Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.2, No.6, 2012

Common fixed point for two weakly compatible pairs of self-maps through property E. A. under an implicit relation

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Abstract

We prove some generalizations of the results of Renu Chug-Sanjay Kumar, the author with necessary corrections, and of Rhoades et al. for weakly compatible self-maps satisfying the property E. A. under an implicit relation.

Key Words: *Compatible and Weakly Compatible Self-maps*, *Property E. A.*, *Contractive Mmodulus*, *Common Fixed Point*

1 INTRODUCTION

Throughout this paper, (X,d) denotes a metric space. If $x \in X$ and A is a self-map on X, we write Ax for the A-image of x, $A(X)$ for the range of A and AS for the composition of self-maps A and S. Also IR_+ will denote the set of all non negative real numbers. A point p of the space X is a fixed point of a self-map A if and only if $Ap = p$. A self-map A on X with the choice $d(Ax,Ay) \le a\,d(x,y)$ for all $x,y \in X$ is known as a *contraction*, provided $0 \le a < 1$. According to the celebrated Banach contraction principle (*BCP*), every contraction on a complete metric space has a unique fixed point. It is easy to see that a discontinuous self-map cannot be a contraction and hence contraction principle cannot ensure a fixed point for it even if X is complete. However, various generalizations of *BCP* have been established by weakening the contraction condition, relaxing the completeness of the underlying space and/or extending to two or more self-maps under additional assumptions. To mention a few are the works of Fisher ([2]-[3]), Fisher and Khan [4], Chang [5], Ciric [6], Das and Naik [7], Jungck [8], Pant [11] and Singh and Singh [18].

Self-maps S and A on X are said to be *weakly commuting* [17] if $d(ASx,SAx) \le a\,d(Sx,Ax)$ for all $x,y \in X$. As a further generalization for *commuting* maps, Gerald Jungck [9] proposed the *compatibility* as in the following lines:

Definition 1.1 Self-maps S and A on X are *compatible* if

$$\lim_{n\to\infty} d(SAx_n, ASx_n) = 0 \tag{1}$$

whenever $\langle x_n\rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \quad \text{for some } t \in X. \tag{2}$$

If there is no $\langle x_n\rangle_{n=1}^{\infty}$ in X with the choice (2), S and T are called *vacuously* compatible.

Remark 1.1 Self-maps S and A on X are not (non-vacuously) compatible if there is a sequence $\langle x_n\rangle_{n=1}^{\infty}$ in

X with choice (2) but $\lim_{n\to\infty} d(SAx_n, ASx_n) \neq 0$ or $+\infty$.
Obviously every commuting pair of self-maps is a weakly commuting one, and a weakly commuting pair is necessarily compatible. But neither reverse implication is true [9].

The following notion is due to Aamri and Moutawakil [1]:

Definition 1.2 Self-maps S and T satisfy the *property E. A.* if there is an $\langle x_n \rangle_{n=1}^{\infty} \subset X$ with the choice (2).

In this paper C_X denotes the class of all non-compatible pairs of self-maps on X, while C_X^*, the class of all pairs of self-maps on X which satisfy the property E. A.

In view of Remark 1.1, non-compatibility implies the existence of the sequence $\langle x_n \rangle_{n=1}^{\infty}$ with the choice (2).
Hence the class C_X^* is potentially <u>*wider*</u> than C_X. We also note that *vacuously compatible self-maps do not satisfy the property E. A.*

We call a point $x \in X$, a coincidence point of self-maps S and A on X if $Tx = Sx$, while $y \in X$ is a point of their coincidence w. r. t. x if $Tx = Sx = y$. Taking $x_n = x$ for all n, from (1) and (2), it follows that $STx = TSx$ whenever $x \in X$ is such that $Tx = Sx$. Hence we have

Definition 1.3 Self-maps which commute at their coincidence points are weakly compatible maps [10] which are also called *coincidentally commuting* or *partially commuting* [16].

Being weakly compatible and possessing property E. A. are independent conditions of each other (Pathak et al. [12])

In this paper $\phi : \mathrm{IR}_+^5 \to \mathrm{IR}_+$ is an upper semi-continuous (written shortly as *usc*) non decreasing in each coordinate variable, and for $\xi > 0$ satisfy the conditions:

(i) $\max\{\phi(\xi,\xi,0,2\xi,0), \phi(\xi,\xi,0,,0,2\xi)\} \le \xi$

(ii) $\max\{\phi(\xi,\xi,0,\alpha\xi,0), \phi(\xi,\xi,0,,0,\alpha\xi)\} < \xi$ when $\alpha < 2$

(iii) $\gamma(t) = \phi(\xi,\xi,\alpha_1\xi,\alpha_2\xi,\alpha_3\xi) < \xi$ if $\alpha_1 + \alpha_2 + \alpha_3 = 4$.

Remark 1.2 $\xi \le \phi(\xi,\xi,\xi,\xi,\xi)$ implies that $\xi = 0$.

With this notion Renu Chug and Sanjay Kumar [14] proved the following result for two weakly compatible pairs of self-maps:

Theorem 1.1 *Let A, B, S and T be self-maps on X satisfying the inclusions*

$$A(X) \subset T(X), \tag{3-a}$$

$$B(X) \subset S(X) \tag{3-b}$$

and the inequality

$$d(Ax,By) \le \phi\big(d(Sx,Ty), d(Ax,Sx), d(By,Ty), d(Ax,Ty), d(By,Sx)\big) \text{ for all } x,y \in X. \tag{4}$$

Suppose that:

(a) *X is complete*

(b) *The pairs* (A,S) *and* (B,T) *are weakly compatible.*

Then all the four maps A, B, S and T have a unique common fixed point.

The authors of Theorem 1.1 asserted that **there is a point** $u \in X$ **such that** $z = Su$ **under the inclusion (3-b)** (Line 4 from the bottom, page 244). We point out that their assertion is not true and that the inclusion (3-b) plays no role to obtain the point *u*. However it will be true if we assume that the map *B* (and hence *S*) is **onto**.

In view of this suggestion we restate Theorem 1.1 as follows:

Theorem 1.2 *Let A, B, S and T be self-maps on X satisfying the inclusions* (3-a), (3-b), *the inequality* (4), (a) *and* (b) *of Theorem* 1.1. *If*

(c) *either A or B is onto*,

then all the four maps A, B, S and T have a unique common fixed point.

In this paper we obtain the conclusion of Theorem 1.1, by relaxing the completeness of the space *X* and using the *property E. A.* (see below). Our result will be a generalization of those some of the authors.

2 MAIN RESULT

First we prove

Theorem 2.1 *Let A, B, S and T be self-maps on X satisfying* (3-a), (3-b), *and the inequality* (4).

Suppose that

(d) *either* $(S,A) \in C_X^*$ *or* $(B,T) \in C_X^*$,

(e) *one of* $S(X), A(X), T(X)$ *and* $B(X)$ *is a complete subspace of X.*

If the condition (*b*) *of Theorem* 1.2 *holds good, then A, B, S and T will have a unique common fixed point.*

Proof. First suppose that $(S,A) \in C_X^*$. Then there is a $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$ for some $p \in X$. By virtue of the inclusion (3-a), we can find another sequence $\langle y_n \rangle_{n=1}^{\infty}$ of points in *X* such that $Ax_n = Ty_n$ for all *n* so that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = p. \quad (5)$$

Observe that $q = \lim_{n\to\infty} By_n$ also equals *p*. In fact, writing $x = x_n$ and $y = y_n$ in the inequality (4), we see that

$$d(Ax_n,By_n) \le \phi(d(Sx_n,Ty_n), d(Ax_n,Sx_n), d(By_n,Ty_n), d(Ax_n,Ty_n), d(By_n,Sx_n))$$

Applying the limit as $n \to \infty$ in this, and using (5) and the usc of ϕ,

$$d(p,q) \le \phi(0,0,d(q,p),0,d(q,p)) \le \phi(d(p,q),d(p,q),d(p,q),d(p,q),d(p,q))$$

so that or $d(p,q) = 0$ or $p = q$, in view of Remark 1.2. Thus

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = p. \quad (6)$$

We first show that *p* is a common coincidence point for (S,A) and (B,T). That is

$$Ap = Sp = Bp = Tp.$$
(7)

<u>**Case (a)**</u>: Suppose that $T(X)$ is complete.

Then $p \in T(X)$ so that $Tu = p$ for some $u \in X$. But then from (4), we see that

$$d(Ax_n, Bu) \le \phi(d(Sx_n, Tu), d(Ax_n, Sx_n), d(Bu, Tu), d(Ax_n, Tu), d(Bu, Sx_n))$$

Applying the limit as $n \to \infty$ in this, and using $Tu = p$ and upper semi-continuity of ϕ followed by its non decreasing nature in each coordinate variable, we get

$$d(p, Bu) \le \phi(0,0, d(Bu, p), 0, d(Bu, p)) \le \phi(d(Bu, p), d(Bu, p), d(Bu, p), d(Bu, p), d(Bu, p))$$

or $d(p, Bu) = 0$ $Bu = p$. Thus u is a coincidence point, and hence p is a point of coincidence for B and T . That is

$$Bu = Tu = p. \tag{8}$$

Again from the inclusion (3-b), we see that $p = Bu = Sr$ for some $r \in X$.

Since ϕ is non decreasing, from (4) and (8) we get

$$d(Ar, Sr) = d(Ar, Bu) \le \phi(d(Sr, Tu), d(Ar, Sr), d(Bu, Tu), d(Ar, Tu), d(Bu, Sr))$$
$$= \phi(0, d(Ar, p), 0, d(Ar, p), 0) \le \phi(d(Ar, p), d(Ar, p), d(Ar, p), d(Ar, p), d(Ar, p))$$

so that $Ar = Sr = p$, in view of Remark 1.2. That is p is a point of coincidence for A and S w. r. t. r. Thus

$$Ar = Sr = Bu = Tu = p. \tag{9}$$

Since (A, S) and (B, T) are weakly compatible pairs, (7) follows from (9).

Case (b): Suppose that $A(X)$ is complete.

Then $p \in A(X) \subset T(X)$, in view of (3-a) and hence (7) follows from Case (a).

Case (c): Suppose that $S(X)$ is complete.

Then $p \in S(X)$ so that $Sw = p$ for some $w \in X$. But then from (4), we see that

$$d(Aw, By_n) \le \phi(d(Sw, Ty_n), d(Aw, Sw), d(By_n, Ty_n), d(Aw, Ty_n), d(By_n, Sw))$$

Applying the limit as $n \to \infty$, and using (6), $Sw = p$ and upper semi-continuity of ϕ, followed by its non decreasing nature in each coordinate variable, this gives

$$d(Aw, p) \le \phi(0, d(Aw, p), 0, d(Aw, p), 0) \le \phi(d(Aw, p), d(Aw, p), d(Aw, p), d(Aw, p), d(Aw, p))$$

or $d(p, Aw) = 0$ or $Aw = p$. Thus w is a coincidence point and p is a point of coincidence for A and S w. r. t. w. That is

$$Aw = Sw = p . \tag{10}$$

Again from the inclusion (3-a), we see that $p = Aw = Tq$ for some $q \in X$.

Since ϕ is non decreasing, from (4) and (10) we get

$$d(p, Bq) = d(Aw, Bq) \le \phi(d(Sw, Tq), d(Aw, Sw), d(Bq, Tq), d(Aw, Tq), d(Bq, Sw))$$
$$= \phi(0,0, d(Bq, p), 0, d(Bq, p)) \le \phi(d(Bq, p), d(Bq, p), d(Bq, p), d(Bq, p), d(Bq, p))$$

so that $p = Bq$ in view of Remark 1.2.

Thus $$Aw = Sw = Bq = Tq = p. \tag{11}$$

Now (7) will follow from (b) and (11).

Case (d): Suppose that $B(X)$ is complete.

Then $p \in B(X) \subset S(X)$, in view of (3-b) and hence (7) follows from Case (c).

To establish that p is a common fixed point for all the four maps, we again use (4) with $x = r$ and $y = p$ so that

$$d(Ar, Bp) \le \phi(d(Sr, Tp), d(Ar, Sr), d(Bp, Tp), d(Ar, Tp), d(Bp, Sr)) .$$

Again since ϕ is non decreasing, this together with (7) and (8) gives

$$d(p, Bp) \le \phi(d(p, Bp), 0, 0, d(p, Bp), d(Bp, p)) \le \phi(d(p, Bp), d(p, Bp), d(p, Bp), d(p, Bp), d(p, Bp))$$

so that from Remark 1.2, it follows that $Bp = p$ and hence p is a common fixed point for A, B, S and T, which in fact is a point of their common coincidence.

Now suppose that $(B,T) \in C_X^*$. Then exchanging the roles of (S,A) and (B,T); of (3-a) and (3-b); and of $T(X)$ and $S(X)$ in the above proof, we can similarly obtain the conclusion.

Uniqueness of the common fixed point follows easily from the choice of ϕ and (4). □

Remark 2.1 If A is onto then $A(X) = X$. Therefore the completeness of X implies the completeness of $A(X)$. Similarly the completeness of X implies the completeness of $B(X)$ whenever B is onto.

Remark 2.2 We can prove that if self-maps A, B, S and T on X satisfy the inclusions (3-a), (3-b) and the inequality (4), and X is complete, then both $(A,S) \in C_X^*$ and $(B,T) \in C_X^*$.

In fact for any $x_0 \in X$, the inclusions (3 a-b) generate a sequence of points $\langle x_n \rangle_{n=1}^{\infty}$ in X with the choice

$$Ax_{2n-2} = Tx_{2n-1}, \quad Bx_{2n-1} = Sx_{2n} \quad \text{for all } n \ge 1. \tag{12}$$

From the proof of Theorem 1.2, it follows that $\langle Ax_{2n} \rangle_{n=1}^{\infty}$ and $\langle Ax_{2n} \rangle_{n=1}^{\infty}$ are Cauchy sequences.

If X is *complete*, these will converge to some $z \in X$. That is

$$\lim_{n\to\infty} Ax_{2n-2} = \lim_{n\to\infty} Tx_{2n-1} = \lim_{n\to\infty} Bx_{2n-1} = \lim_{n\to\infty} Sx_{2n} = p. \tag{13}$$

With $x_{2n} = x_n^*$ and $x_{2n-1} = y_n^*$ from (13), we see that

$$\lim_{n\to\infty} Ax_n^* = \lim_{n\to\infty} Sx_n^* = p \quad \text{and} \quad \lim_{n\to\infty} By_n^* = \lim_{n\to\infty} Ty_n^* = p,$$

proving that the pairs (S,A) and (B,T) satisfy the property E. A.

Remark 2.3 In view of Remarks 2.1 and 2.2, we see that Theorem 1.2 follows as a particular case of our result (Theorem 2.1).

It is possible to relax the condition (b), and drop the inclusions in Theorem 2.1 for three self-maps A, B and T.

In deed we prove the following

Theorem 2.2 *Let A, B and T be self-maps on X satisfying the inequality*

$$d(Ax,By) \le \phi\big(d(Tx,Ty), d(Ax,Tx), d(By,Ty), d(Ax,Ty), d(By,Tx)\big)$$
$$\text{for all } x, y \in X. \tag{14}$$

Suppose that $T(X)$ is complete subspace of X and

(f) *either of the pairs (A,T) and (B,T) is both (weakly compatible and belongs to the class C_X^*).*

Then A, B and T will have a unique common fixed point.

Proof. Suppose that the pair $(A,T) \in C_X^*$. Then there exists a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = p \text{ for some } p \in X. \tag{15}$$

Let $q = \lim_{n\to\infty} Bx_n$. Then $q = p$. In fact, writing $x = y = x_n$ in the inequality (14),

$$d(Ax_n, Bx_n) \le \phi(d(Tx_n,Tx_n), d(Ax_n,Tx_n), d(Bx_n,Tx_n), d(Ax_n,Tx_n), d(Bx_n,Tx_n))$$

Applying the limit as $n \to \infty$ in this and using (15) and upper semi-continuity of ϕ,

$$d(p,q) \le \phi(0,0,d(q,p),0,d(q,p)) \le \phi(d(p,q), d(p,q), d(p,q), d(p,q), d(p,q))$$

so that $d(p,q) = 0$ or $p = q$, in view of Remark 1.2. Thus

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Bx_n = p. \tag{16}$$

Suppose that $T(X)$ is complete. Then $p \in T(X)$ so that $Tu = p$ for some $u \in X$. But then from (14),

$$d(Ax_n, Bu) \le \phi(d(Tx_n, Tu), d(Ax_n, Tx_n), d(Bu, Tu), d(Ax_n, Tu), d(Bu, Tx_n)).$$

Applying the limit as $n \to \infty$ in this, and using $Tu = p$ and upper semi-continuity of ϕ followed by its non decreasing nature in each coordinate variable, we get

$d(p, Bu) \le \phi(0,0,d(Bu,p),0,d(Bu,p)) \le \phi(d(Bu,p),d(Bu,p),d(Bu,p),d(Bu,p),d(Bu,p))$ or $d(p,Bu) = 0$

or $Bu = p$. Thus u is a coincidence point of B and T. That is $Bu = Tu = p$.

Since ϕ is non decreasing, from (14), we get

$$\begin{aligned} d(Au,Bu) &\le \phi(d(Tu,Tu),d(Au,Su),d(Bu,Tu),d(Au,Tu),d(Bu,Tu)) \\ &= \phi(0,d(Au,p),0),d(Au,p),0) \le \phi(d(Au,p),d(Au,p),d(Au,p),d(Au,p),d(Au,p)) \end{aligned}$$

so that $Au = Tu$ in view of Remark 1.2.

Thus $Au = Bu = Tu = p$. (17)

Since (A, T) is weakly compatible pair, from (17) we get that $Ap = Tp$.

Again since ϕ is non decreasing, from (14) and using $Ap = Tp$, we get

$$\begin{aligned} d(Ap,Bp) &\le \phi(d(Tp,Tp),d(Ap,Tp),d(Bp,Tp),d(Ap,Tp),d(Bp,Tp)) \\ &= \phi(0,0,d(Ap,Bp),0,d(Ap,Bp)) \le \phi(d(Ap,Bp),d(Ap,Bp),d(Ap,Bp),d(Ap,Bp),d(Ap,Bp)) \end{aligned}$$

so that $Ap = Bp$ in view of Remark 1.2.

Thus $Ap = Bp = Tp$. (18)

Finally p will be a common fixed point for the three maps. For, writing $x = u$ and $y = p$ in (14) and then using (17) and (18) we get

$$\begin{aligned} d(p,Tp) = d(Au,Bp) &\le \phi(d(Tu,Tp),d(Au,Tp),d(Bp,Tp),d(Au,Tp),d(Bp,Tu)) \\ &= \phi(d(p,Tp),d(p,Tp),0,d(p,Tp),d(Tp,p)) \le \phi(d(p,Tp),d(p,Tp),d(p,Tp),d(Tp,p),d(p,Tp)) \end{aligned}$$

Again since ϕ is non decreasing, this with Remark 1.2, implies that $Bp = p$ and hence p is a common fixed point for A, B, and T.

Similarly if (A,T) is a weakly compatible pair satisfying the property E.A., it follows that A, B, and T have a common fixed point. Uniqueness of the common fixed point follows easily from the choice of ϕ and (14).

□

Taking $A = B$ in Theorem 2.2 we have

Corollary 2.1 *Let A and T be self-maps on X satisfying the inequality*

$$d(Ax,Ay) \le \phi\big(d(Tx,Ty),d(Ax,Tx),d(Ay,Ty),d(Ax,Ay),d(Ay,Tx)\big) \text{ for all } x,y \in X. \quad (19)$$

If $T(X)$ *is complete subspace of X,* (A,T) *is weakly compatible and* $(A,T) \in C_X^*$*, then A and T will have a unique common fixed point.*

Given $x_0 \in X$ and self-maps A and T on X, if there exist points $x_1, x_2, x_3, \ldots$ in X with $Ax_{n-1} = Tx_n$ for $n \ge 1$, the sequence $\langle Ax_n \rangle_{n=1}^{\infty}$ defines a (A,T)*-orbit* or simply an *orbit* at x_0. The space X is (A,T)*-orbitally complete* [13] at $x_0 \in X$ if every Cauchy sequence in some orbit at x_0 converges in X.

Note that every complete metric space is orbitally complete at each of its points. However there are incomplete metric spaces which are orbitally complete at some point of it [13].

We give the following generalization of Corollary 2.1, the proof of which can be obtained on similar lines of that of Theorem 2.2.

Theorem 2.3 *Let A and T be self-maps on X satisfying the inequality* (14) *and* $(A,T)\in C_X^*$. *Suppose that any one of the following conditions holds good:*

(g) *the subspace* $A(X)$ *is orbitally complete at* $x_0 \in X$

(h) $T(X)$ *is orbitally complete at* $x_0 \in X$.

Then there is a coincidence point z for A and T. Further if A and T are weakly compatible, then the point of coincidence of A and T w. r. t. z will be a unique common fixed point for them.

In the remainder of the paper $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ represents a *contractive modulus*, due to Solomon Leader [19] with the choice $\psi(t) < t$ for $t > 0$.

Corollary 2.2 (Theorem 2, [13]) *Let A and T be self-maps on X satisfying the inclusion*

$$A(X) \subset T(X) \tag{20}$$

and the inequality

$$d(Ax,Ay) \le \psi\left(\max d(Tx,Ty), d(Ax,Tx), d(Ay,Ty) \tfrac{1}{2}[d(Ax,Ty)+d(Ay,Tx)]\right) \text{ for all } x,y\in X, \tag{21}$$

where ψ *is non decreasing and upper semi continuous contractive modulus.*

Suppose that one of the conditions (g), (h) *and* (i) *holds good, where*

(i) *the space X is orbitally complete at some* $x_0 \in X$ *and T is onto*

Then there is a coincidence point z for A and T. Further if A and T are weakly compatible, then the point of coincidence of A and T w. r. t. z will be a unique common fixed point for them.

Proof. We note that the condition (*i)* implies (*g*). We set

$$\phi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = \psi\left(\max\{\xi_1,\xi_2,\xi_3,\tfrac{1}{2}[\xi_4+\xi_5]\}\right) \text{ for all } \xi_i,\ i=1,2,\ldots,5 .$$

Then (14) reduces to (21).

Let $x_0 \in X$ be arbitrary. By virtue of the inclusion (20), we can construct an (A,T) - *orbit* x_0 with choice $Ax_{n-1} = Tx_n$ for $n \ge 1$. Then the sequence $\langle Ax_n \rangle_{n=1}^{\infty}$ is a Cauchy sequence in this orbit at *X*.

In view of Remark 2.2, *A* and *T* satisfy the property E. A. Hence the conclusion follows from Theorem 2.3.

Now write $B = A$ in Theorem 2.1, we get

Theorem 2.4 *Let A, S and T be self-maps on X satisfying the inclusion*

$$A(X) \subset [S(X) \cap T(X)] \tag{22}$$

and the inequality

$$d(Ax,Ay) \le \phi\left(d(Sx,Ty), d(Ax,Sx), d(Ay,Ty), d(Ax,Ty), d(Ay,Sx)\right) \text{ for all } x,y\in X. \tag{23}$$

Suppose that

(j) *either of the pairs* (A,S) *and* (A,T) *satisfies the property E. A.*

(k) *one of* $A(X), B(X)$ *and* $T(X)$ *is a compete subspace of X.*

If (A,S) *and* (A,T) *are weakly compatible, then A, B, S and T will have a unique common fixed point.*

Since every compatible pair is weakly compatible, we have the following sufficiency part of Main theorem of [15].

Corollary 2.3 *Let A, S and T be self-maps on X satisfying* (20) *and the inequality*

$$d(Ax,Ay) \le \max\{d(Sx,Ty), d(Ax,Sx), d(Ay,Ty), \tfrac{1}{2}[d(Ax,Ty)+d(Ay,Sx)]\}$$
$$-\omega\left(\max\{d(Sx,Ty), d(Ax,Sx), d(Ay,Ty), \tfrac{1}{2}[d(Ax,Ty)+d(Ay,Sx)]\}\right) \text{ for all } x,y\in X, \quad (24)$$

where $\omega: \mathrm{IR}_+ \to \mathrm{IR}_+$ *is continuous and* $0<\omega(t)<t$ *for* $t>0$. *Suppose that X is complete, A is continuous and* (A,S) *and* (A,T) *are compatible. Then A, S and T will have a unique common fixed point.*

Note that (23) reduces to (24) with

$$\phi(\xi_1,\xi_2,\xi_3,\xi_4,\xi_5) = \max\left\{\xi_1,\xi_2,\xi_3,\frac{1}{2}[\xi_4+\xi_5]\right\} - \omega\left(\max\left\{\xi_1,\xi_2,\xi_3,\frac{1}{2}[\xi_4+\xi_5]\right\}\right) \text{ for all } \xi_i,\ i=1,2,\dots,5.$$

Given $x_0 \in X$, due to the inclusion (22), we can construct the sequence $\langle Ax_n\rangle_{n=1}^{\infty}$ with choice

$$Ax_{2n-2} = Tx_{2n-1}, \quad Ax_{2n-1} = Tx_{2n} \quad \text{for } n\ge 1,$$

which is a Cauchy sequence in the complete space *X* and hence converges in it.

In view of Remark 2.2, *A* and *T* satisfy the property E. A.

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