

Application of Galerkin Weighted Residual Method to 2nd, 3rd and 4th order Sturm-Liouville Problems

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Abstract
The aim of this paper is to compute the eigenvalues for a class of linear Sturm-Liouville problems (SLE) with *Dirichlet* and *mixed* boundary conditions applying Galerkin Weighted Residual methods. We use Legendre polynomials over [0,1] as trial functions to approximate the solutions of second, third and fourth order SLE problems. We derive rigorous matrix formulations and special attention is given about how the polynomials satisfy the corresponding homogeneous form of *Dirichlet* boundary conditions of Sturm-Liouville problems. The obtained approximate eigenvalues are compared with the previous computational studies by various methods available in literature.

Keywords: Sturm-Liouville problems, eigenvalue, Legendre polynomials, Galerkin method.

1. Introduction:

Geneneralized eigenvalue problem arise in connection with many problems in mechanics, theory of vibrations and stability, optimal control, mathematical and theoretical physics, hydrodynamics, acoustics,dynamics of atmosphere and ocean, elasticity etc. In Physics they describe BVP corresponding to simple harmonic standing waves. For the solution of SLE's, some studies have been carried out .

Chawla and Shivakumar (1993) presented fourth-order finite-difference method for computing eigenvalues of fourth-order two-point boundary value problems. The differential Transform method is applied to compute eigenvalues and eigenfuctions of second order regular SLE's by (Chen and Ho, 1996). Chanane (1998, 2002, 2010) introduced a novel series representation for the boundary/characteristic function associated with fourth-order Sturm-Liouville problems using the concepts of Fliess series, iterated integrals and also Extended Sampling method. The Weighted residual collocation method using Chebyshev points are investigated for approximate eigenvalues of second order SLE's by (Ibrahim, 2005).

In the recent years numerical solution of eigenvalues and eigenfunctions of the fourth-order Sturm-Liouville problems have studied by many researchers. Different algorithms have applied to reduce the convergence rates. Jia, Song and Li (2005) approximated the eigenvalues of fourth order BVP for a class of crosswise vibration equation of beam using Galerkin method and obtained the estimation of errors using the trigonometric polynomials that satisfies all the boundary conditions directly. Attili and Lesnic (2006) used the Adomian decomposition method (ADM) to solve fourth-order eigenvalue problems.

In recent years (Al Quran and Al-Khaled, 2010) have presented a comparative study of Sinc galerkin and Differential Tranform method to solve second order SLE's. Recently, Abbasbandy and Shirzadi (2011) applied the homotopy analysis method (HAM) to numerically approximate the eigenvalues of the second and fourth order SLE problems. Ycel, and Boubaker (2012) applied differential quadrature method (DQM) and Boubaker polynomial expansion scheme (BPES) for efficient computation of the eigenvalues of fourth-order SLE problems. Finally (Gamel and Sameeh, 2012) applied Chebychev method for finding eigenvalues of fourth order nonsingular Sturm-Liouville problems compared the results to the other methods available in the literature. Very recently (Taher, Malek, and Mousuleh, 2013) applied Chebychev spectral collocation method where Chebychev differentiation matrix is defined to compute the eigenvalues of SLE's.

Orthogonal polynomials are incredibly useful mathematical tools as they are simply defined, can be calculated quickly on computer system and can be modified to any desired form so as to satisfy the essential boundary conditions of BVP's. This motivates our attention to compute the eigenvalues of the SLE's using Legendre polynomials.

However, in section 2 of this paper, Legendre polynomials together with properties are exemplified in brief. In section 3, matrix formulations of Galerkin WRM are presented for solving linear Sturm-Liouville problems.

Convergence analysis is given in section 4. Numerical examples and results for the second, third and fourth order eigenvalue problems are regarded as to verify the efficiency of proposed method and results are compared with the existing methods available in the literature in section 5. Conclusions is given in section 6.

2. Legendre polynomials

The Legendre polynomials of degree n is defined on [-1, 1] as follows:

$$L_n(x)=\sum_{r=0}^{N}(-1)^r\frac{(2n-2r)!}{2^n r!\,(n-r)!\,(n-2r)!}x^{n-2r}, \tag{1a}$$

$$\text{where } N=\begin{cases}\dfrac{n}{2}, & \text{when } n \text{ is even}\\ \dfrac{n-1}{2}, & \text{when } n \text{ is odd}\end{cases} \tag{1b}$$

The Rodrigues' Formula of degree n is defined as:

$$L_n(x)=\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1),\ \text{where}\quad n\geq 1 \tag{2}$$

The n -th order Legendre differential equation is given by ,

$$\left((1-x^2)L_n'(x)\right)'=n(n+1)L_n(x)\ \text{ on } (-1,1) \tag{3}$$

provided L_n is bounded on[-1,1] i.e, $|L_n(x)|\leq 1$. (4)

Properties of Legendre polynomials

(i) $L_n(\pm 1)=(\pm 1)^n$,

(ii) $L_n'(\pm 1)=\frac{1}{2}(\pm 1)^{n-1}n(n+1)$

(iii) $\int_{-1}^{1}L_n(x)\,L_r(x)dx=\dfrac{\delta_{nr}}{n+\frac{1}{2}}\quad\forall\quad r,n\geq 0$

The Legendre polynomials are orthogonal with.respect.to the L^2 (-1,1) inner product. Also these polynomials are complete in the sense that for any

$$v(x)=\sum_{n=0}^{\infty}\tilde{v}_n L_n(x) \tag{5a}$$

$$\tilde{v}=(n+\frac{1}{2})\int_{-1}^{1}v(x)\,L_n(x)dx \tag{5b}$$

where the sum converges to $L^2(-1,1)$ norm. Legendre polynomial which are orthogonal in the interval [-1,1] satisfy the following recurrence relation.

$$L_{n+1}(x)=\frac{2n+1}{n+1}x\,L_n(x)-\frac{n}{n+1}L_{n-1}(x)\ ,\ n\geq 1 \tag{6}$$

We modified the above basis as

$$\tilde{p}_n(x)=\left[\frac{1}{n!}\frac{d^n}{dx^n}(x^2-x)^n-(-1)^n\right]\times(x-1) \tag{7}$$

so as to satisfy the homogeneous form of the Dirichlet boundary conditions to derive the matrix formulations of the fourth order Sturm-Liouville problems over the interval [0,1].

3. Matrix Formulation

(i) SLE with Dirichlet boundary conditions:

Consider the following general fourth order nonsingular Sturm-Liouville problem (SLE)

$$Lu_n := \frac{d^2}{dx^2}\left[p(x)\frac{d^2u}{dx^2}\right] - \frac{d}{dx}\left[q(x)\frac{du}{dx}\right] + r(x)u = \lambda\omega(x)u \quad x \in (\xi,\eta) \tag{8}$$

here a,b are finite numbers; $p(x), q(x), r(x)$ and $\omega(x)$ are all piecewise continuous functions and $p(x)$, $\omega(x) \ge 0$ subject to some specified conditions and at these conditions mean that equation (8) is regular, i.e, nonsingular.

We can rewrite the equation (8) in the following form as a general second order Sturm-Liouville problem as

$$\frac{d^4u}{dx^4} + a_3\frac{d^3u}{dx^3} + a_2\frac{d^2u}{dx^2} + a_1\frac{du}{dx} + a_0u = \lambda\omega u, \quad \xi \le x \le \eta \tag{9}$$

where ,

$$a_3(x) = \frac{2p'(x)}{p(x)},\ a_0(x) = \frac{r(x)}{p(x)},\ a_2(x) = \frac{p''(x)-q(x)}{p(x)},\ a_1(x) = -\frac{q'(x)}{p(x)} \quad \omega(x) = \frac{\phi(x)}{p(x)}$$

Let us consider the fourth order SLE (9) subject to the two types of homogeneous boundary conditions:

Type 1:

$$u(\xi) = 0, \quad u(\eta) = 0 \quad u'(\xi) = 0 \quad u'(\eta) = 0. \tag{9a}$$

Type 2:

$$u(\xi) = 0, \quad u(\eta) = 0 \quad u''(\xi) = 0 \quad u''(\eta) = 0. \tag{9b}$$

To approximate the solution of SLE (9), we express in terms of Legendre polynomials basis as

$$\tilde{u}(x) = \psi_0(x) + \sum_{i=1}^{n} c_i L_i(x) \tag{10}$$

where $\psi_0(x)$ is specified by the Dirichlet boundary conditions and $L_i(\xi) = 0$ and $L_i(\eta) = 0$ for each $i = 1,2,3,\ldots\ldots,n$.

Using (10) into equation (9), considering boundary conditions of Type 1, the Galerkin weighted residual equations are:

$$\int_{\xi}^{\eta}\left[\frac{d^4\tilde{u}}{dx^4} + a_3\frac{d^3\tilde{u}}{dx^3} + a_2\frac{d^2\tilde{u}}{dx^2} + a_1\frac{d\tilde{u}}{dx} + a_0\tilde{u} - \lambda\omega\tilde{u}\right]L_j \; dx = 0 \tag{11}$$

Now integrating each term of (11) by parts, we have

$$\begin{aligned}
\int_{\xi}^{\eta}\frac{d^4\tilde{u}}{dx^4}L_j(x)dx &= \left[L_j(x)\frac{d^3\tilde{u}}{dx^3}\right]_{\xi}^{\eta} - \int_{\xi}^{\eta}\frac{dL_j(x)}{dx}\frac{d^3\tilde{u}}{dx^3}dx \quad , \quad j = 1,2,3,\cdots\cdots,n. \\
&= -\left[\frac{dL_j(x)}{dx}\frac{d^2\tilde{u}}{dx^2}\right]_{\xi}^{\eta} + \int_{\xi}^{\eta}\frac{d^2L_j(x)}{dx^2}\frac{d^2\tilde{u}}{dx^2}dx \\
&= -\left[\frac{dL_j(x)}{dx}\frac{d^2\tilde{u}}{dx^2}\right]_{\xi}^{\eta} + \left[\frac{d^2L_j(x)}{dx^2}\frac{d\tilde{u}}{dx}\right]_{\xi}^{\eta} - \int_{\xi}^{\eta}\frac{d^3L_j(x)}{dx^3}\frac{d\tilde{u}}{dx}dx \\
&= -\left[\frac{dL_j(x)}{dx}\frac{d^2\tilde{u}}{dx^2}\right]_{\xi}^{\eta} - \int_{\xi}^{\eta}\frac{d^3L_j(x)}{dx^3}\frac{d\tilde{u}}{dx}dx
\end{aligned} \tag{12}$$

$$\begin{aligned}
\int_{\xi}^{\eta}a_3(x)\frac{d^3\tilde{u}}{dx^3}L_j(x)dx &= \left[a_3(x)L_j(x)\frac{d^2\tilde{u}}{dx^2}\right]_{a}^{b} - \int_{\xi}^{\eta}\frac{d}{dx}\left[a_3(x)L_j(x)\right]\frac{d^2\tilde{u}}{dx^2}dx \\
&= -\left[\frac{d}{dx}\left[a_3(x)L_j(x)\right]\frac{d\tilde{u}}{dx}\right]_{\xi}^{\eta} + \int_{\xi}^{\eta}\frac{d^2}{dx^2}\left[a_3(x)L_j(x)\right]\frac{d\tilde{u}}{dx}dx \\
&= -\left[\frac{d}{dx}\left[a_3(x)L_j(x)\right]\frac{d\tilde{u}}{dx}\right]_{x=\eta} + \left[\frac{d}{dx}\left[a_3(x)L_j(x)\right]\frac{d\tilde{u}}{dx}\right]_{x=\xi}
\end{aligned}$$

$$+\int_{\xi}^{\eta}\frac{d^2}{dx^2}\left[a_3(x)L_j(x)\right]\frac{d\tilde{u}}{dx} \tag{13}$$

$$\int_{\xi}^{\eta}a_2(x)\frac{d^2\tilde{u}}{dx^2}L_j(x)dx=\left[L_j(x)\frac{d\tilde{u}}{dx}\right]_{\xi}^{\eta}-\int_{\xi}^{\eta}\frac{dL_j(x)}{dx}\frac{d\tilde{u}}{dx}dx$$

$$=-\int_{\xi}^{\eta}\frac{dL_j(x)}{dx}\frac{d\tilde{u}}{dx}dx \tag{14}$$

$\left[L_j(x)\frac{d\tilde{u}}{dx}\right]_{\xi}^{\eta}=0$, by the the Dirichlet b.c's. $L_j(\xi)=L_j(\eta)=0$

$$\int_{\xi}^{\eta}a_1(x)\frac{d\tilde{u}}{dx}L_j(x)dx=\left[a_1(x)L_j(x)\tilde{u}(x)\right]_{\xi}^{\eta}-\int_{\xi}^{\eta}\frac{d}{dx}\left[a_1(x)L_j(x)\right]\tilde{u}(x)dx$$

$$=-\int_{\xi}^{\eta}\frac{d}{dx}\left[a_1(x)L_j(x)\right]\tilde{u}(x)dx \tag{15}$$

Substituting equations (12),(13),(14) and (15) into equation (11) and after rearranging the terms we have

$$\int_{\xi}^{\eta}\left[-\frac{d^3L_j(x)}{dx^3}\frac{d\tilde{u}}{dx}+\frac{d^2}{dx^2}\left[a_3(x)L_j(x)\right]\frac{d\tilde{u}}{dx}-\frac{d}{dx}\left[a_2(x)L_j(x)\right]\frac{d\tilde{u}}{dx}\right.$$

$$\left.-\frac{d}{dx}\left[a_1(x)L_j(x)\right]\tilde{u}+a_0(x)L_j\tilde{u}-\lambda\ \omega(x)L_j\tilde{u}\right]dx-\left[\frac{dL_j(x)}{dx}\frac{d^2\tilde{u}}{dx^2}\right]_{\xi}^{\eta}=0 \tag{16}$$

Also from equation (10), we have

$$\tilde{u}(\xi)=\sum_{i=1}^{n}c_i\ L_i(\xi),\quad \tilde{u}(\eta)=\sum_{i=1}^{n}c_i\ L_i(\eta) \tag{17a}$$

$$\frac{d^2\tilde{u}}{dx^2}=\sum_{i=1}^{n}c_i\frac{d^2B_i}{dx^2} \tag{17b}$$

Using equations (17a) and (17b) into equation (16) we obtain

$$\sum_{i=1}^{n}\left[\int_{\xi}^{\eta}\left[-\frac{d^3L_j}{dx^3}\frac{dL_i}{dx}+\frac{d^2}{dx^2}\left[a_3(x)L_j(x)\right]\frac{dL_i}{dx}-\frac{d}{dx}\left[a_2(x)L_j(x)\right]\frac{dL_i}{dx}\right.\right.$$

$$\left.\left.-\frac{d}{dx}\left[a_1(x)L_j(x)\right]L_i(x)+a_0(x)L_i(x)L_j(x)\right]dx\right]c_i$$

$$+\sum_{i=1}^{n}\left\{-\left[\frac{d}{dx}\left[L_j(x)\right]\frac{d^2L_i}{dx^2}\right]_{x=\eta}+\left[\frac{d}{dx}\left[L_j(x)\right]\frac{d^2L_i}{dx^2}\right]_{x=\xi}\right\}c_i$$

$$-\lambda\sum_{i=1}^{n}\int_{\xi}^{\eta}\left[\omega(x)L_iL_jdx\right]c_i=0 \tag{18}$$

Finally, the eigenvalues are obtained in matrix form as below

$$\sum_{i=1}^{n}\left[A_{i,j}-\lambda D_{i,j}\right]c_i=0 \tag{19}$$

where,

$$A_{i,j}=\int_{\xi}^{\eta}\left\{-\frac{d^3B_j}{dx^3}\frac{dL_i}{dx}+\frac{d^2}{dx^2}\left[a_3(x)L_j(x)\right]\frac{dL_i}{dx}-\frac{d}{dx}\left[a_2(x)L_j(x)\right]\frac{dL_i}{dx}\right\}dx$$

$$+\int_{\xi}^{\eta}\left\{-\frac{d}{dx}\left[a_1(x)L_j(x)\right]L_i(x)+a_0(x)L_i(x)L_j(x)\right\}dx-\left[\frac{d}{dx}\left[L_j(x)\right]\frac{d^2L_i}{dx^2}\right]_{x=\eta}$$

$$+\left[\frac{d}{dx}\left[L_j(x)\right]\frac{d^2B_i}{dx^2}\right]_{x=\xi} \qquad (19a)$$

$$D_{i,j}=\int_{\xi}^{\eta}\left[\omega(x)\,L_iL_j\,dx\right] \qquad (19b)$$

Hence, the eigenvalues can be obtained by solving the determinant of the coefficient matrix in equation, such that,

$$det\left(A_{i,j}-\lambda\,D_{i,j}\right)=0. \qquad (20)$$

Similarly for the boundary conditions of the type 2 , the formulation can be obtained easily.

(ii) SLE with Robin or mixed boundary conditions

We Consider the second order linear Strum-Liouville problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+\left[\lambda\,r(x)-q(x)\right]u=0\,,\quad \xi<x<\eta \qquad (21a)$$

$$\left.\begin{array}{l}\alpha_0u(\xi)+\alpha_1u'(\xi)=\gamma_1\\ \beta_0u(\eta)+\beta_1u'(\eta)=\gamma_2\end{array}\right\} \qquad (21b)$$

where $p(x)$, $q(x)$ and $r(x)$ are specified continuous functions, and $\alpha_0,\alpha_1,\beta_0,\beta_1,\gamma_1,\gamma_2$ are specified numbers.We use Legendre polynomials as trial functions which are derived over the interval [0,1] so as to SLE (21a) is to be converted to an equivalent problem on [0,1]. This exertion is performed by placing x by $(\eta-\xi)x+\xi$, and thus we have:

$$-\frac{d}{dx}\left(\tilde{p}(x)\frac{du}{dx}\right)+\left[\lambda\,\tilde{r}(x)-\tilde{q}(x)\right]u=0\,,\qquad 0<x<1 \qquad (22a)$$

$$\left\{\begin{array}{l}\alpha_0u(0)+\dfrac{\alpha_1}{\eta-\xi}u'(0)=\gamma_1\\[2ex] \beta_0u(1)+\dfrac{\beta_1}{\eta-\xi}u'(1)=\gamma_2\end{array}\right. \qquad (22b)$$

where, $$\left\{\begin{array}{l}\tilde{p}(x)=\dfrac{1}{(\eta-\xi)^2}\,p\left[(\eta-\xi)x+\xi\right]\\ \tilde{q}(x)=q\left[(\eta-\xi)x+\xi\right]\\ \tilde{r}(x)=r\left[(\eta-\xi)x+\xi\right]\end{array}\right. \qquad (22c)$$

We assume the trial terms of Legendre polynomials, $L_j(x)$ as

$$\tilde{u}(x)=\sum_{i=1}^{n}c_iL_i(x)\,,\qquad n\geq 1 \qquad (23)$$

where c_i's are unknown parameters.

Now the weighted residual equations corresponding to the equation (22a) given by

$$\int_0^1\left[-\frac{d}{dx}\left(\tilde{p}(x)\frac{d\tilde{u}}{dx}\right)+\tilde{q}(x)\tilde{u}-\lambda\tilde{r}(x)\tilde{u}\right]L_j(x)dx=0\,,\qquad j=1,2,3,\ldots\ldots\ldots\ldots,n \qquad (24)$$

Again from equation (23), we have

$$\tilde{u}(0)=\sum_{i=1}^{n}c_iL_i(0)\qquad \tilde{u}(1)=\sum_{i=1}^{n}c_iL_i(1) \qquad (25a)$$

$$\frac{d\tilde{u}}{dx}=\sum_{i=1}^{n}c_i\frac{dL_i}{dx} \qquad (25b)$$

Integrating each term of equation (24) by parts and using equations (25) we obtain the Galerkin Weighted residual equations :

$$\sum_{i=1}^{n}\left[\int_0^1\left[\tilde{p}(x)\frac{dL_i}{dx}\frac{dL_j}{dx}+\tilde{q}(x)L_i(x)L_j(x)-\lambda\tilde{r}(x)L_i(x)L_j(x)\right]dx+\frac{\beta_0(\eta-\xi)\tilde{p}(1)L_i(1)L_j(1)}{\beta_1}\right.$$

$$\left.-\frac{\alpha_0(\eta-\xi)\tilde{p}(0)L_i(0)L_j(0)}{\alpha_1}\right]c_i=\frac{\gamma_2(\eta-\xi)\tilde{p}(1)L_j(1)}{\beta_1}-\frac{\gamma_1(\eta-\xi)\tilde{p}(0)L_j(0)}{\alpha_1} \tag{26}$$

or, equivalently in matrix form

$$\sum_{i=1}^{n}\left(A_{i,j}-\lambda B_{i,j}\right)c_i=D_j\ ,\ j=1,2,3,\ldots\ldots\ldots\ldots,n \tag{27a}$$

where,

$$A_{i,j}=\int_0^1\left[\tilde{p}(x)\frac{dL_i}{dx}\frac{dL_j}{dx}+\tilde{q}(x)L_i(x)L_j(x)\right]dx+\frac{\beta_0(\eta-\xi)\tilde{p}(1)L_i(1)L_j(1)}{\beta_1}$$
$$-\frac{\alpha_0(\eta-\xi)\tilde{p}(0)L_i(0)L_j(0)}{\alpha_1} \tag{27b}$$

$$B_{i,j}=\int_0^1\tilde{r}(x)L_i(x)L_j(x)\,dx \tag{27c}$$

$$D_j=\frac{\gamma_2(\eta-\xi)\tilde{p}(1)L_j(1)}{\beta_1}-\frac{\gamma_1(\eta-\xi)\tilde{p}(0)L_j(0)}{\alpha_1} \tag{27d}$$

$i,j=1,2,3\ldots\ldots,n$

Equivalently eigenvalues can be obtained by solving the determinant of this coefficient matrix of equ. (27)

$$det(A_{i,j}-D_jI-\lambda B_{i,j})=0. \tag{28}$$

Solving this determinant we find the values of λ.

4. Convergence and error estimation

Consider the exact and be approximate solutions be $u(x)$ and $\tilde{u}_n$ respectively.

where,

$$\tilde{u}_n(x,c)=\psi_0(x)+\sum_{i=1}^{n}c_i\psi_i(x) \tag{29}$$

Completeness condition states that the sequence of approximate eigenfunctions will converge to the exact solution if degree polynomials increase indefinitely.

Mathematically,

$$|u(x)-\tilde{u}_n(x)|\to 0\ \text{ as }\ n\to\infty\qquad \xi<x<\eta, \tag{30}$$

A set of trial functions form a complete set of functions and they are complete in a space if any function in the space can be expanded in terms of the set of functions , for sufficiently large *n* [Finlayson (1972)].

$$\left\|u-\sum_{i=1}^{\infty}c_i\psi_i\right\|<\varepsilon \tag{31}$$

The convergence theorem required the trial functions to satisfy the two conditions in equ. (29) and equ.(31).
To assess convergence, we observe uniform convergence that requires the maximum value of $|u(x)-\tilde{u}_n(x)|$ in the domain vanish as $n\to\infty$. This requires a uniform rate of convergence at every point in the domain. Besides, we minimize the residual error with eigensolution defined as

$$E(x)=u-\tilde{u} \tag{32}$$

The approximate eigensolutions are arbitrarily close to the exact solutions which is measured by the energy error which involves derivatives of *u*. Convergence in energy requires $|u-\tilde{u}|<\varepsilon$

$$\text{Energy error} = \int_\Omega E(x)L[E(x)]^{1/2}dx \tag{33a}$$

where $E(x) = u - \tilde{u}$ and $L = \dfrac{d^2}{dx^2}\left(p(x)\dfrac{d^2}{dx^2}\right)$ (33b)

Therefore, from the above arguments we examine that our polynomial trial solution in equ. (29) will converge in energy as $n \to \infty$ because the powers of Legendre polynomials and their derivatives (n=1, 2, 3,......) have sufficient continuity and they form an infinite sequence of functions which is complete in energy for equ.(33a) and its boundary conditions. The convergence of the eigenvalues by Galerkin WRM method is measured by the relative error

$$\varepsilon_k = \left|\frac{\lambda^{Exact} - \lambda^{(Approx)}}{\lambda^{Exact}}\right| < \delta \tag{34}$$

where $\lambda^{(Approx)}$ denotes the approximate solution using n-th polynomials and $\delta \le 10^{-10}$ depends upon the of the problems.

4. Numerical Examples :

In this section we will present six numerical examples of second order SLE problems, using the method outlined in the previous section. The convergence of the our existing method is measured by the two errors

$$\text{Absolute error } \delta_k = \left|\lambda^{exact} - \lambda^{(Gal.)}\right| \text{ , Relative error } \varepsilon_k = \left|\frac{\lambda^{exact} - \lambda^{(Gal..)}}{\lambda^{exact}}\right|. \tag{35}$$

Example 1: We compute the eigenvalues of the Sturm-Liouville Problem work out by Al-Quran and Al-Khaled (2010) given below.

Consider,

$$\begin{cases} -\dfrac{d^2u}{dx^2} + cos^2\, x\, y(x) = \lambda\, y(x) \\ y(0) = y(\pi) = 0 \end{cases} \tag{36}$$

Table 1: Comparison of absolute errors of Galerkin Weighted Residual method with the sinc Galerkin and differential Transform method for example 3.

Exact eigevalues	Absolute error Sinc Gal. N=32	Absolute error Legn.Gal. N=20	Absolute error Diff.Transform N=10	Absolute error Legn.Gal. N=10
1.24242	1.42e-005	8.826e-006	3.19e-005	8.826e-006
4.49479	4.85e-004	3.079e-006	1.70e-004	3.122e-006
9.50366	9.91e-003	4.867e-006	1.83e-003	2.365e-005
16.50208	1.31e-001	1.901e-006	3.81e-001	1.336 e-004

Table 1, lists the first four eigenvalues. From Table 1, it has been noticed that the absolute errors for N=10 in case of our present method showing the better accuracy than that of the Differential Transform method by Chen and Ho (1996), for *N=10*. Also when we use n=32 grid points, the maximum absolute error is 10^{-6} which is more accurate than Galerkin method for *N=32*.This proves that our present method is much more efficient than that of Sinc Galerkin method and Differential Transform method (2010).

Example 2: Consider the second order SLE with constant co-efficients boundary conditions are of mixed types.

$$\begin{cases} \dfrac{d^2u}{dx^2} = \lambda\, u \\ u(0) - u'(0) = 0 \\ u(1) + u'(1) = 0 \end{cases} \tag{37a}$$

$$\left.\begin{aligned} &4\frac{d^2y}{dx^2}=\lambda\, y \\ &y(-1)-y'(-1)=0 \\ &y(1)+y'(1)=0 \end{aligned}\right\} \qquad (37b)$$

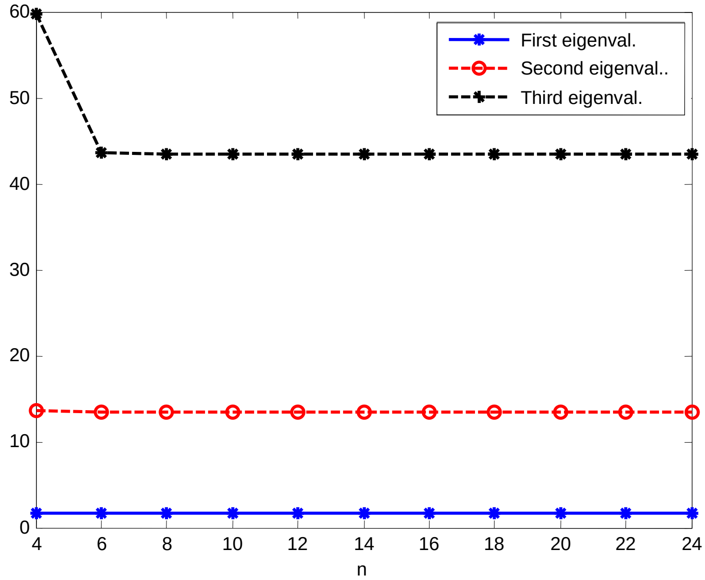

Figure 1: Convergence of eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

Here λ_k^n is *k*-th estimated eigenvalue corresponding to *n* and the differences between the *k*-th and *(k-1)*-th eigenvalues are given by $\left|\lambda_i^k - \lambda_i^{k-1}\right| < \varepsilon$, where ε is very small and $\varepsilon \to 0$.

Table 2: Absolute errors between the successive eigenvalues for example 2

i	Exact eigevalues Chen and Ho (1996)	Absolute error present $\left\lvert\lambda_i^6 - \lambda_i^5\right\rvert$	Absolute error present $\left\lvert\lambda_i^{12} - \lambda_i^{11}\right\rvert$
1	1.71	3.66e-004	4.80e-012
2	13.49	2.32e-003	3.22e-008
3	43.36	8.70e-001	1.70e-004

It is noticed from table 2, that the errors decreased with the increasing degree of *n* and differences between successive eigenvalues converge to zero as the node number inceased and is given as follows:

$\left|\lambda_1^{12} - \lambda_1^{11}\right| \le 0.00000000001$, $\left|\lambda_2^{12} - \lambda_2^{11}\right| \le 0.0000000001$ and $\left|\lambda_3^{12} - \lambda_3^{11}\right| \le 0.000001$

Example 3: Consider the SLP studied by Celik (2005)is given below

$$\left.\begin{aligned} &\frac{d^2y}{dx^2}+\lambda\, u=(x+0.1)^{-2}u \\ &u(0)=u(\pi)=0 \end{aligned}\right\} \qquad (38)$$

Table 3: Relative errors of the smallest eigenvalues in present method for example 3 for different values of *n*.

k	Exact eigenvalues	Rel. err *n*=30	Rel. err *n*=35	Rel. err *n*=40	Rel. err *n*=45
1	1.51986582	7.332e-09	8.333e-09	1.278e-08	1.310e-08
2	4.9433098	1.662e-08	4.342e-09	7.486e-09	8.010e-09
3	10.284663	1.010e-08	1.979e-08	3.783e-08	2.9707-08
4	17.559958	4.251e-09	7.497e-08	1.418e-09	8.416e-09
5	26.782863	2.391e-08	2.322e-08	3.252e-08	1.872e-08
6	37.964426	1.343e-09	8.356e-09	8.025e-09	3.977e-09
7	51.113358	7.851e-010	9.999e-09	1.085e-09	3.024e-09
8	66.236448	2.169e-09	2.350e-09	5.443e-09	5.951e-09
9	83.338962	1.348e-09	1.859e-09	8.711e-09	5.517e-09
10	102.42499	2.178e-07	1.655e-08	1.031e-08	9.059e-09
11	123.49771	9.131 e-07	6.853e-07	5.111e-07	1.984e-09
12	146.55961	1.797e-05	5.681e-05	8.022e-06	7.318e-06
13	171.61264	6.080e-04	5.551e-04	9.201e-05	5.481e-05
14	198.65837	2.585e-03	2.050e-03	1.406e-03	5.140e-04
15	227.69803	2.401e-02	1.342e-02	3.657e-03	2.819e-03

Table 3 lists the first 15 eigenvalues for different values of n. It is clearly noticed that the convergence rate does not improve much with the increasing degree of *n*. For the range *n*=30 to 40 , the first 11 eigenvalues converge at the same rate. For *n*=45, the convergence rate slightly improves but not as much as we expect. Our computed results for the first 11 eigenvalues are correct up to 8 significant figures which is much compatible with the result of Celik (2005).This indicates that the existing method is efficient for obtaining smaller harmonics than that of larger.

Example 4:

$$u'' + \left[\lambda(1+x)^{-2} - (1+x)^2\right]u = 0 \tag{39a}$$

$$u(0) - u'(0) = 0 \ \ ; u(1) + u'(1) = 0 \tag{39b}$$

Table 4: Relative errors of the smallest eigenvalues in present method for example for different values of *n*.

Exact smallest eigenvalue Chawla(1983)	*x*	No of grid points Chawla (1983) *N*	rel. error Chawla (1983)	Degree of polynomials present method *n*	rel. error present
	1	8	1.65e-004	8	4.548e-009
5.833767621	2	16	1.97e-005	12	4.097e-009
	3	32	2.32-006	16	4.098e-009
	4	64	2.81-007	20	4.098e-009

Relative error norms are listed in table 4 .We computed approximate eigenvalues by symmetric finite difference method for the smallest eigenvalue of the problem above by the present method. The maximum absolute error achieved by the present method is about 4.54×10^{-9} ,whereas error attained by Chawla and Shivakumar(1993) of order 10^{-7} .Besides using only eight Legendre polynomials the accuracy is achieved and finite difference method attained the less accuracy applying 64 grids. It is also observed that, relative errors remain constant and not decreasing with the increased number of degree of polynomials.

Example : 5

We calculate the ten eigenvalues of third order problem as illustrated in Gheorghiu (2007).

$$\begin{cases} \dfrac{d^3u}{dx^3} + \lambda u = 0 \\ u(-1) = u(1) = 0 \\ u'(-1) = 0 \end{cases} \tag{40a}$$

To Change the boundary points from -1 to 1 into -1 to 1, we transfer the equation (40a) by changing the variables $x = 2t - 1$,the Sturm-Liouville problem transforms leads the SLE as follows:

$$\begin{cases} \dfrac{1}{8}\dfrac{d^3u}{dx^3} + \lambda u = 0 \\ u(0) = u(1) = 0 \\ u'(0) = 0 \end{cases} \tag{40b}$$

Using eigencondition $e^{3\lambda^{\frac{1}{3}}} - 2\sin\left(\sqrt{3}\lambda^{\frac{1}{3}} + \frac{\pi}{6}\right) = 0$, (41a)

The exact eigenvalues are found from the relation

$$\lambda_k = \left\{\left(k + \frac{1}{6}\right)\frac{\pi}{\sqrt{3}}\right\}^3 , \quad k = 1,2,3,\ldots\ldots\ldots\ldots,n \tag{41b}$$

We calculate the ten eigenvalues of third order problem as illustrated in Gheorghiu(2007).

Table 5: Comparison of relative errors of Legendre spectral collocation with the results of Gheorghiu (2007) for example 5.

Exact eigenvalues	Gal. Legn (present).	Spect. Tau (2007)	Rel. error Error (present) n=29	Rel.error (Tao) Error (2007) N=64
-9.47563023219617	-9.48240693552001	−9.48240693549165e+ 00	7.152e-004	7.152e-004
-60.6937598254664	-60.6936584118949	−6.06936584119385e+ 01	1.671e-006	1.671e-006
-189.484978899806	-189.484979838545	-1.89484979838051e+ 02	4.954e-009	4.952e-009
-431.652251831094	-431.652251824051	−4.31652251829144e+ 02	1.632e-011	4.517e-012
-822.998542995207	-822.998542995237	-8.22998542950849e+ 02	3.633e-014	5.389e-011
-1399.32681676803	-1399.32681676847	-1.39932681708419e+ 03	3.113e-013	2.259e-010
-2196.44003752543	- 2196.44003751879	-2.19644003549006e+ 03	3.025e-012	9.267e-010
-3250.14116964328	-3250.14117085548	-3.25014118218594e+ 03	3.730e-010	3.859e-009
−4596.23310058846	-4596.23385177037	-4.59623310058847e+ 03	4.440e-007	1.673e-008
-6270.51902546391	- 6270.51347261484	-6.27051950319929e+ 03	8.856e-007	7.619e-008

From table 5, it has been examined that the first eight computed eigenvalues using Legendre polynomials are very close to the exact results and the values for the lower eigenvalues have a better accuracy than those for the higher eigenvalues Gheorghiu (2007). Smaller eigenvalues converge more rapidly than those of larger ones.

Example 6: We first consider the Sturm-Liouville BVP Yucel and Boubaker (2012), Gamel and Sameeh (2012) and Chebychev spectral collocation method of Taher et al (2013).

$$\frac{d^4u}{dx^4} - \lambda u(x) = 0, \quad 0 < x < 1 \tag{42a}$$

$$\left.\begin{aligned} u(0) = u'(0) = 0 \\ u(1) = u''(1) = 0 \end{aligned}\right\}, \tag{42b}$$

which corresponds to the case $a_0(x) = a_1(x) = a_2(x) = a_3(x) = 0$, $a = 0$ and $b = 1$ in equation (9).

The exact solution of (42a) can be obtained by solving $tanh(\sqrt{\lambda}) - tan(\sqrt{\lambda}) = 0$.

Table 6: Observed relative errors of the eigenvalues for example 6

Exact eigenvalues	Relative error present n=26	Relative error (2013)	Relative errors (2012)	Relative errors (2012) PDQ N=20	Relative errors PDQ (2012) N=30
237.72106753	4.6974e-012	2.03E-009	4.697 E-012	7.59E-009	7.59 E-09
2496.48743786	2.703E-012	7.93E-010	3.046E-12	4.44E-008	4.45E-08
10867.58221698	2.139E-013	2.33E-010	5.104E-12	1.94E-009	1.71E-08
31780.09645408	3.388E-014	8.605E-09	8.605E-09	4.50E-008	2.36E-08
74000.849349156	6.686E-015	7.51E-011	…………	3.97E-005	2.99E-08
148634.47728577	1.958E-015	2.24E-010	………..	1.43E-004	4.77E-08
269123.43482664	9.517E-015	……………	……….….	4.08E-003	9.61E-10
451247.99471928	2.933E-013	…………..	…………	1.11E-002	1.74E-08
713126.24789600	1.920E-010	…………	………..	9.02E-002	3.16E-06
1075214.10347396	4.809E-009	….....	. ……….	2.06E-002	9.31E-06

It is observed in Table 6, that all 10 eigenvalues obtained using Legendre polynomials converge more rapidly are very close to the exact results than those obtained by the other methods. In fact relative error decreases as the degree of polynomials increase from $n = 20$ to $n = 26$ in the case of Legendre basis.

Example 7: Consider the Sturm-Liouville EVP (2006, 2009, 2010,2012, 2013)

$$\frac{d^4 y}{dx^4} = 0.02\, x^2 \frac{d^2 y}{dx^2} + 0.04\ x\ \frac{dy}{dx} - (0.0001\, x^4 - 0.02)\, u(x) + \lambda\, u(x) \qquad (43a)$$

$$\left.\begin{aligned} u(0) = u'(0) = 0 \\ u(5) = u''(5) = 0 \end{aligned}\right\} \qquad (43b)$$

Table 7 **:** Comparison of eigenvalues for example 7

$\lambda_k^{(galerkin)}$ Present (n=18)	Result of Spect. Coll.(2013)	Result of coll.(2012)	Result of ADM(2006)	Result of (2012)	Result of (2010)
0.2150508643697	0.2150508643160	0.2150508643697	0.2150508643697	0.21505086437	0.21505086437
2.7548099346830	2.7548099336169	2.7548099346829	2.7548099346829	2.75480993468	2.75480993468
13.215351540558	13.215351540581	13.215351540416	13.215351540558	13.2153515406	13.2153515406
40.950819759161	40.950819758144	40.950820029821	40.950819759137	40.9508197591	40.9508193487
99.053478067698	99.053478038354	………………	99.053478138138	99.0534780633	………………
204.355734247 63	204.35573547934	………………..	204.35449348957	204.355732256	………………

Table7, illustrates the comparison of our result obtained using n=18, for Legendre polynomials and the first six eigenvalues of the problem with the results of various methods. From table 7, it is noticed that using Legendre polynomials the eigenvalues obtained using the present method show better performance and fairly close to the results of the other available methods.

6. Conclusions

In this paper a novel formulation of the Galerkin method using Legendre polynomials is proposed. The main reason why the Galerkin method is chosen, are its flexibility and simple implementation. Excellent agreement and better accuracy is achieved even with small number of basis polynomials for some SLE which sometimes minimizes the cost of computational time for some problems. The disadvantage of the current method is that, in case of huge number of eigenvalues computation, higher eigen modes are less convergent than the lower modes and with increasing of the degree of polynomials the computational time highly increases. In spite of this disadvantage, we can conclude that for a relatively small n, i.e., n = 18, fairly accurate numerical results are obtained using the proposed method.

The results of the previous section shown in all the tables signify that Galerkin method using Legendre polynomials shows very accurate results compared to the other numerical methods with degree of polynomials not greater than 10. Furthermore, the smallest eigenvalue which characterizes potentially the most visual structures of the dynamical systems arises in vibration of a deformable bodies can be computed very accurately applying WRM .Our proposed method is much superior in the sense of accuracy and applicability specially for higher order problems.

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