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Hybrid and Non-hybrid Implicit Schemes for Solving Third Order ODEs Using Block Method as Predictors

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Abstract

This work considers the direct solution of general third order ordinary differential equation of the form $y^{'''}(x) = f(x, y, y', y''), y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2$. The method is derived by collocating and interpolating the approximate solution in power series. Two methods were derived-the hybrid three-step and four-step non-hybrid methods using the block method to generate the independent solution at selected grid and off grid points. The order, error constant, zero stability and convergence of the methods were investigated. The two schemes derived are tested on standard problems; the hybrid scheme is more superior to the non-hybrid scheme in terms of accuracy and efficiency.

Keywords: collocation, interpolation, power series, hybrid, linear multistep method, block method and zero stability

1. Introduction

Differential equations which are applicable to our day to day life are frequently encountered in areas of sciences, social sciences and engineering. In this research work, general third order of initial value problems (IVP) of ordinary differential equation of the form:

$$y^{'''}(x) = f(x, y, y', y'') \qquad 1$$

is considered. Customarily, third order ordinary differential equations are solved by reducing it to a system of first order ordinary differential equations and then an appropriate numerical method for first order will be used to solve the system.

This reduction approach has been extensively discussed by several authors such as Lambert (1973), Fatunla (1988), Awoyemi (1999) and Jator (2001). In spite of the success of this approach, there are several setbacks. Writing computer programs for these methods is often complicated especially when subroutines are incorporated to supply starting values required for the methods. The consequence is in longer computer time and more human effort. Many authors have developed methods for the direct solution of (1) without reducing it to systems of first order ordinary differential equations which will be of the form

$$y^{'''}(x) = f(x, y, y', y''), y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2 \qquad 2$$

However, this approach has its own disadvantages; some of which are non-economization of computer time, implementation cost and computational burden. Methods of linear multistep method (LMM) have been considered by Brown (1977), Lambert (1973), Awoyemi (1999, 2001, 2003,), Olabode (2007), Adesanya et al. (2008) for problem (1). They independently proposed linear multistep methods with continuous coefficients to solve (2) in the predictor-corrector mode based on collocation method and used Taylor's series expansion to supply starting values. Ibijola, Skwame and Kumleng (2011) also worked on hybrid-block method but with application to first order ordinary differential equations.

Despite the improvement by different authors in developing schemes that can directly solve (1), there is still the setback of lower accuracy when used to solve problems due to the low order of the methods developed. As a result of the above challenges, we propose a direct method based on collocation and interpolation of power series approximate solution to derive a three-step hybrid block method and a four-step block method for direct solution of general third order ordinary differential equations.

2. Methodology

We consider power series as an approximate solution to the general third order ordinary differential equation:

$$y'''(x) = f(x, y, y', y''), y(x_0) = y_0, y'(x_0) = y_1, y''(x_0) = y_2 \qquad 2$$

To be of the form

$$y(x) = \sum_{j=0}^{c+i} a_j x^j \qquad 3$$

where c is the collocation points and i is the interpolation points

The third derivative of (3) gives

$$y'''(x) = \sum_{j=0}^{c+i} j(j-2)(j-1)a_j x^{j-3} = f(x, y, y', y'') \qquad 4$$

Collocate equation (4) at $x = x_{n+j}$, where $j = 0(1)k, v_c$ and interpolate equation (3) at $x = x_{n+j}$, where $j = 0(1)k-1, v_i . v_i, v_c$ are the off-step interpolation and collocation point. Collocation and interpolation equation at some selected grid and off-grid points is consider generating system of non-linear equations which can be solved using Gaussian elimination method. The resulting values generated is substituted back to the power series to give a continuous linear multi step method (LMM) of the form

$$y(x) = \sum_{j=0}^{k-1} \alpha_j(x) y_{n+j} + h^3 \left(\sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_v(x) f_{n+v} \right) \qquad 5$$

Where $y(x)$ is the numerical solution of the initial value problem and $v = \frac{3}{2}$. α_j and β_j are constant.

$$f_{n+j} = y\left(x_{n+j}, y_{n+j}, y'_{n+j}, y''_{n+j}\right) \quad j = 0(1)k$$

And that α_o and β_o are not both zero since (4) is continuous and differentiable; hence it is evaluated along with its derivatives at all the grid points. This generate a block method for the general third order ordinary differential equation of the form

$$A^{(0)}Y_m = A^{(i)}Y_{m-1} + h^{\mu}[B^{(i)}F_m + B^{(0)}F_{m-1}]$$

where

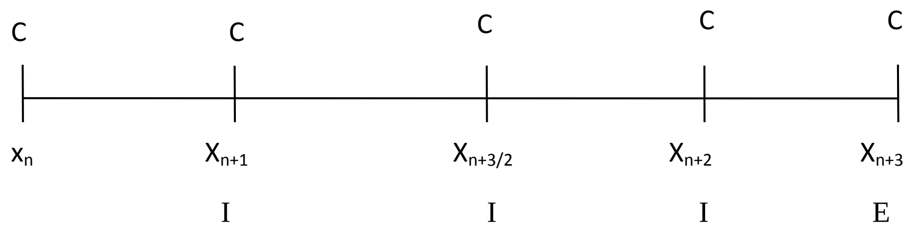
$Y_m = [y_{n+1}, y_{n+2}, ..., y_{n+r}]^T$, $Y_{m-1} = [y_{n-1}, y_{n-2}, ..., y_n]^T$, $F_m = [F_n, f_{n+1}, f_{n+2}, ..., f_{n+k}]^T$

$F_{m-1} = [f_{n-1}, f_{n-2}, f_{n-3}, ..., f_n]^T$, $\mu = 3$ which is the order of the differential equation.

This gives the independent solution $\{y_{n+j}\}$, $i = 1(1)k$ without overlapping

3. Derivation of the Three-Step Hybrid Scheme

A three-step single hybrid implicit method is developed. Here below is the sketch of the scheme.



Where,

$C =$ Points of collocation, $I =$ Points of interpolation, $E =$ Point of evaluation

Thus we consider an approximate solution for (2) in the form

$$y(x) = \sum_{j=0}^{c+i} a_j x^j \qquad 3$$

Where a_j-parameters to be determined and k=3 which is the step length
The first, second and third derivatives are

$$y^{'}(x) = \sum_{j=1}^{c+i} j a_j x^{j-1} \qquad 6$$

$$y^{''}(x) = \sum_{j=2}^{c+i} j(j-1) a_j x^{j-2} \qquad 7$$

$$y'''(x) = \sum_{j=3}^{c+i} j(j-2)(j-1) a_j x^{j-3} \qquad 8$$

Substituting (1.8) into (1.2) we obtain

$$y'''(x) = \sum_{j=0}^{c+i} j(j-2)(j-1) a_j x^{j-3} = f(x, y, y^{'}, y^{''}) \qquad 9$$

Collocating (9) at x_{n+c}, $c = 0, 1, 3/2, 2, 3$ and interpolating (3) at $x_{n+i}, i = 1, 3/2, 2$ give

$$\begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+3/2} & x_{n+3/2}^2 & x_{n+3/2}^3 & x_{n+3/2}^4 & x_{n+3/2}^5 & x_{n+3/2}^6 & x_{n+3/2}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 \\ 0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+3/2} & 60x_{n+3/2}^2 & 120x_{n+3/2}^3 & 210x_{n+3/2}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+2} & 60x_{n+2}^2 & 120x_{n+2}^3 & 210x_{n+2}^4 \\ 0 & 0 & 0 & 6 & 24x_{n+3} & 60x_{n+3}^2 & 120x_{n+3}^3 & 210x_{n+3}^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} = \begin{bmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \\ f_n \\ f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} \qquad 10$$

Solving for $a_j, j = 0(1)7$ in equation (10) using Gaussian elimination method and substituting into (3) gives a linear multistep method with continuous coefficients in the form;

$$y(x)=\sum_{j=0}^{k-1}\alpha_j(x)y_{n+j}+h^3\left(\sum_{j=0}^{k}\beta_j(x)f_{n+j}+\beta_v(x)f_{n+v}\right) \qquad 11$$

where $y(x)$ is the numerical solution of the initial value problem and $v=\dfrac{3}{2}$. $_j$ and $_j$ are constants.

$$f_{n+j}=f(x_{n+j},y_{n+j},y'_{n+j},y''_{n+j}) \qquad 12$$

Using the transformation $t=\dfrac{x-x_{n+2}}{h},\dfrac{dt}{dx}=\dfrac{1}{h}$ 13

The coefficients of y_{n+j} and f_{n+j} are obtained as:

$$\alpha_1(t)=2t^2+t$$

$$\alpha_{3/2}(t)=-(4t^2+4t)$$

$$\alpha_2(t)=(1+3t+2t^2)$$

$$\beta_0(t)=\frac{h^3}{241920}\left[47t+175t^2-560t^4-448t^5+112t^6+128t^7\right]$$

$$\beta_1(t)=\frac{h^3}{241920}\left[-61t-7t^2+112t^4+672t^5-336t^6-128t^7\right] \qquad 14$$

$$\beta_{3/2}(t)=\frac{h^3}{945}\left[62t+154t^2-35t^4-28t^5+28t^6+8t^7\right]$$

$$\beta_2(t)=\frac{h^3}{26880}\left[541t+2653t^2+4480t^3+2800t^4-560t^6-128t^7\right]$$

$$\beta_3(t)=\frac{h^3}{241920}\left[-79t-287t^2+1120t^4+1568t^5+112t^6+128t^7\right]$$

Evaluating (14) at the non-interpolation points i.e. $t=1,-2$, gives

Evaluating (14) at the non-interpolation points i.e. $t=1$, gives

$$y_{n+3}-6y_{n+2}+8y_{n+3/2}-3y_{n+1}=\frac{1}{5760}\left[77f_{n+3}+2097f_{n+2}+512f_{n+3/2}+207f_{n+1}-13f_n\right] \qquad (15)$$

The order of the scheme (15) is $p=6$ with error constant $c_{p+2}=-0.000439453125$. And also evaluating (14) at $t=-1$ gives

$$y_n-6y_{n+1}+8y_{n+\frac{3}{2}}-3y_{n+2}=\frac{1}{5760}[13f_{n+3}-207f_{n+2}-512f_{n+3/2}-2097f_{n+1}-77f_n] \qquad (16)$$

Finding the first derivative of (14) gives:

$$\alpha_1^{'}(t) = 4t + 1$$
$$\alpha_{3/2}^{'}(t) = -8t - 4$$
$$\alpha_2^{'}(t) = 4t + 3$$
$$\beta_0^{'}(t) = \frac{h^3}{241920}\left[47 + 350t - 2240t^3 - 2440t^4 + 672t^5 + 896t^6\right]$$
$$\beta_1^{'}(t) = \frac{h^3}{26880}\left[-61 - 14t + 4490t^3 + 3360t^4 - 2016t^6\right]$$
$$\beta_{3/2}^{'}(t) = \frac{h^3}{945}\left[62 + 308t - 140t^3 - 140t^4 + 168t^5 + 56t^6\right]$$
$$\beta_2^{'}(t) = \frac{h^3}{26880}\left[541 + 5306t + 13440t^2 + 11200t^3 - 3360t^5 - 896t^6\right]$$
$$\beta_3^{'}(t) = \frac{h^3}{241920}\left[-79 - 574t + 4480t^3 + 7840t^4 + 672t^5 + 896t^6\right]$$

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While the second derivative of (14) gives:

$$\alpha_1^{''}(t) = 4$$
$$\alpha_{3/2}^{''}(t) = -8$$
$$\alpha_2^{''}(t) = 4$$
$$\beta_0^{''}(t) = \frac{h^3}{241920}\left[350 - 6720t^2 - 9760t^3 + 3360t^4 + 5376t^5\right]$$
$$\beta_1^{''}(t) = \frac{h^3}{26880}\left[-14 + 13470t^2 + 13440t^3 - 10080t^4 - 5376t^5\right]$$
$$\beta_{3/2}^{''}(t) = \frac{h^3}{945}\left[308 - 420t^2 - 560t^3 + 840t^4 + 336t^5\right]$$
$$\beta_2^{''}(t) = \frac{h^3}{26880}\left[5306 + 26880t + 33600t^2 - 16800t^4 - 5376t^5\right]$$
$$\beta_3^{''}(t) = \frac{h^3}{241920}\left[-574 + 13440t^2 + 31360t^3 3360t^4 + 5376t^5\right]$$

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Evaluating (17) at the entire grid and off-grid points i.e at $t = -2, -1, -1/2, 0, 1$ gives the following equations:

$$241920hy'_n + 1693440y_{n+1} - 2903040y_{n+3/2} + 1209600y_{n+2} = h^2\begin{bmatrix}17267f_n + 236079f_{n+1} - 27136f_{n+3/2}\\ +38385f_{n+2} - 2515f_{n+3}\end{bmatrix} (19)$$

$$241920hy'_{n+1} + 725760y_{n+1} - 967680y_{n+3/2} + 241920y_{n+2} = h^2\begin{bmatrix}-79f + 4869f_{n+1} + 15872f_{n+3/2}\\ -549f_{n+2} + 47f_{n+3}\end{bmatrix} (20)$$

$$1734082560hy'_{n+3/2} + 1734082560y_{n+1} - 1734082560y_{n+2} = h^2\begin{bmatrix}5f_n - 549f_{n+1} - 8992f_{n+3/2}\\ -549f_{n+2} + 5f_{n+3}\end{bmatrix} (21)$$

$$241920hy'_{n+2} - 241920y_{n+1} + 967680y_{n+3/2} - 725760y_{n+2} = h^2\begin{bmatrix}47f_n - 549f_{n+1} + 15872f_{n+3/2}\\ +4869f_{n+2} - 79f_{n+3}\end{bmatrix} (22)$$

$$241920hy'_{n+3} - 1209600y_{n+1} + 2903040y_{n+3/2} - 1693440y_{n+2} = h^2\begin{bmatrix}-2515f_n + 38385f_{n+1} - 27136f_{n+3/2}\\ +236079f_{n+2} + 17267f_{n+3}\end{bmatrix} (23)$$

While evaluating equation (18) at all the grid and off grid points i.e at $t = -2, -1, -1/2, 0, 1$ gives the following equations:

$$5760h^2y''_n - 23040y_{n+1} + 46080y_{n+3/2} - 23040y_{n+2} = h\begin{bmatrix}-1741f_n - 9345f_{n+1} + 4608f_{n+3/2}\\ -2319f_{n+2} + 157f_{n+3}\end{bmatrix} (24)$$

$$17280h^2y''_{n+1} - 69120y_{n+1} + 138240y_{n+3/2} - 69120y_{n+2} = h\begin{bmatrix}41f_n - 3411f_{n+1} - 5632f_{n+3/2}\\ -6525f_{n+2} - 25f_{n+3}\end{bmatrix} (25)$$

$$1440h^2y''_{n+3/2} - 5760y_{n+1} + 11520y_{n+3/2} - 5760y_{n+2} = h\left[-f_n + 33f_{n+1} - 24f_{n+2} + f_{n+3}\right] (26)$$

$$17280h^2y''_{n+2} - 69120y_{n+1} + 138240y_{n+3/2} - 69120y_{n+2} = h\begin{bmatrix}25f_n - 387f_{n+1} + 5632f_{n+3/2}\\ +3411f_{n+2} - 41f_{n+3}\end{bmatrix} (27)$$

$$5760h^2y''_{n+3} - 23040y_{n+1} + 46080y_{n+3/2} - 23040y_{n+2} = h\begin{bmatrix}-157f_n + 2319f_{n+1} - 46080f_{n+3/2}\\ +11649f_{n+2} + 1741f_{n+3}\end{bmatrix} (28)$$

4. Derivation of Block for Three-Step Method

Combining equations (15), (16), (19) and (24) give the block below.

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$$\begin{bmatrix} -17280 & 46080 & -34560 & 5760 \\ -34560 & 46080 & -17280 & 0 \\ 1693440 & -2903040 & 1209600 & 0 \\ -23040 & 46080 & -23040 & 0 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5760 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-3/2} \\ y_{n-1} \\ y_n \end{bmatrix}$$

$$+h \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -241920 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y'_{n-2} \\ y'_{n-3/2} \\ y'_{n-1} \\ y'_n \end{bmatrix} + h^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5760 \end{bmatrix} \begin{bmatrix} y''_{n-2} \\ y''_{n-3/2} \\ y''_{n-1} \\ y''_n \end{bmatrix}$$

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$$+h^3 \begin{bmatrix} 207 & 512 & 2097 & 77 \\ -2097 & -512 & -207 & 13 \\ 23079 & -27136 & 38385 & -2515 \\ -9345 & 4608 & -2319 & 157 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} + h^3 \begin{bmatrix} 0 & 0 & 0 & -13 \\ 0 & 0 & 0 & -77 \\ 0 & 0 & 0 & 17267 \\ 0 & 0 & 0 & -1741 \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-3/2} \\ f_{n-1} \\ f_n \end{bmatrix}$$

Using matrix inversion

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+3/2} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_n \end{bmatrix} + h \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3/2 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y'_{n-1} \\ y'_{n-2} \\ y'_{n-3} \\ y'_n \end{bmatrix} + h^2 \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 9/8 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 9/2 \end{bmatrix} \begin{bmatrix} y''_{n-1} \\ y''_{n-2} \\ y''_{n-3} \\ y''_n \end{bmatrix}$$

$$+h^3 \begin{bmatrix} \frac{67}{336} & \frac{-188}{945} & \frac{11}{140} & \frac{-83}{15120} \\ \frac{880}{1213} & \frac{-9}{14} & \frac{145}{578} & \frac{-157}{9061} \\ \frac{58}{35} & \frac{-1216}{945} & \frac{11}{21} & \frac{-34}{945} \\ \frac{2673}{560} & \frac{-108}{35} & \frac{243}{140} & \frac{-9}{112} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+3/2} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} + h^3 \begin{bmatrix} 0 & 0 & 0 & \frac{88}{945} \\ 0 & 0 & 0 & \frac{337}{1368} \\ 0 & 0 & 0 & \frac{449}{945} \\ 0 & 0 & 0 & \frac{81}{70} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ f_n \end{bmatrix}$$

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Writing out (30) explicitly, we have

$$y_{n+1} = y_n + hy'_n + \left(\frac{1}{2}\right)h^2 y''_n + h^3\left(\frac{88}{945} f_n + \frac{67}{336} f_{n+1} - \frac{118}{945} f_{n+3/2} + \frac{11}{140} f_{n+2} - \frac{83}{15120} f_{n+3}\right)$$

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$$y_{n+3/2} = y_n + \frac{3}{2}hy'_n + \frac{9}{8}h^2 y''_n + h^3\left(\frac{337}{1368} f_n + \frac{880}{1213} f_{n+1} - \frac{9}{14} f_{n+3/2} + \frac{145}{578} f_{n+2} - \frac{157}{9061} f_{n+3}\right)$$

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$$y_{n+2} = y_n + 2hy_n' + 2h^2 y_n'' + h^3\left(\frac{449}{945}f_n + \frac{58}{35}f_{n+1} - \frac{1216}{945}f_{n+3/2} + \frac{11}{21}f_{n+2} - \frac{34}{945}f_{n+3}\right)$$

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$$y_{n+3} = y_n + 3hy_n' + \frac{9}{2}h^2 y_n'' + h^3\left(\frac{81}{70}f_n + \frac{2673}{560}f_{n+1} - \frac{108}{35}f_{n+3/2} + \frac{243}{140}f_{n+2} - \frac{9}{112}f_{n+3}\right)$$

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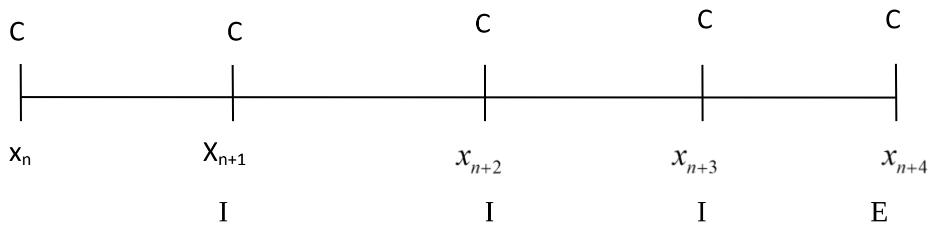
Substituting (31)-(34) into (19)-(23) gives the following

$$\begin{aligned}
y'_{n+1} &= y'_n + hy''_n + \frac{h^2}{360}\left[83f_n + 240f_{n+1} - 224f_{n+3/2} + 87f_{n+2} - 6f_{n+3}\right]\\
y'_{n+3/2} &= y'_n + \frac{3h}{2}y''_n + \frac{h^2}{1280}\left[489f_n + 1863f_{n+1} - 1440f_{n+3/2} + 567f_{n+2} - 39f_{n+3}\right]\\
y'_{n+2} &= y'_n + 2hy''_n + \frac{h^2}{45}\left[24f_n + 102f_{n+1} - 64f_{n+3/2} + 30f_{n+2} - 2f_{n+3}\right]\\
y'_{n+3} &= y'_n + 3hy''_n + \frac{h^2}{40}\left[33f_n + 162f_{n+1} - 96f_{n+3/2} + 81f_{n+2}\right]
\end{aligned} \qquad 35$$

While substituting (31)-(34) into (24)-(28) gives the following equations:

$$\begin{aligned}
y''_{n+1} &= y''_n + \frac{h}{1080}\left[329f_n + 1539f_{n+1} - 1216f_{n+3/2} + 452f_{n+2} - 31f_{n+3}\right]\\
y''_{n+3/2} &= y''_n + \frac{h}{640}\left[193f_n + 1053f_{n+1} - 512f_{n+3/2} + 243f_{n+2} - 17f_{n+3}\right]\\
y''_{n+2} &= y''_n + \frac{h}{135}\left[41f_n + 216f_{n+1} - 64f_{n+3/2} + 81f_{n+2} - 4f_{n+3}\right]\\
y''_{n+3} &= y''_n + \frac{h}{40}\left[11f_n + 81f_{n+1} - 64f_{n+3/2} + 81f_{n+2} + 11f_{n+3}\right]
\end{aligned} \qquad 36$$

5. Derivation of Four-Step Non-Hybrid Scheme



Collocating and interpolating (1) and (8) respectively at the points indicated in the sketch above yields a system of equations with unknown $a_{i,} i = 0(1)7$. Using the same method of solution of scheme one, we obtained the second scheme as:

$$y_{n+4} - 3y_{n+3} + 3y_{n+2} - y_{n+1} = \frac{h^3}{240}\left[f_{n+4} + 116 f_{n+3} + 126 f_{n+2} - 4 f_{n+1} + f_n \right] \quad 37$$

With the order $p = 6,$ Error constant $C_{p+2} = 2.08333 \times 10^{-3}$.

Evaluating at x_n also gives the scheme

$$y_n - 3y_{n+1} + 3y_{n+2} - y_{n+3} = \frac{h^3}{240}\left[- f_{n+4} + 4 f_{n+3} - 126 f_{n+2} - 116 f_{n+1} - f_n \right] \quad 38$$

Evaluating the first derivative of the continuous scheme at all the grid points gives the following schemes:

$$y'_n = \frac{1}{h}\left(-\frac{5}{2} y_{n+1} + 4y_{n+2} - \frac{3}{2} y_{n+3} \right) + \frac{h^2}{10080}\left[677 f_n + 10480 f_{n+1} + 7254 f_{n+2} + 64 f_{n+3} + 5 f_{n+4} \right] \quad 39$$

$$y'_{n+1} = \frac{1}{h}\left(-\frac{3}{2} y_{n+1} + 2y_{n+2} - \frac{1}{2} y_{n+3} \right) + \frac{h^2}{5040}\left[-29 f_n + 452 f_{n+1} + 1296 f_{n+2} - 52 f_{n+3} + 13 f_{n+4} \right] \quad 40$$

$$y'_{n+2} = \frac{1}{h}\left(-\frac{1}{2} y_{n+1} + \frac{1}{2} y_{n+3} \right) + \frac{h^2}{10080}\left[5 f_n - 104 f_{n+1} - 1482 f_{n+2} - 104 f_{n+3} + 5 f_{n+4} \right] \quad 41$$

$$y'_{n+3} = \frac{1}{h}\left(\frac{1}{2} y_{n+1} - 2y_{n+2} + \frac{3}{2} y_{n+3} \right) + \frac{h^2}{5040}\left[13 f_{n+} - 52 f_{n+1} + 1296 f_{n+2} + 452 f_{n+3} - 29 f_{n+4} \right] \quad 42$$

$$y'_{n+4} = \frac{1}{h}\left(\frac{3}{2} y_{n+1} - 4y_{n+2} + \frac{5}{2} y_{n+3} \right) + \frac{h^2}{10080}\left[5 f_n + 64 f_{n+1} + 7254 f_{n+2} + 10480 f_{n+3} + 677 f_{n+4} \right] \quad 43$$

While the evaluation of the second derivatives of the continuous scheme at all the grid points yields the following schemes:

$$y''_n = \frac{1}{h^2}\left(y_{n+1} - 2y_{n+2} + 3y_{n+3} \right) + \frac{h}{360}\left[-118 f_n - 477 f_{n+1} - 96 f_{n+2} - 35 f_{n+3} + 6 f_{n+4} \right] \quad 44$$

$$y''_{n+1} = \frac{1}{h^2}\left(y_{n+1} - 2y_{n+2} + y_{n+3} \right) + \frac{h}{720}\left[15 f_n - 308 f_{n+1} - 456 f_{n+2} + 36 f_{n+3} - 7 f_{n+4} \right] \quad 45$$

$$y''_{n+2} = \frac{1}{h^2}\left(y_{n+1} - 2y_{n+2} + y_{n+3} \right) + \frac{h}{360}\left[-2 f_n + 19 f_{n+1} - 19 f_{n+3} + 2 f_{n+4} \right] \quad 46$$

$$y''_{n+3} = \frac{1}{h^2}\left(y_{n+1} - 2y_{n+2} + y_{n+3} \right) + \frac{h}{720}\left[7 f_n - 36 f_{n+1} + 456 f_{n+2} + 308 f_{n+3} - 15 f_{n+4} \right] \quad 47$$

$$y''_{n+4} = \frac{1}{h^2}\left(y_{n+1} - 2y_{n+2} + y_{n+3} \right) + \frac{h}{360}\left[-6 f_n + 35 f_{n+1} + 96 f_{n+2} + 477 f_{n+3} 118 f_{n+4} \right] \quad 48$$

The combination of equation (37), (38), (39) and (44), gives the block which after using matrix inversion yields the following equations:

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2} y''_n + h^3 \left[\frac{113}{1120} f_n + \frac{107}{1008} f_{n+1} - \frac{103}{1680} f_{n+2} + \frac{43}{1680} f_{n+3} - \frac{47}{10080} f_{n+4} \right]$$

$$y_{n+2} = y_n + 2hy'_n + 2h^2 y''_n + h^3 \left[\frac{331}{630} f_n + \frac{332}{315} f_{n+1} - \frac{8}{21} f_{n+2} + \frac{52}{315} f_{n+3} - \frac{19}{630} f_{n+4} \right] \qquad 49$$

$$y_{n+3} = y_n + 3hy'_n + \frac{9}{2} h^2 y''_n + h^3 \left[\frac{1431}{1120} f_n + \frac{1683}{560} f_{n+1} - \frac{243}{560} f_{n+2} + \frac{45}{112} f_{n+3} - \frac{81}{1120} f_{n+4} \right]$$

$$y_{n+4} = y_n + 4hy'_n + 8h^2 y''_n + h^3 \left[\frac{248}{105} f_n + \frac{2176}{315} f_{n+1} + \frac{32}{105} f_{n+2} + \frac{128}{105} f_{n+3} - \frac{8}{63} f_{n+4} \right]$$

Substituting equation (49) into equations (39)-(43) and (44)-(48) yield the following equations:

$$y'_{n+1} = y'_n + hy''_n + \frac{h^2}{1440} \left[367 f_n + 540 f_{n+1} - 282 f_{n+2} + 116 f_{n+3} - 21 f_{n+4} \right]$$

$$y'_{n+2} = y'_n + 2hy''_n + \frac{h^2}{90} \left[35 f_n + 144 f_{n+1} - 30 f_{n+2} + 16 f_{n+3} - 3 f_{n+4} \right]$$

$$y'_{n+3} = y'_n + 3hy''_n + \frac{h^2}{160} \left[147 f_n + 468 f_{n+2} + 54 f_{n+2} + 60 f_{n+3} - 9 f_{n+4} \right]$$

$$y'_{n+4} = y'_n + 4hy''_n + \frac{h^2}{45} \left[56 f_n + 192 f_{n+1} + 48 f_{n+2} + 64 f_{n+3} \right]$$

$$y''_n = y''_n + \frac{h}{720} \left[251 f_n + 646 f_{n+1} - 264 f_{n+2} + 106 f_{n+3} - 19 f_{n+4} \right] \qquad 51$$

$$y''_{n+2} = y''_n + \frac{h}{90} \left[29 f_n + 124 f_{n+1} + 24 f_{n+2} + 4 f_{n+3} - f_{n+4} \right]$$

$$y''_{n+3} = y''_n + \frac{h}{80} \left[27 f_n + 102 f_{n+1} + 72 f_{n+2} + 42 f_{n+3} - 3 f_{n+4} \right]$$

$$y''_{n+4} = y''_{n+} \frac{h}{45} \left[14 f_n + 64 f_{n+1} + 24 f_{n+2} + 64 f_{n+3} + 14 f_{n+4} \right]$$

Finally, we use Taylor series expansion to calculate the values of $y_{n+i,} i = 1(1)3$ and their first and second derivatives

at $x = x_n$ In (31)-(34), (35) and (36);

$$y_{n+i} = y(xn+h) = y_{(xn)} + ihy'_{(xn)} + \frac{(ih)^2}{2!}y''_{(xn)} + \frac{(ih)^3}{3!}f_n + \frac{(ih)^4}{4!}f'_n + \frac{(ih)^5}{5!}f''_n + \frac{(ih)^6}{6!}f'''_n + ...$$
$$y'_{n+i} = y'(xn+ih) = y'_{(xn)} + ihy''_{(xn)} + \frac{(ih)^2}{2!}f_n + \frac{(ih)^3}{3!}f'_n + \frac{(ih)^4}{4!}f''_n + \frac{(ih)^5}{5!}f''' + \frac{(ih)^6}{6!}f^{iv}_n + ...$$
$$y''_{n+i} = y''(xn+ih) = y''_{(xn)} + ihf_n + \frac{(ih)^2}{2!}f'_n + \frac{(ih)^3}{3!}f''_n + \frac{(ih)^4}{4!}f'''_n + \frac{(ih)^5}{5!}f^{iv}_n + \frac{(ih)^6}{6!}f^{v}_n + ...$$
$$f_{n+i} = y'''(xn+ih) = f_n + ihf'_n + \frac{(ih)^2}{2!}f''_n + \frac{(ih)^3}{3!}f'''_n + \frac{(ih)^4}{4!}f^{iv}_n + \frac{(ih)^5}{5!}f^{v}_n + \frac{(ih)^6}{6!}f^{iv}_n + ... \quad 52$$

6. Analysis of the Block

6.1 Order of the Block

In this section, we discuss the estimation of the order and error constant of the block with the difference equation of the form:

$$L[y(x),h] = \sum_{j=1}^{k} a_j y(x+jh) - h^3 \sum_{j=0}^{k} b_j y^m(x+jh) \quad 53$$

If we assume that $y(x)$ has as many higher derivatives as we require, we can expand the terms in (45) as a Taylor series about the point x to obtain the expansion;

$$L[y(x),h] = C_0 y(x_n) + C_1 hy'(x_n) + C_2 h^2 y''(x_n) + C_3 h^3 y'''(x_n) + ... + C_p h^p y^{(p)}(x_n) \quad 54$$

Where the constant coefficient $c_q, q = 0,1,.......$ are given as follows

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j$$

$$C_q = \frac{1}{q!}[\sum_{j=1}^{k} j^q \alpha_j - q(q-1) \sum_{j=1}^{k} j^{q-3}\beta_j]$$

Definition: The method (53) are said to be of order p if, in (54) $C_0 = C_1 = C_2 = C_3 = ...C_P = C_{P+1} = 0$ and $C_{p+2} \neq 0$. Thus C_{P+2} is the error constant.

For our hybrid method, expanding (4.1)-(4.4) in Taylor series expansion gives

$$\begin{bmatrix} \sum_q^{\infty}\left(\frac{h^q}{q!}y^q\right)-\left(y_{n+1}-y_n-hy'_n-\frac{h^2}{2}y''-\frac{88}{945}h^3y'''\right) \\ -\sum_q^{\infty}\left(\frac{h^{q+3}}{q!}y^{q+3}\right)\left(-\frac{67}{336}(1)^q+\frac{188}{945}(3/2)^q-\frac{11}{140}(2)^q+\frac{83}{15120}(3)^q\right) \\ \sum_q^{\infty}\left(\frac{(3/2\,h)^q}{q!}y^q\right)-\left(y_{n+3/2}-y_n-\frac{3}{2}hy'_n-\frac{9}{8}h^2y''_n-\frac{882}{35840}h^3y'''_n\right) \\ -\sum_q^{\infty}\left(\frac{h^{q+3}}{q!}y^{q+3}\right)\left(-\frac{26001}{35840}(1)^q+\frac{9}{14}(3/2)^q-\frac{8991}{35840}(2)^q+\frac{621}{35840}(3)^q\right) \\ \sum_q^{\infty}\left(\frac{(2h)^q}{q!}y^q\right)-\left(y_{n+2}-y_n-2hy'_n-2h^2y''_n-\frac{449h^3}{945}y'''_n\right) \\ -\sum_q^{\infty}\left(\frac{h^{q+3}}{q!}y^q\right)\left(-\frac{58}{35}(1)^q+\frac{1216}{945}(3/2)^q-\frac{11}{21}(2)^q+\frac{34}{945}(3)^q\right) \\ \sum_q^{\infty}\left(\frac{(3h)^q}{q!}y^q\right)-\left(y_{n+3}-y_n-3hy'_n-\frac{9h^2}{14}-\frac{81h^3}{70}\right) \\ -\sum_q^{\infty}\left(\frac{h^{q+3}}{q!}y^q\right)\left(-\frac{2673}{560}(1)^q+\frac{108}{35}(3/2)^q-\frac{243}{140}(2)^q+\frac{9}{112}(3)^q\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the block is of order 6, with error constant of $\left[-\frac{47}{1400},\frac{3461}{1000000},-\frac{38}{175},\frac{1161}{1960}\right]^T$

While for the non-hybrid method, we expand equation (49) in Taylor series as done above. The block is also of order 6, with error constant $\left[\frac{139}{40320},\frac{1}{45},\frac{243}{4480},\frac{32}{315}\right]^T$

6.2 Zero Stability of the Block

Definition: The block is said to be zero stable if the roots $z_{s,}s=1,2,3,\ldots,n$ of the characteristics polynomial $\rho(z)$ defined by $\rho(z)=\det\left(zA-E\right)$ satisfies $|z_s|\le 1$ and the roots $|z_s|=1$ is simple.

For our hybrid method,

$$A=\left[z\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right]=0$$

$$A=z^4-z^3=0, z=0,0,0,1$$

Hence the block is zero stable. The non-hybrid method is also zero stable.

Theorem 1: Convergence (Lambert (1973)
The necessary and sufficient condition for a linear multistep method to be convergent is for it to be

consistent and zero stable. From the theorem above, the two block methods are convergent

7. Numerical Examples

Our methods are adopted on some initial value problems of general third order ordinary differential equations to test the accuracy of the scheme and our results are compared with the results of other researchers. Our scheme is found to compare favorably with the existing ones.

Problem 1: We consider the non-linear IVP which was solved by Awoyemi (2003) for the step-size h=0.1

$$y''' = -y,\ y(0)=1,\ y'(0)=-1, y''(0)=1, h=0.1$$

Exact solution: $y(x)=e^{-x}$

In this example, our method of order p=6 is compared with the method in Awoyemi (2003). In terms of accuracy, our result performs better than those given in Awoyemi (2003). The details of the numerical result at some selected points are given in table 1 below

Table 1: Comparison of the numerical results of hybrid and non-hybrid method with Awoyemi (2003) for problem 1.

X	Exact-Solution	Computed-Solution	Error in Hybrid K=3, h=0.1 Order=6,	Error in Non-hybrid K=4, h=0.1, Order=6	Error in Awoyemi (2003) K=3,Order=5, h=0.1 (P-C) method
0.10	0.904837418035959520	0.904837418026245840	9.713674E-12	4.038414E-11	4.165922496E-09
0.20	0.818730753077981820	0.818730753015859850	6.212197E-11	2.576643E-10	9.577831017E-08
0.30	0.740818220681717770	0.740818220522834860	1.588829E-10	5.283249E-10	3.991507930E-07
0.40	0.670320046035639330	0.670320045749979720	2.856596E-10	2.272178E-10	1.036855911E-06
0.50	0.606530659712633420	0.606530659251041330	4.615921E-10	2.416383E-10	2.128500409E-06
0.60	0.548811636094026390	0.548811635406465050	6.875613E-10	1.468228E-09	3.789530170E-06
0.70	0.496585303791409470	0.496585302838955580	9.524539E-10	4.085009E-09	6.130076711E-06
0.80	0.449328964117221560	0.449328962847427850	1.269794E-09	6.779826E-09	9.253856792E-06
0.90	0.406569659740599050	0.406569658101199830	1.639399E-09	1.099445E-08	1.325713611E-05
1.00	0.367879441171442220	0.367879439119434160	2.052008E-09	1.651192E-08	1.822776743E-05
1.10	0.332871083698079500	0.332871081181678460	2.516401E-09	2.337009E-08	2.424431283E-05
1.20	0.301194211912202030	0.301194208881219510	3.030983E-09	3.159917E-08	3.137525869E-05

Problem 2:

$$y''' = y'\left(2xy''+y'\right),\ y(0)=1,\ y'(0)=\frac{1}{2}, y''(0)=0, h=0.01$$

Exact solution: $y(x)=1+\frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Our result was compared with Adesanya 2011 which is of order 6. Using thesame step size (h=0.01), it is observed that our result performs better. The details of the numerical result at some selected points are in table 2 below:

Table 2: Comparison of the numerical results of hybrid and non-hybrid method with Adesanya (2011) for problem 2.

X	Exact Solution	Computed Solution	Error in Hybrid k=3, h=0.01, Order=6	Error in Non-hybrid K=4, h=0.01, Order, P=6	Error in Adesanya 2011 K=3,Order =6, h=0.01 (block method)
0.21	1.105388447838498800	1.105388447838482300	1.643130E-14	2.042810E-14	8.037948 E – 11
0.31	1.156259497799360100	1.156259497799276400	8.371082E-14	1.023626E-13	6.043090 E – 10
0.41	1.207946365635211800	1.207946365634930500	2.813305E-13	3.421707E-13	2.581908 E – 09
0.51	1.260753316593162600	1.260753316592395900	7.667200E-13	9.259260E-13	8.158301E – 09
0.61	1.315023237096001100	1.315023237094148600	1.852518E-12	2.214229E-12	2.141286 E – 08
0.71	1.371153208259014500	1.371153208254851700	4.162892E-12	4.922729E-12	4.969641 E – 08
0.81	1.429615588111108300	1.429615588102143500	8.964829E-12	1.047873E-11	1.620387 E – 07

Problem 3:

$$y''' = 3\sin x\,,\; y(0)=1,\; y'(0)=0, y''(0)=-2, h=0.1$$

Exact solution: $y(x)=3\cos x+\frac{x^2}{2}-2$

Our result was compared with Olabode 2007 which is of order 5. Using thesame step size (h=0.1), it is observed that our result is more accurate. The details of the numerical result at some selected points are in table 2 below:

Table 3: comparison of the numerical results of hybrid and non-hybrid method with Olabode (2007) for problem 3.

X	Exact-Solution	Computed-Solution	Error in Hybrid K=3, h=0.1, Order, P=6	Error in Non-hybrid K=4, h=0.1, Order, P=6	Error in Olabode (2007) block method K=3,h=0.1, Order P=5
0.10	0.990012495834077020	0.990012495800901000	3.317602E-11	1.017524E-10	1.6592250E-10
0.20	0.960199733523725120	0.960199733311422500	2.123026E-10	6.558604E-10	4.7627491E-10
0.30	0.911009467376818090	0.911009466839369340	5.374488E-10	1.601112E-09	6.2318195E-10
0.40	0.843182982008655380	0.843182981081771250	9.268841E-10	2.998699E-09	2.9134462E-10
0.50	0.757747685671118280	0.757747684220156700	1.450962E-09	5.670582E-09	8.7111829E-10
0.60	0.656006844729034810	0.656006842619148880	2.109886E-09	9.092473E-09	3.9290352E-09
0.70	0.539526561853465480	0.539526559026371920	2.827094E-09	1.347564E-08	9.5534655E-09
0.80	0.410120128041496560	0.410120124384088290	3.657408E-09	1.950064E-08	1.8041497E-08
0.90	0.269829904811993430	0.269829900210842190	4.601151E-09	2.673002E-08	3.0311993E-08
1.00	0.120906917604418850	0.120906912010654510	5.593764E-09	3.533854E-08	4.7304419E-08
1.10	0.034211635723267797	0.034211642392915083	6.669647E-09	4.580460E-08	7.0036732E-08
1.20	0.19292673656997961	0.192926744399186780	7.829207E-09	5.781661E-08	9.9630021E-08

8. Discussion of Result

We have proposed two direct methods for the solution of general third order initial value problems of ordinary differential equations using the three-step hybrid block method and the four-step block method. In table 1, the results of the three-step hybrid block method and the four-step block method are more accurate than that of Awoyemi 2003 which was executed by predictor-corrector method. It was also seen that the three-step hybrid block method has the best result.

In table 2, the two methods also perform better than Adesanya 2011 block method implemented scheme. Though we used the same parameters with that of Adesanya 2011 that is, order, P=6, K=3 and h=0.01 except for the step length, K=4 for scheme two, our methods are still more accurate. The three-step hybrid block method also has the best performance.

Table 3 shows the comparison of the results of our two methods with that of Olabode 2007 block method implemented scheme. It is only the three-step hybrid block method that has better accuracy than that of Olabode 2007.

9. Conclusion

In this study, we have shown that the hybrid scheme solves third order ordinary differential equation more accurately than non-hybrid four step scheme. Hence, the hybrid block method is recommended for the general solution of third order initial value problem ordinary differential equation.

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