) + \frac{a_1 + a_3}{2} G(g(y), g(x), g(x)).$$

Therefore

$$G(g(x), g(y), g(y)) + G(g(y), g(x), g(x)) \le \frac{2a_1 + a_2 + 3a_3}{2} [G(g(x), g(y), g(y)) + G(g(y), g(x), g(x))]$$

$$< G(g(x), g(y), g(y)) + G(g(y), g(x), g(x)).$$

which is a contradiction. So $g(x) = g(y)$. We conclude that $T(x, y) = g(x) = g(y) = T(y, x)$. Thus, $(g(x), g(x))$ is a coupled point of coincidence of mappings T and g. Now, if there is another $x_1 \in X$ such that $(g(x_1), g(x_1))$ is a coupled point of coincidence of mappings T and g, then

$$G(g(x), g(x_1), g(x_1)) = G(T(x, x), T(x_1, x_1), T(x_1, x_1))$$

$$\le \frac{a_1}{2} [G(g(x), g(x_1), g(x_1)) + G(g(x), g(x_1), g(x_1))]$$

$$+ \frac{a_2}{2} [G(g(x), T(x, x), T(x, x)) + G(g(x), T(x_1, x_1), T(x_1, x_1)) + G(g(x), g(x_1), g(x_1))]$$

$$+ \frac{a_3}{2} [G(g(x), T(x_1, x_1), T(x_1, x_1)) + G(g(x_1), T(x, x), T(x, x)) + G(g(x), g(x_1), g(x_1))]$$

$$= (a_1 + a_2 + a_3) G(g(x), g(x_1), g(x_1)) + \frac{a_3}{2} G(g(x_1), g(x), g(x)).$$

Similarly, we may show that

$$G(g(x_1), g(x), g(x)) \le (a_1 + a_2 + a_3) G(g(x_1), g(x), g(x)) + \frac{a_3}{2} G(g(x), g(x_1), g(x_1)).$$

Therefore

$$G(g(x), g(x_1), g(x_1)) + G(g(x_1), g(x), g(x)) \le \frac{2a_1 + 2a_2 + 3a_3}{2} [G(g(x), g(x_1), g(x_1)) + G(g(x_1), g(x), g(x))].$$

It implies that $G(g(x), g(x_1), g(x_1)) = G(g(x_1), g(x), g(x)) = 0$ and so $g(x) = g(x_1)$. Hence, $(g(x), g(x))$ is a unique coupled point of coincidence of mappings T and g. Now, we show that T and g have common coupled fixed point. For this, let $u = g(x)$. Then, we have $u = g(x) = T(x, x)$. By $w^* -$ compatibility of T and g, we have

$$g(u) = g(g(x)) = g(T(x, x)) = T(g(x), g(x)) = T(u, u).$$

Then, $(g(u), g(u))$ is a coupled point of coincidence of mappings T and g. By the uniqueness of coupled point of coincidence, we have $g(x) = g(u)$. Therefore, (u,u) is the common coupled fixed point of T and g. To prove the uniqueness, let $v \in X$ with $v \neq u$ such that (v,v) is the common coupled fixed point of T and g. Then, using (2.1),

$$G(u,v,v) = G(T(u,u),T(v,v),T(v,v)) \leq \frac{a_1}{2}[G(gu,gv,gv) + G(gu,gv,gv)]$$
$$+\frac{a_2}{2}[G(gu,T(u,u),T(u,u)) + G(gv,T(v,v),T(v,v)) + G(gu,gv,gv)]$$
$$+\frac{a_3}{2}[G(gu,T(v,v),T(v,v)) + G(gv,T(u,u),T(u,u)) + G(gu,gv,gv)]$$
$$= (a_1 + \frac{a_2}{2} + a_3)G(u,v,v) + \frac{a_3}{2}G(v,u,u).$$

Similarly, we may show that

$$G(v,u,u) \leq (a_1 + \frac{a_2}{2} + a_3)G(v,u,u) + \frac{a_3}{2}G(u,v,v).$$

Hence,

$$G(u,v,v) + G(v,u,u) \leq \frac{2a_1 + a_2 + 3a_3}{2}[G(u,v,v) + G(v,u,u)].$$

Since $\frac{2a_1 + a_2 + 3a_3}{2} < 1$, so that $G(u,v,v) = G(v,u,u) = 0$ and $u = v$. Thus T and g have a unique common coupled fixed point. In Theorem 2.1, take $w = u$ and $z = v$, to obtain the following corollary.

Corollary 2.2 *Let* (X,G) *be a* $G-$*metric space. Set* $T: X \times X \to X$ *and* $g: X \to X$. *Assume there exist* $a_1, a_2, a_3 \geq 0$ *with* $2a_1 + 3a_2 + 3a_3 < 2$ *such that*

$$G(T(x,y),T(u,v),T(u,v)) \leq \frac{a_1}{2}[G(gx,gu,gu) + G(gy,gv,gv)] \tag{2.6}$$
$$+\frac{a_2}{2}[G(gx,T(x,y),T(x,y)) + G(gu,T(u,v),T(u,v)) + G(gy,gv,gv)]$$
$$+\frac{a_3}{2}[G(gx,T(u,v),T(u,v)) + G(gu,T(x,y),T(x,y)) + G(gy,gv,gv)],$$

for all $x,y,u,v,w,z \in X$. If $T(X \times X) \subseteq g(X)$, $g(X)$ is a $G-$complete subset of X, then T and g have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with g, then T and g have a unique common coupled fixed point.

Now, putting $g = I_X$ (the identity map of X) in the Theorem 2.1, we obtain

Corollary 2.3 *Let* (X,G) *be a complete* $G-$*metric space. Assume* $T: X \times X \to X$ *be a function* *satisfying (2.1)(with* $g = I_X$ *)for all* $x, y, u, v, w, z \in X$. *Then* T *has a unique fixed point.*

By choosing a_1, a_2 and a_3 suitably, one can deduce some corollaries from Theorem 2.1.

For example, if $a_1 = 2k$ and $a_2 = a_3 = 0$ in Theorem 2.1, then the following corollary is obtained which extends and generalizes the comparable results of [10].

Corollary 2.4 *Let* (X, G) *be a* $G-$*metric space. Set* $T : X \times X \to X$ *and* $g : X \to X$. *Assume there exist* $k \in [0, \frac{1}{2})$ *such that*

$$G(T(x,y),T(u,v),T(w,z)) \le k[G(gx,gu,gw)+G(gy,gv,gz)], \tag{2.7}$$

for all $x, y, u, v, w, z \in X$. If $T(X \times X) \subseteq g(X)$, $g(X)$ is a $G-$complete subset of X, then T and g have a unique common coupled coincidence point. Moreover, if T is w^*-compatible with g, then T and g have a unique common coupled fixed point.

Now, we introduce an example to support the usability of our results.

Example 2.1. Let $X = [0,1]$. Define $T : X \times X \to X$ by $T(x, y) = \frac{1}{16} x^2 y^2$ and define $g : X \to X$ by $g(x) = \frac{1}{2} x^2$.

Define a $G-$metric on X by $G(x, y, z) = |x - y| + |x - z| + |y - z|$ for all $x, y, z \in X$.

By routine calculations, the reader can easily verify that the following assumptions hold:

(1) $T(X \times X) \subseteq g(X)$;

(2) $g(X)$ is a $G-$complete subset of X;

(3) T is w^*-compatible with g.

Here, we show only that T and g are condition (2.1) in Theorem 2.1 is satisfied for all real numbers a_1, a_2, a_3 with $0 \le 2a_1 + 3a_2 + 3a_3 < 2$. Since $|xy - uv| \le |x - u| + |y - v|$ holds for all $x, y, u, v \in X$, we have

$$G(T(x,y),T(u,v),T(w,z)) = G(\frac{1}{16}x^2y^2, \frac{1}{16}u^2v^2, \frac{1}{16}w^2z^2)$$
$$= \frac{1}{16}|x^2y^2 - u^2v^2| + \frac{1}{16}|x^2y^2 - w^2z^2| + \frac{1}{16}|u^2v^2 - w^2z^2|$$

$$\leq \frac{1}{16}[|x^2-u^2|+|y^2-v^2|+|x^2-w^2|+|y^2-z^2|+|u^2-w^2|+|v^2-z^2|]$$

$$\leq \frac{1}{8}[|\frac{1}{2}x^2-\frac{1}{2}u^2|+|\frac{1}{2}x^2-\frac{1}{2}w^2|+|\frac{1}{2}u^2-\frac{1}{2}w^2|+|\frac{1}{2}y^2-\frac{1}{2}v^2|+|\frac{1}{2}y^2-\frac{1}{2}z^2|+$$

$$|\frac{1}{2}v^2-\frac{1}{2}z^2|]+\frac{1}{16}[\frac{1}{2}x^2-\frac{1}{16}x^2y^2|+|\frac{1}{2}u^2-\frac{1}{16}u^2v^2|+|\frac{1}{2}y^2-\frac{1}{2}v^2|+$$

$$|\frac{1}{2}y^2-\frac{1}{2}z^2|+|\frac{1}{2}v^2-\frac{1}{2}z^2|]+\frac{1}{16}[\frac{1}{2}x^2-\frac{1}{16}u^2v^2|+|\frac{1}{2}u^2-\frac{1}{16}x^2y^2|+$$

$$|\frac{1}{2}y^2-\frac{1}{2}v^2|+|\frac{1}{2}y^2-\frac{1}{2}z^2|+|\frac{1}{2}v^2-\frac{1}{2}z^2|]$$

$$\leq \frac{\frac{1}{4}}{2}[G(gx,gu,gw)+G(gy,gv,gz)]$$

$$+\frac{\frac{1}{8}}{2}[G(gx,T(x,y),T(x,y))+G(gu,T(u,v),T(u,v))+G(gy,gv,gz)]$$

$$+\frac{\frac{1}{8}}{2}[G(gx,T(u,v),T(u,v))+G(gu,T(x,y),T(x,y))+G(gy,gv,gz)].$$

Thus, (2.1) is satisfied with $a_1=\frac{1}{4}$ and $a_2=a_3=\frac{1}{8}$ where $2a_1+3a_2+3a_3<2.$ Hence, all the conditions of Theorem 2.1 are satisfied. Moreover, $(0,0)$ is the unique common coupled fixed point of T and g. References

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