Optimal Control Approach to Robust Control of Linear and Non-linear Systems

M.O. OKE
Department of Mathematical Sciences, Ekiti State University, P.M.B. 5363, Ado-Ekiti, Nigeria
E-mail: femioke91@gmail.com

Abstract
This paper presents an optimal control approach to robust control of linear and non-linear systems. The approach translates the robust control problems into optimal control problems of minimizing cost functionals. Because of the uncertainty bound that is reflected in the cost functionals, the solution of the optimal control problems are solutions of the robust control problems. The optimal control approach allows us to solve some non-linear robust control problems that cannot be easily solved otherwise.

Keywords: Robust Control, Optimal Control, Cost Functional, Dynamic Equation, Riccati equation, Uncertainty.

1. Introduction

Robust control is concerned with modeling uncertainties. Nearly all models in engineering and applied sciences have errors and uncertainties. A controller is said to be robust if it works under uncertainties and performs even when the actual system deviates from its ideal model on which the controller's design is based. Robust control design has a very good tolerance to modeling errors and uncertainties. There are several approaches available for the design of robust control. These include parametric, fuzzy control, optimal control and H_∞/H_2 approaches. In the optimal control approach, we do not attempt to solve the robust control problems directly; rather we translate the robust control problems into optimal control problems. The solutions of the optimal control problems are now taking as the solutions of the robust control problems because of the uncertainty bound that is reflected in the cost functionals. To use the optimal control approach, the matching condition must be satisfied. We applied the optimal control approach to both linear and non-linear systems. Optimal control is a particular control technique in which the control signal optimizes a certain cost index. After stabilizing a system, the next thing is to optimize the system performance.
A lot of research work had been conducted on robust control systems. Lin et al. (1992) considered the compensation for uncertainty in the robust control of non-linear systems while Kuo and Wang (1991) worked on the robust position control of robotics manipulator in cartesian coordinates. Kehinde (2000) applied the fuzzy control to robust control problems while Mohammed and Hamson (2008) looked the Kharitorov and Hankel norms approach to some robust control problems. Daniel and Kuo (2010) studied the application of fuzzy logic to the robust control of linear systems while Landau et al. (1996) worked on the robust control of a 360 flexible arm by using the combined pole placement/sensitivity function shaping method. Lin and Olbrot (1996) used a linear quadratic regulator approach for robust control of linear systems with uncertainty parameters while Lin and Zhang (1993) concentrated on the

robust control of non-linear systems without matching condition. None of these researchers considered the optimal control approach to the robust control of linear and non-linear systems, hence the need for this research paper.

2. Materials and Methods

The optimal control problem to be considered in this paper is of the form

$$x = f(x,u) \tag{1}$$

which minimizes the cost functional

$$J(x,t) = \int_{t_0}^{t_f} L(x,u)d\tau \tag{2}$$

where t_0 is the initial time, t_f is the final time, $x = x(t)$ is the current state and $L(x,u)$ characterizes the cost objectives.

Applying the principle of optimality to the problems in (1) and (2) above and considering a time $t_0 + \Delta t_0$ in between $t_0$and t_f, then the cost functional $J(x,t)$ becomes

$$J(x,t) = \int_{t_0}^{t_f} L(x,u)d\tau = \int_{t_0}^{t_0+\Delta t} L(x,u)d\tau + \int_{t_0+\Delta t}^{t_f} L(x,u)d\tau \tag{3}$$

which gives

$$J(x,t) = \int_{t_0}^{t_0+\Delta t} L(x,u)d\tau + J(x+\Delta x, t+\Delta t) \tag{4}$$

where $x + \Delta x$ is the state at $t + \Delta t$ and $\Delta x = f(x,u)\Delta t$. Let $J^*(x,t)$ denote the minimal cost functional. Then, by the principle of optimality we have

$$J^*(x,t) = Min_{U(\tau)\epsilon R^m}\left[\int_{t_0}^{t_0+\Delta t} L(x,u)d\tau + J^*(x+\Delta x, t+\Delta t)\right] \tag{5}$$

Approximating and using the Taylor's series expansion, we get

$$J^*(x,t) = Min_{U(\tau)\epsilon R^m}\left[L(x,u)\Delta x + J^*(x,t) + (\frac{\partial J^*}{\partial x})^T\Delta x + (\frac{\partial J^*}{\partial t})^T\Delta t\right] \tag{6}$$

Now since $J^*(x,t)$ and $(\frac{\partial J^*}{\partial t})\Delta t$ are independent of $U(\tau)\epsilon R^m$. (6) can therefore be written as

$$J^*(x,t) = J^*(x,t) + (\frac{\partial J^*}{\partial t})\Delta t + Min_{U(\tau)\epsilon R^m}\left[L(x,u)\Delta x + (\frac{\partial J^*}{\partial x})^T\Delta x\right] \tag{7}$$

which implies that

$$-(\frac{\partial J^*}{\partial t})\Delta t = Min_{U(\tau)\epsilon R^m}\left[L(x,u)\Delta x + (\frac{\partial J^*}{\partial x})^T\Delta x\right] \tag{8}$$

For a time-invariant system with an infinite horizon (i.e.$t_f = \infty$), we obtain, the Hamilton-Jacobi-Bellman equation

$$-(\frac{\partial J^*}{\partial t})\Delta t = Min_{U(\tau)\epsilon R^m}\left[L(x,u) + \left(\frac{\partial J^*}{\partial x}\right)^T f(x,u)\right] \tag{9}$$

For a linear quadratic time invariant regulator problem, we have

$$\dot{x} = Ax + Bu \tag{10}$$

which minimizes the cost functional

$$J(x,t) = \int_{t_0}^{t_f} (x^T Q x + u^T R u) d\tau \tag{11}$$

where $Q = Q^T \geq 0$ is a symmetric, positive semi-definite matrix and $R = R^T \geq 0$ is a symmetric, positive definite matrix. To find the solution to the linear quadratic regulator problem, we assume the minimal cost to be quadratic in the form

$$J^*(x,t) = S^T S(t) x \tag{12}$$

where $S(t) = S(t)^T \geq 0$ is a symmetric, positive semi-definite matrix function of t. By the Hamilton-Jacobi-Bellman equation, the optimal control u^* satisfies (9) above or

$$S^T S(t) x = Min_{U(\tau) \epsilon R^m} [x^T Q x + u^T R u + 2x^T S(t)(Ax + Bu)] \tag{13}$$

To find this minimum, we set the derivative of $[x^T Q x + u^T R u + 2x^T S(t)(Ax + Bu)]$ with respect to u to be equal to zero to get

$$2Ru^* + 2B^T S(t) x = 0 \tag{14}$$

Therefore the optimal control is given by

$$u^* = -R^{-1} B^T S(t) x \tag{15}$$

Now $S^T S(t) x = Min_{U(\tau) \epsilon R^m} [x^T Q x + u^T R u + 2x^T S(t)(Ax + Bu)]$ can be therefore calculated as

$$S^T S(t) x = x^T Q x + u^{*T} R u^* + 2x^T S(t)(Ax + Bu^*) \tag{16}$$

which now gives

$$S^T S(t) x = x^T Q x + (-R^{-1} B^T S(t) x)^T R(-R^{-1} B^T S(t) x) + 2x^T S(t)(Ax + B(-R^{-1} B^T S(t) x)) \tag{17}$$

Simplifying this we have

$$S^T S(t) x = x^T Q x - x^T S(t) B R^{-1} B^T S(t) x + x^T S(t) A x + x^T A^T S(t) x \tag{18}$$

That is

$$S^T S(t) x = x^T x [Q - S(t) B R^{-1} B^T S(t) + S(t) A + A^T S(t)] \tag{19}$$

By the Hamilton-Jacobi-Bellman equation, we have

$$-x^T S(t) x = x^T x [Q - S(t) B R^{-1} B^T S(t) + S(t) A + A^T S(t)] \tag{20}$$

In other words $S(t)$ satisfies

$$S(t) = -[S(t) A + A^T S(t) + Q - S(t) B R^{-1} B^T S(t)] \tag{21}$$

(21) is the Riccati equation.
If the cost functional is quadratic with infinite horizon, then we have our optimal control problem as

$$\dot{x} = Ax + Bu \qquad (22)$$

which minimizes the cost functional

$$J(x,t) = \int_{t_0}^{\infty} (x^T Q x + u^T R u) d\tau \qquad (23)$$

The optimal control of this linear quadratic regulator problem with an infinite horizon will only exist if the system is stabilizable and the Riccati equation will further reduce to the following algebraic equation

$$SA + A^T S + Q - SBR^{-1}B^T S = 0 \qquad (24)$$

3. Robust Control of Linear Systems

A system (network) is said to be linear if the principle of superposition and the principle of proportionality holds. By the superposition principle, if the excitation of the system is given by $e(t) = e_1(t) + e_2(t)$, then the response would be given by $r(t) = r_1(t) + r_2(t)$. That is, the responses produced by the simultaneous application of two different forcing functions are the sum of the two individual responses. By the proportionality principle, if the excitation is given by $Ke(t)$, where K is a constant of proportionality, then the response would be $Kr(t)$. That is, the constant of proportionality (K) is preserved by the linear network, Franklin (1966).

Let us consider the robust control of linear time invariant systems of the form

$$\dot{x} = Ax + Bu \qquad (25)$$

where $x \in R^n$ and $u \in R^m$ are state and control variables respectively. Matrices A and B are used to model the uncertainty in the system dynamics. The system to be controlled is therefore given by

$$\dot{x} = A(\rho)x + BD(\rho)u \qquad (26)$$

where $D(\rho)$ is an $m \times m$ matrix representing the uncertainty in the system. Our goal is to design a state feedback to stabilize the system for all possible values of ρ within a specified bound. The solution of this robust problem depends on whether the uncertainty satisfies a matching condition which requires that the uncertainty is within the range of B. If the uncertainty satisfies the matching condition, then the solution to the robust control problem exists and can be obtained by solving the associated linear quadratic regulator problem. The associated linear quadratic regulator problem is obtained by just including the bounds on the uncertainty in the cost functional. If the uncertainty does not satisfy the matching condition, then we first decompose the uncertainty into the matched and unmatched parts. We then use an augmented control to deal with the unmatched uncertainty, Lin (1997) and Lin et al. (1992).

For a matching condition to be satisfied, the uncertainty in A can be written as

$$A(\rho) - A(\rho_0) = BD\varphi(\rho) \tag{27}$$

For an m$\times n$ matrix $\varphi(\rho)$ which is bounded, where $\rho_0 \epsilon \rho$ is a nominal value of ρ, the system dynamics can therefore be written as

$$\dot{x} = A(\rho)x + BD(u + E(\rho)u) + BD\varphi(\rho)x \tag{28}$$

where $E(\rho) = D^{-1}D(\rho) - I \geq 0$

To solve the robust control problem of stabilizing the system under uncertainty, we find a control law $u = kx$ such that the closed-loop system

$$\dot{x} = A(\rho_0)x + BD(u + E(\rho)u) + BD\varphi(\rho)x = A(\rho_0)x + BD(kx + E(\rho)kx) + BD\varphi(\rho)x \tag{29}$$

is asymptotically stable for all $\rho_0 \epsilon \rho$, Mohammed and Hamson (2008). For the auxiliary system $\dot{x} = A(\rho_0)x + BDu$, we find a feedback control law $u = kx$ that minimizes the cost functional

$$\int_0^\infty (x^T Fx + x^T x + u^T u)dt \tag{30}$$

where F is an upper bound on the uncertainty $\varphi(\rho)^T\varphi(\rho)$. To solve the linear quadratic regulator problem, we just solve the algebraic Riccati equation given by

$$A(\rho_0)^T S + SA(\rho_0) + F + I - SBDD^T B^T S = 0 \tag{31}$$

to get

$$u = -D^T B^T Sx \tag{32}$$

For an unmatched uncertainty, we consider a system

$$\dot{x} = A(\rho_0)x + BD(\rho)u \tag{33}$$

The system dynamics can therefore be written as

$$\dot{x} = A(\rho)x + BD(u + E(\rho)u) \tag{34}$$

where $E(\rho) = D^{-1}D(\rho) - I \geq 0$

We now find a control law $u = kx$ such that the closed-loop system

$$\dot{x} = A(\rho)x + BD(u + E(\rho)u) = A(\rho)x + BD(kx + E(\rho)kx) \tag{35}$$

is asymptotically stable. In order to solve this robust control problem, we first decompose the uncertainty $A(\rho) - A(\rho_0)$ into the sum of matched and unmatched components. This can be done by using pseudo-inverse $BD^+$of BD given by $BD^+ = (BD^T BD)^{-1}BD^T$, Lin and Olbrot (1996).

Let

$$A(\rho) - A(\rho_0) = (BD)(BD)^+\big(A(\rho) - A(\rho_0)\big) + (1 - (BD)(BD)^+)\big(A(\rho) - A(\rho_0)\big) \quad (36)$$

Then $(BD)(BD)^+\big(A(\rho) - A(\rho_0)\big)$ is the matched component and $(1 - (BD)(BD)^+)(A(\rho) - A(\rho_0))$ is the unmatched component. If the matching condition is satisfied, then the unmatched part will be equal to zero.
Now let

$$\varphi(\rho) = (BD)^+\big(A(\rho) - A(\rho_0)\big) \quad (37)$$

So that

$$A(\rho) - A(\rho_0) = BD\varphi(\rho) \quad (38)$$

we also have

$$\big(A(\rho) - A(\rho_0)\big)^T (BD^+)^T (BD)^+\big(A(\rho) - A(\rho_0)\big) \leq F \quad (39)$$

$$\alpha^{-2}\big(A(\rho) - A(\rho_0)\big)^T\big(A(\rho) - A(\rho_0)\big) \leq H \quad (40)$$

where F and H are upper bounds on the uncertainties and α is a design parameter. We now solve the robust control problem indirectly by translating it into a linear quadratic regulator problem. For the auxiliary system given by

$$\dot{x} = A(\rho_0)x + BDu + \alpha(1 - (BD)(BD^+)v \quad (41)$$

we will find feedback control laws $u = kx$ and $v = lx$ that minimizes the cost functional

$$\int_0^\infty (x^T(F + \rho^2 H + \beta^2 I)x + u^T u + \rho^2 v^T v)dt \quad (42)$$

where $\alpha \geq 0, \rho \geq 0$ and $\beta \geq 0$ are design parameters. In this linear quadratic regulator problem, v is an augmented control that is used to deal with the unmatched uncertainty. If the matching condition is satisfied, then the design parameters are taken to be $\alpha = 0, \rho = 0$ and $\beta = 1$ and the solution of the linear quadratic regulator problem is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = -R^{-1}B^T Sx \quad (43)$$

where S is the unique positive definite solution to the following algebraic Riccati equation

$$SA + A^T S + Q - S\dot{B}R^{-1}B^T S = 0 \quad (44)$$

4. Robust Control of Non-Linear Systems

Robust control design is more complex for non-linear systems. In fact many approaches to robust control problems are applicable only to linear systems. But the optimal control approach is applicable to both linear and non-linear systems.

Let us consider a non-linear system

$$\dot{x} = A(x) + B(x)u + h(x)u + B(x)f(x) \tag{45}$$

where $h(x)$ is an $m \times m$ positive semi-definite matrix representing the uncertainty in the input matrix. $A(x)$ and $B(x)$ are non-linear (matrix) function of x. $B(x)$ and $f(x)$ are used to model the uncertainty in the system dynamics. Since the uncertainty is in the range of $B(x)$, the matching condition is satisfied. For the robust control problem, we find a feedback control law $u = u_0(x)$ such that the closed loop system given by

$$\dot{x} = A(x) + B(x)u_0(x) + h(x)u_0(x) + B(x)f(x) \tag{46}$$

is globally asymptotically stable for all uncertainties $f(x)$ satisfying $\|f(x)\| \leq f_{max}(x)$, Lin and Zhang (1993).

For the nominal system

$$\dot{x} = A(x) + B(x)u \tag{47}$$

we find a feedback control law $u = u_0(x)$ that minimizes the following cost functional

$$\int_0^\infty (f_{max}x^2 + x^T x + u^T u)dt \tag{48}$$

If the uncertainty is not in the range of $B(x)$, then we consider the following non-linear system

$$\dot{x} = A(x) + B(x)u + h(x)u) + C(x)f(x) \tag{49}$$

where $f(x)$ models the uncertainty in the system dynamics and it does not satisfy the matching condition. $C(x)$ is a matrix introduced to make $f(x)$ flexible.

We now find a feedback control law $u = u_0(x)$ such that the closed loop system given by

$$\dot{x} = A(x) + B(x)u_0(x) + h(x)u_0(x) + C(x)f(x) \tag{50}$$

is globally asymptotically stable for all uncertainties $f(x)$.

For the auxiliary system given by

$$\dot{x} = A(x) + B(x)u + \alpha(I - B(x)(Bx)^+)C(x)v \tag{51}$$

we find a feedback control law $(u_0(x), v_0(x))$ that minimizes the following cost functional

$$\int_0^\infty (f_{max}x^2 + \rho^2 g_{max}x^2 + \beta^2\|x\|^2 + \|u\|^2 + \rho^2\|v\|^2)dt \tag{52}$$

where $\alpha \geq 0$, $\beta \geq 0$, and $\rho \geq 0$ are design parameters. f_{max} and g_{max} are non-negative functions such that

$$\|B(x)^+C(x)f(x)\| < f_{max}(x) \tag{53}$$

and

$$\|\alpha^{-1}f(x)\| < g_{max}(x) \tag{54}$$

Conclusion

We have used the optimal control approach to represent the robust control of linear and non-linear systems in this paper. We accomplished this by translating the robust control problems into optimal control problems of minimizing cost functionals. The uncertainties of the robust control systems are reflected in the performance index of the optimal control problems. This approach allows us to solve some non-linear robust control problems that cannot be easily solved by the use of other methods.

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