

Generalized Jordan (σ,τ) - Higher Homomorphisms of a ring R into a ring R'

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Abstract

Let R , R' be two prime rings and σ^n,τ^n be two higher homomorphisms of a ring R for all n $\in$ N , in the present paper we show that under certain conditions of R , every generalized Jordan (σ,τ)-higher homomorphism of a ring R into a prime ring R' is either generalized (σ,τ)-higher homomorphism or (σ,τ)-higher anti homomorphism.

Key Words: prime ring , homomorphism , generalized Jordan higher homomorphism.
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1.Introduction

A ring R is called a prime if aRb = (0) implies a = 0 or b = 0, where a , b $\in$ R, this definition is due to [2] , [5].

A -ring R is called semiprime if aRa = (0) implies = 0, such that $a \in$ R, this definition is due to [2] , [5].

Let R be a 2-torsion free semiprime ring and suppose that a , b $\in$ R if $a$rb + br$a$ = 0 , for all r $\in$ R, then $a$rb = br$a$ = 0 , this definition is due to [2].

Let R be a ring then R is called 2-torsion free if 2a = 0 implies a = 0, for every $a \in$ R, this definition is due to [2].

Let R be a ring and d: R $\longrightarrow$ R be an additive mapping (that is d(a + b) = d(a) + d(b)) of a ring R into itself then d is called a derivation on R if :

d(ab) = d(a) b + ad(b), for all a , b $\in$ R.

d is called a Jordan derivation on R if $d(a^2) = d(a)\,a + a\,d(a)$, for all $a \in$ R, [2], [4].

Let R be a ring and f: R $\longrightarrow$ R be an additive map (that is f (a + b) = f (a) + f (b)) Then f is called a generalized derivation if there exists a derivation d: R $\longrightarrow$ R such that

f (ab) = f (a) b + ad(b), for all a , b $\in$ R.

And f is called a generalized Jordan derivation if there exists a Jordan derivation d: R$\longrightarrow$ R such that

$f(a^2) = f(a)\,a + a\,d(a)$, for all a $\in$ R, [2].

Let θ be an additive mapping of a ring R into a ring R', θ is called a homomorphism if $\theta(a\,b) = \theta(a)\,\theta(b)$.

And θ is called a Jordan homomorphism if for all a, b $\in$ R

$\theta(a\,b + b\,a) = \theta(a)\,\theta(b) + \theta(b)\,\theta(a)$ for all a , b $\in$ R , [2],[3] .

Let F be an additive mapping of a ring R into a ring R', F is called a generalized homomorphism if there exists a homomorphism θ from a ring R into a ring R', such that $F(ab) = F(a)\,\theta(b)$, for all $a, b \in R$, where θ is called the relating homomorphism .

F is called a generalized Jordan homomorphism if there exists a Jordan homomorphism θ from a ring R into a ring R', such that

$F(ab + ba) = F(a)\,\theta(b) + F(b)\,\theta(a)$, for all $a, b \in R$, where θ is called the relating Jordan homomorphism , [1] .

Let $F = (f_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R' and there exists a higher homomorphism $\theta = (\phi_i)_{i\in N}$ from a ring R into a ring R' then f is said to be a generalized higher homomorphism (resp. generalized Jordan higher homomorphism) on a ring R into a ring R' if for all n$\in$N, (resp. Jordan higher homomorphism) on a ring R into a ring R' if for all n$\in$N , $f_n(ab) = \sum_{i=1}^{n} f_i(a)\,\phi_i(b)$ $\left(\text{resp. } f_n(ab + ba) = \sum_{i=1}^{n} f_i(a)\,\phi_i(b) + \sum_{i=1}^{n} f_i(b)\,\phi_i(a) \right)$, for all a, b $\in$ R , [1].

Now, the main purpose of this paper is that every generalized Jordan (σ,τ)-higher homomorphism of a ring R into a prime ring R' is either generalized (σ,τ)-higher homomorphism or (σ,τ)- higher anti homomorphism and every generalized Jordan (σ,τ)-higher homomorphism from a ring R into a 2-torsion free ring R' is a generalized Jordan triple (σ,τ)-higher homomorphism.

2. Generalized Jordan (σ,τ)- Higher Homomorphisms On Rings

Definition(2.1):

Let $F = (f_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R' and σ,τ be two homomorphisms of a ring R. F is called a **generalized (σ,τ)-higher homomorphism** if there exists a (σ,τ)-higher homomorphism $\theta = (\phi_i)_{i\in N}$ from a ring R into a ring R', such that for all $a, b \in M$ and n $\in$ N

$$f_n(ab) = \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(b))$$

Where θ is called the **relating (σ,τ)-higher homomorphism**.

Example (2.2):

Let S_1, S_2 be two rings and $f = (f_i)_{i\in N}$ be a generalized (σ,τ)-higher homomorphism from a ring S_1 into a ring S_2, then there exists a higher homomorphism $\theta = (\theta_i)_{i\in N}$ from a ring S_1 into a ring S_2, such that for all n∈ N $f_n(ab) = \sum_{i=1}^{n} f_i(\sigma^i(a))\theta_i(\tau^i(b))$, for all $a, b \in S_1$.

Let $R= S_1\oplus S_1$ and $R'= S_2\oplus S_2$. Let $F = (F_i)_{i\in N}$ be a family of additive mappings from a ring R into a ring R', such that for all $a, b \in R$

$F_n((a,b)) = (f_n(a) \quad f_n(b))$.

Then there exists a (σ,τ)-higher homomorphism $\phi = (\phi_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R', such that

$\phi_n((a,b)) = (\theta_n(a),\theta_n(b))$, for all $(a,b) \in R$.

Let σ_1^n, τ_1^n be two homomorphism of a ring R, such that

$\sigma_1^n((a,b)) = (\sigma^n(a),\sigma^n(b))$, $\tau_1^n((a,b)) = (\tau^n(a),\tau^n(b))$.

be two homomorphism of a ring R, such that

$\sigma_1^n((a,b)) = (\sigma^n(a),\sigma^n(b))$, $\tau_1^n((a,b)) = (\tau^n(a),\tau^n(b))$.

Then F generalized (σ,τ)-higher homomorphism .

Definition(2.3):

Let $F = (f_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R' and σ,τ be two homomorphisms of a ring R . F is called a **generalized Jordan (σ,τ)-higher homomorphism** if there exists a Jordan (σ,τ)-higher homomorphism $\theta = (\phi_i)_{i\in N}$ from a ring R into a ring R', such that for all $a, b \in R$ and $n \in N$

$$f_n(ab + ba) = \sum_{i=1}^{n} f_i(\sigma^i(a))\,\phi_i(\tau^i(b)) + \sum_{i=1}^{n} f_i(\sigma^i(b))\,\phi_i(\tau^i(a))$$

Where θ is called the **relating Jordan (σ,τ)-higher homomorphism**.

Definition (2.4):

Let $F = (f_i)_{i\in N}$ be a family of additive mappings of a ring R into a ring R' and σ,τ be two homomorphisms of a ring R. F is called a **generalized Jordan triple (σ,τ)-higher homomorphism** if there exists a Jordan triple (σ,τ)-higher homomorphism $\theta = (\phi_i)_{i\in N}$ from a ring R into a ring R', such that for all $a, b \in R$ and $n \in N$

$$f_n(aba) = \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$$

Where θ is called the **relating Jordan triple (σ,τ)-higher homomorphism**.

Definition (2.5):

Let F = (f_i)$_{i\in N}$ be a family of additive mappings of a ring R into a ring R' and σ,τ be two homomorphisms of a ring R. F is called a **generalized (σ,τ)-higher anti homomorphism** if there exists a (σ,τ)-anti higher homomorphism θ = $(\phi_i)_{i\in N}$ from a ring R into a ring R', such that for all *a*, *b* ∈ R and n ∈ N, we have:

$$f_n(ab) = \sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\tau^i(a))$$

Where θ is called the **relating (σ,τ)-higher anti homomorphism**.

Lemma (2.6):

Let F = (f_i)$_{i\in N}$ be a generalized Jordan (σ,τ)-higher homomorphism of a ring R into a ring R', then for all *a*, *b*, *c* ∈ R and for every n ∈ N

(i)
$$f_n(abc + cba) = \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) + \sum_{i=1}^{n} f_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$$

(ii) In particular , if R, R' are commutative rings and R' is a 2-torsion free ring

$$f_n(abc) = \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$$

Proof:

(i) Replace $a + c$ for a in definition (2.4), we get :

$$f_n((a+c)b(a+c)) = \sum_{i=1}^{n} f_i(\sigma^i(a+c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a+c))$$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a) + \sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a) + \tau^i(c))$$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a)\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(a)\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$$
$$= \sum_{i=1}^{n} f_i(\sigma^i(c)\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(c)\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))$$

…(1)

On the other hand:

$f_n((a+c)b(a+c)) = f_n(aba + abc + cba + cbc)$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)) + f_n(abc + cba)$$

…(2)

Compare (1) and (2), we get:

$$f_n(abc+cba)=\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c))+$$
$$\sum_{i=1}^{n} f_i(\sigma^i(c))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(a))$$

(ii) By (i) and since R , R' are commutative rings and R' is a 2-torsion free ring

$$f_n(abc+abc)=2f_n(abc)=\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(b))\phi_i(\tau^i(c)).$$

Definition (2.7):

Let F = (f_i)$_{i\in N}$ be a generalized Jordan (σ,τ)-higher homomorphism from a ring R into a ring R', then for all a, b ∈ R and n ∈ N, we define $\delta_n(a,b)$:R×R ⟶R' by:

$$\delta_n(a,b)=f_n(ab)-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(b)).$$

Lemma (2.8):

Let F = (f_i)$_{i\in N}$ be a generalized Jordan (σ,τ)-higher homomorphism from a ring R into a ring R', then for all a, b, c ∈ R and n ∈ N:

(i) $\delta_n(a,b) = -\delta_n(b,a)$

(ii) $\delta_n(a+b,c) = \delta_n(a,c) + \delta_n(b,c)$

(iii) $\delta_n(a,b+c) = \delta_n(a,b) + \delta_n(a,c)$

Proof:

(i) By Definition(2.3)

$$f_n(ab+ba)=\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(b))+\sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\tau^i(a))$$

$$f_n(ab)-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(b))=-(f_n(ba)+\sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\tau^i(a)))$$

$$\delta_n(a,b) = -\delta_n(b,a)$$

(ii) $$\delta_n(a+b,c)=f_n((a+b)c)-\sum_{i=1}^{n} f_i(\sigma^i(a+b))\phi_i(\tau^i(c))$$

$$=f_n(ac+bc)-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(c))-\sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\tau^i(c))$$

$$=f_n(ac)-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(c))+f_n(bc)-\sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\tau^i(c))$$

$= \delta_n(a,c) + \delta_n(b,c)$

(iii) $$\delta_n(a,b+c)=f_n(a(b+c))-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(b+c))$$

$$=f_n(ab+ac)-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(b))-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(c))$$

$$=f_n(ab)-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(b))+f_n(ac))-\sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(c))$$

$= \delta_n(a,b) + \delta_n(a,c)$

Remark (2.9):

Note that $F = (f_i)_{i\in N}$ is a generalized (σ,τ)-higher homomorphism from a ring R into a ring R' if and only if $\delta_n(a,b) = 0$ for all $a, b \in R$ and $n \in N$.

Lemma (2.10):

Let $F = (f_i)_{i\in N}$ be a generalized Jordan (σ,τ) - higher homomorphism of a ring R into a ring R', such that $\sigma^{n^2} = \sigma^n$, $\tau^n\sigma^n = \sigma^n$, $\sigma^i\tau^{n-i} = \tau^i\sigma^i$ and $\sigma^i\tau^i = \tau^i\sigma^i$ for all $i \in N$, then for all $a, b, m \in R$ and $n \in N$

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$\delta_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) = 0$$

Proof: We prove by using the induction, we can assume that:

$$\delta_s(\sigma^s(a),\sigma^s(b))\phi_s(\sigma^s(m))G_s(\tau^s(b),\tau^s(a)) +$$
$$\delta_s(\sigma^s(b),\sigma^s(a))\phi_s(\sigma^s(m))G_s(\tau^s(a),\tau^s(b)) = 0$$

for all $a, b, m \in R$, and $s, n \in N$, $s < n$.

Let $w = abmba + bamab$, since F is a generalized Jordan (σ,τ)-higher homomorphism

$$f_n(w) = f_n(a(bmb)a + b(ama)b)$$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(bmb))\phi_i(\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(ama))\phi_i(\tau^i(b))$$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\left(\sum_{j=1}^{i}\phi_j(\sigma^j(\sigma^j\tau^{n-j}(b)))\phi_j(\sigma^j\tau^{n-j}(\sigma^j\tau^{n-j}(m)))\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\right)\phi_i(\tau^i(a)) +$$
$$\sum_{i=1}^{n} f_i(\sigma^i(b))\left(\sum_{j=1}^{i}\phi_j(\sigma^j(\sigma^j\tau^{n-j}(a)))\phi_j(\sigma^j\tau^{n-j}(\sigma^j\tau^{n-j}(m)))\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\right)\phi_i(\tau^i(b))$$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\phi_i(\tau^i(\sigma^i\tau^{n-i}(b)))\phi_i(\tau^i(a)) +$$
$$\sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\phi_i(\tau^i(\sigma^i\tau^{n-i}(a)))\phi_i(\tau^i(b))$$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\phi_j(\tau^j(a)) +$$
$$\sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(m)))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\phi_j(\tau^j(b))$$

$$= f_n(\sigma^n(a))\phi_n(\sigma^n(\sigma^n(b)))\phi_n(\sigma^n(\sigma^n(\mathrm{m})))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\phi_j(\tau^j(a)) +$$

$$\sum_{i=1}^{n-1} f_i(\sigma^i(a))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(b)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(\mathrm{m})))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\phi_j(\tau^j(a)) +$$

$$f_n(\sigma^n(b))\phi_n(\sigma^n(\sigma^n(a)))\phi_n(\sigma^n(\sigma^n(\mathrm{m})))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\phi_j(\tau^j(b)) +$$

$$\sum_{i=1}^{n-1} f_i(\sigma^i(b))\phi_i(\sigma^i(\sigma^i\tau^{n-i}(a)))\phi_i(\sigma^i\tau^{n-i}(\sigma^i\tau^{n-i}(\mathrm{m})))\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\phi_j(\tau^j(b))\ldots(1)$$

On the other hand:

$f_n(\mathrm{w}) = f_n((ab)\mathrm{m}(ba) + (ba)\mathrm{m}(ab))$

$$= \sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ba)) + \sum_{i=1}^{n} f_i(\sigma^i(ba))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab))$$

$$= \sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\left(\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\phi_j(\tau^{j^2}(b)) + \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\phi_j(\tau^{j^2}(a)) - \phi_i(\tau^i(ab))\right)$$

$$+\sum_{i=1}^{n}\left(\sum_{j=1}^{i} f_j(\sigma^{j^2}(a))\phi_j(\tau^j\sigma^j(b)) + \sum_{j=1}^{i} f_j(\sigma^{j^2}(b))\phi_j(\tau^j\sigma^j(a)) - f_i(\sigma^i(ab))\right)\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab))$$

$$= \sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\phi_j(\tau^{j^2}(b)) + \sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))$$

$$\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\phi_j(\tau^{j^2}(a)) - \sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n} f_i(\sigma^{i^2}(a))$$

$$\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab)) + \sum_{i=1}^{n} f_i(\sigma^{i^2}(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab))$$

$$-\sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab))$$

$$= -\sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))(\phi_i(\tau^i(ab)) - \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\phi_j(\tau^{j^2}(b))) -$$

$$\sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))(\phi_i(\tau^i(ab)) - \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\phi_j(\tau^{j^2}(a))) +$$

$$\sum_{i=1}^{n} f_i(\sigma^{i^2}(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab)) +$$

$$\sum_{i=1}^{n} f_i(\sigma^{i^2}(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab))$$

$$= -\sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))G_i(\tau^i(a),\tau^i(b)) -$$

$$\sum_{i=1}^{n} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))G_i(\tau^i(b),\tau^i(a)) +$$

$$\sum_{i=1}^{n} f_i(\sigma^{i^2}(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab)) +$$

$$\sum_{i=1}^{n} f_i(\sigma^{i^2}(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(\mathrm{m}))\phi_i(\tau^i(ab))$$

$$= -f_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) -$$
$$\sum_{i=1}^{n-1} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) -$$
$$f_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) -$$
$$\sum_{i=1}^{n-1} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(b),\tau^i(a)) +$$
$$f_n(\sigma^{n^2}(a))\phi_n(\tau^n\sigma^n(b))\phi_n(\sigma^n(m))\phi_n(\tau^n(ab)) +$$
$$\sum_{i=1}^{n-1} f_i(\sigma^{i^2}(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab)) +$$
$$f_n(\sigma^{n^2}(b))\phi_n(\tau^n\sigma^n(a))\phi_n(\sigma^n(m))\phi_n(\tau^n(ab)) +$$
$$\sum_{i=1}^{n-1} f_i(\sigma^{i^2}(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))\phi_i(\tau^i(ab))\ldots(2)$$

Compare (1), (2) and since $\sigma^{n^2} = \sigma^n$, $\tau^n\sigma^n = \sigma^n$, $\sigma^i\tau^{n-i} = \tau^i\sigma^i$,$\sigma^i\tau^i = \tau^i\sigma^i$

$$0 = -f_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) - f_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$f_n(\sigma^n(a))\phi_n(\sigma^n(b))\phi_n(\sigma^n(m))(\phi_n(\tau^n(ab)) - \sum_{i=1}^{n}\phi_i(\tau^i(\sigma^i\tau^{n-i}(b)))\phi_i(\tau^i(a))) +$$
$$f_n(\sigma^n(b))\phi_n(\sigma^n(a))\phi_n(\sigma^n(m))(\phi_n(\tau^n(ab)) - \sum_{i=1}^{n}\phi_i(\tau^i(\sigma^i\tau^{n-i}(a)))\phi_i(\tau^i(b))) -$$
$$\sum_{i=1}^{n-1} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) - \sum_{i=1}^{n-1} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))$$
$$G_i(\tau^i(b),\tau^i(a)) + \sum_{i=1}^{n-1} f_i(\sigma^i(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))(\phi_i(\tau^i(ab)) -$$
$$\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(b)))\phi_j(\tau^j(a))) + \sum_{i=1}^{n-1} f_i(\sigma^i(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))(\phi_i(\tau^i(ab)) -$$
$$-\sum_{j=1}^{i}\phi_j(\tau^j(\sigma^j\tau^{n-j}(a)))\phi_j(\tau^j(b)))$$

$$0 = -f_n(\sigma^n(ab))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) - f_n(\sigma^n(ab))\phi_n(\sigma^n(m))$$
$$G_n(\tau^n(b),\tau^n(a)) + f_n(\sigma^n(a))\phi_n(\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) +$$
$$f_n(\sigma^n(b))\phi_n(\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) - \sum_{i=1}^{n-1} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))$$
$$G_i(\tau^i(a),\tau^i(b)) - \sum_{i=1}^{n-1} f_i(\sigma^i(ab))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(b),\tau^i(a)) + \sum_{i=1}^{n-1} f_i(\sigma^i(a))\phi_i(\tau^i\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))$$
$$G_i(\tau^i(b),\tau^i(a)) + \sum_{i=1}^{n-1} f_i(\sigma^i(b))\phi_i(\tau^i\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b))$$

$$0 = -\delta_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) - \delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) -$$
$$\sum_{i=1}^{n-1}\delta_i(\sigma^i(b),\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) - \sum_{i=1}^{n-1}\delta_i(\sigma^i(a),\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(b),\tau^i(a))$$
$$0 = -(\delta_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) +$$
$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))) -$$
$$(\sum_{i=1}^{n-1}\delta_i(\sigma^i(b),\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(a),\tau^i(b)) +$$
$$\sum_{i=1}^{n-1}\delta_i(\sigma^i(a),\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(m))G_i(\tau^i(b),\tau^i(a)))$$

By our hypothesis, we have:

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$\delta_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b)) = 0.$$

Lemma (2.11):

Let F = (f_i)$_{i\in N}$ be a generalized Jordan (σ,τ)-higher homomorphism from a ring R into a 2- torsion free prime ring R', then for all a, b, m ∈ R and n ∈ N

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))=$$
$$\delta_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b))=0$$

Proof:

By Lemma (2.10) , we have:

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$\delta_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b))=0$$

And by Lemma (Let R be a 2-torsion free semiprime ring and suppose that a, b ∈ R if $arb + bra = 0$, for all a ∈ R, then $arb = bra = 0$) .

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a)) =$$
$$\delta_n(\sigma^n(b),\sigma^n(a))\phi_n(\sigma^n(m))G_n(\tau^n(a),\tau^n(b))=0$$

Theorem (2.12):

Let F = (f_i)$_{i\in N}$ be a generalized Jordan (σ,τ)-higher homomorphism from a ring R into a prime ring R', then for all a, b, c, d, m ∈ R, and n ∈ N

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(c))=0$$

Proof:

Replacing $a+c$ for a in Lemma (2.11), we get :

$$\delta_n(\sigma^n(a+c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a+c))=0$$

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))+$$
$$\delta_n(\sigma^n(c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$\delta_n(\sigma^n(c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))=0$$

By Lemma (2.11), we get :

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))+$$
$$\delta_n(\sigma^n(c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))=0$$

Therefore, we get:

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))\phi_n(\sigma^n(m))$$
$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))=0$$

$$=-\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))\phi_n(\sigma^n(m))$$
$$\delta_n(\sigma^n(c),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))=0$$

Since R' is a prime ring and therefore:

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))=0 \qquad \ldots(1)$$

Replacing $b+d$ for b in Lemma (2.11) , we get :

$$\delta_n(\sigma^n(a),\sigma^n(b+d))\phi_n(\sigma^n(m))G_n(\tau^n(b+d),\tau^n(a))=0$$

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))+$$
$$\delta_n(\sigma^n(a),\sigma^n(d))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$\delta_n(\sigma^n(a),\sigma^n(d))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))=0$$

By Lemma (2.11) , we get :

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))+$$
$$\delta_n(\sigma^n(a),\sigma^n(d))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))=0$$

Therefore, we get:

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))_\alpha\,\phi_n(\sigma^n(m))$$
$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))=0$$

$$=-\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))\phi_n(\sigma^n(m))$$
$$\delta_n(\sigma^n(a),\sigma^n(d))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))=0$$

Since R' is a prime ring and therefore:

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))=0 \qquad \ldots(2)$$

Now, $\delta_n(\sigma^n(a),\sigma^n(b))_n(\sigma^n(m))G_n(\tau^n(b+d),\tau^n(a+c))=0$

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(a))+$$
$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(b),\tau^n(c))+$$
$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(a))+$$
$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(c))=0$$

Since by Lemma (2.11) and (1), (2) , we get:

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(c))=0\cdot$$

3.The Main Result

Theorem (3.1):

Every generalized Jordan (σ,τ)-higher homomorphism from a ring R into a prime ring R' is either generalized (σ,τ)-higher homomorphism or (σ,τ)-higher anti homomorphism.

Proof:

Let F = (f_i)$_{i\in N}$ be a generalized Jordan (σ,τ)-higher homomorphism of a ring R into a prime ring R', then by Theorem (2.12):

$$\delta_n(\sigma^n(a),\sigma^n(b))\phi_n(\sigma^n(m))G_n(\tau^n(d),\tau^n(c))=0\cdot$$

Since R' is a prime ring therefore either $\delta_n(\sigma^n(a),\sigma^n(b))=0$

or $G_n(\tau^n(d),\tau^n(c))=0$, for all $a, b, c, d \in$ R and n $\in$ N.

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If $G_n(\tau^n(d),\tau^n(c)) \neq 0$ for all $c, d \in$ R and n $\in$ N then $\delta_n(\sigma^n(a),\sigma^n(b)) = 0$. Hence, we get F is a generalized (σ,τ)-higher homomorphism.

But if $G_n(\tau^n(d),\tau^n(c)) = 0$ for all $c, d \in$ R and n $\in$ N then we get F is a (σ,τ)-higher anti homomorphism.

Proposition (3.2):

Let F = $(f_i)_{i\in N}$ be a generalized Jordan (σ,τ)-higher homomorphism from a ring R into a 2-torsion free ring R', $\sigma^{i^2} = \sigma^i$, $\tau^{i^2} = \tau^i$, $\sigma^i\tau^i = \sigma^i\tau^{n-i}$ and $\sigma^i\tau^i = \tau^i\sigma^i$ for all i$\in$ N, then F is a generalized Jordan triple (σ,τ)-higher homomorphism.

Proof:

Replace $ab + ba$ for b in Definition (2.3), we get :

$$f_n(a(ab+ba)+(ab+ba)a)$$
$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(ab+ba)) + \sum_{i=1}^{n} f_i(\sigma^i(ab+ba))\phi_i(\tau^i(a))$$
$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\tau^i(a)\tau^i(b)+\tau^i(b)\tau^i(a)) + \sum_{i=1}^{n} f_i(\sigma^i(a)\sigma^i(b)+\sigma^i(b)\sigma^i(a))\phi_i(\tau^i(a))$$
$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\left(\sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(a))\phi_j(\tau^{j^2}(b)) + \sum_{j=1}^{i}\phi_j(\sigma^j\tau^j(b))\phi_j(\tau^{j^2}(a))\right) +$$
$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i} f_j(\sigma^{j^2}(a))\phi_j(\tau^j\sigma^j(b)) + \sum_{j=1}^{i} f_j(\sigma^{j^2}(b))\phi_j(\tau^j\sigma^j(a))\right)\phi_i\tau^i(a))$$

$\sigma^{i^2} = \sigma^i$, $\tau^{i^2} = \tau^i$, $\sigma^i\tau^i = \sigma^i\tau^{n-i}$ and $\sigma^i\tau^i = \tau^i\sigma^i$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\ \phi_i(\sigma^i\tau^{n-i}(a))\ \phi_i(\tau^i(b)) + 2\sum_{i=1}^{n} f_i(\sigma^i(a))\ \phi_i(\sigma^i\tau^{n-i}(b))\ \phi_i(\tau^i(a)) +$$
$$\sum_{i=1}^{n} f_i(\sigma^i(b))\ \phi_i(\sigma^i\tau^{n-i}(a))\ \phi_i(\tau^i(a)) \quad \ldots(1)$$

On the other hand:

$f_n(a(ab + ba) + (ab + ba)) = f_n(aab + aba + aba + baa)$

$$= \sum_{i=1}^{n} f_i(\sigma^i(a))\phi_i(\sigma^i\tau^{n-i}(a))\phi_i(\tau^i(b)) + \sum_{i=1}^{n} f_i(\sigma^i(b))\phi_i(\sigma^i\tau^{n-i}(a))\phi_i(\tau^i(a)) + 2f_n(aba) \quad \ldots(2)$$

Compare (1) and (2), we get:

$$2f_n = 2\sum_{i=1}^{n} f_i(\sigma^i(a))\ \phi_i(\sigma^i\tau^{n-i}(b))\ \phi_i(\tau^i(a)).$$

Since R'is a 2-torsion free ring , then F is a generalized Jordan triple (σ,τ)-higher homomorphism .

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