Connected domination in Block subdivision graphs of Graphs

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Abstract

A dominating set $D \subseteq V[BS(G)]$ is called connected dominating set of a block subdivision graph $BS(G)$ if the induced subgraph $\langle D \rangle$ is connected in $BS(G)$. The connected domination number $\gamma_c[BS(G)]$ of a graph $BS(G)$ is the minimum cardinality of a connected dominating set in $BS(G)$. In this paper, we study the connected domination in block subdivision graphs and obtain many bonds on $\gamma_c[BS(G)]$ in terms of vertices, blocks and other different parameters of *G* but not members of $BS(G)$. Also its relationship with other domination parameters were established.

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Introduction

All graphs considered here are simple, finite, nontrivial, undirected and connected. As usual, *p*, *q* and *n* denote the number of vertices, edges and blocks of a graph *G* respectively. In this paper, for any undefined term or notation can be found in Harary [6], Chartrand [3] and T.W.Haynes et al.[7]. The study of domination in graphs was begun by Ore [12] and Berge [2].

As usual, the maximum degree of a vertex in *G* is denoted by $\Delta(G)$. A vertex *v* is called a cut vertex if removing it from *G* increases the number of components of *G*. For any real number *x*, $\lceil x \rceil$ denotes the smallest integer not less than *x* and $\lfloor x \rfloor$ denotes the greatest integer not greater than *x*. The complement $\bar{G}$ of a graph *G* has

V as its vertex set, but two vertices are adjacent in $\bar{G}$ if they are not adjacent in *G*. A graph *G* is called trivial if it has no edges. If *G* has at least one edge then *G* is called a nontrivial graph. A nontrivial connected graph *G* with at least one cut vertex is called a separable graph, otherwise a non-separable graph.

A vertex cover in a graph *G* is a set of vertices that covers all edges of *G*. The vertex covering number $\alpha_0(G)$ is a minimum cardinality of a vertex cover in *G*. An edge cover of a graph *G* without isolated vertices is a set of edges of *G* that covers all vertices of *G*. The edge covering number $\alpha_1(G)$ of a graph *G* is the minimum cardinality of an edge cover of *G*. A set of vertices in a graph *G* is called an independent set if no two vertices in the set are adjacent. The vertex independence number $\beta_0(G)$ of a graph *G* is the maximum cardinality of an independent set of vertices in *G*. The edge independence number $\beta_1(G)$ of a graph *G* is the maximum cardinality of an independent set of edges.

Now coloring the vertices of any graph. By a proper coloring of a graph *G*, we mean an assignment of colors to the vertices of *G*, one color to each vertex, such that adjacent vertices are colored differently. The smallest number of colors in any coloring of a graph *G* is called the chromatic number of *G* and is denoted by $\chi(G)$. Two graphs *G* and *H* are isomorphic if there exists a one-to-one correspondence between their point sets which preserves adjacency. A subgraph *F* of a graph *G* is called an induced subgraph $\langle F \rangle$ of *G* if whenever *u* and *v* are vertices of *F* and *uv* is an edge of *G*, then *uv* is an edge of *F* as well.

A nontrivial connected graph with no cut vertex is called a block. A subdivision of an edge *uv* is obtained by removing an edge *uv*, adding a new vertex *w* and adding edges *uw* and *wv*. For any (*p, q*) graph *G*, a subdivision graph *S*(*G*) is obtained from *G* by subdividing each edge of *G*. A block subdivision graph *BS*(*G*) is the graph whose vertices correspond to the blocks of *S*(*G*) and two vertices in *BS*(*G*) are adjacent whenever the corresponding blocks contain a common cut vertex of *S*(*G*).

A set $D \subseteq V(G)$ of a graph $G = (V, E)$ is a dominating set if every vertex in *V* – *D* is adjacent to some vertex in *D*. The domination number $\gamma(G)$ of *G* is the minimum cardinality of a minimal dominating set in *G*. The domination number $\gamma[BS(G)]$ of $BS(G)$ is the minimum cardinality of a minimal dominating set in $BS(G)$. A dominating set *D* in a graph $G = (V, E)$ is called restrained dominating set if every vertex in

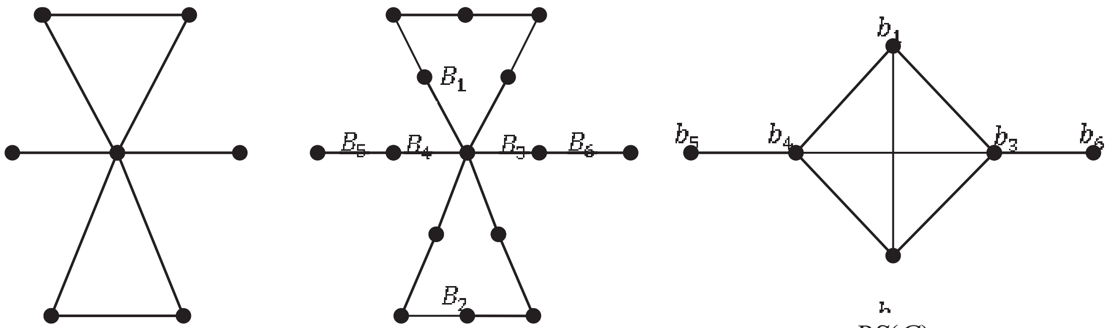
$V - D$ is adjacent to a vertex in *D* and to a vertex in $V - D$. The restrained domination number of a graph *G* is denoted by $\gamma_{re}(G)$, is the minimum cardinality of a restrained dominating set in *G*. The restrained domination number of a block subdivision graph $\gamma_{re}[BS(G)]$ is the minimum cardinality of a restrained dominating set in $BS(G)$. This concept was introduced by G.S.Domke et al. in [5].

A dominating set *D* is a total dominating set if the induced subgraph $\langle D \rangle$ has no isolated vertices. The total domination number $\gamma_t(G)$ of a graph *G* is the minimum cardinality of a total dominating set in *G*. This concept was introduced by Cockayne, Dawes and Hedetniemi in [4].

A set *F* of edges in a graph $G(V, E)$ is called an edge dominating set of *G* if every edge in $E - F$ is adjacent to at least one edge in *F*. The edge domination number $\gamma'(G)$ of a graph *G* is the minimum cardinality of an edge dominating set of *G*. Edge domination number was studied by S.L. Mitchell and Hedetniemi in [10].

A dominating set *D* is called connected dominating set of *G* if the induced subgraph $\langle D \rangle$ is connected. The connected domination number $\gamma_c(G)$ of a graph *G* is the minimum cardinality of a connected dominating set in *G*. The connected domination number $\gamma_c[BS(G)]$ of a graph $BS(G)$ is the minimum cardinality of a connected dominating set in $BS(G)$. E. Sampathkumar and Walikar[13] defined a connected dominating set. For any connected graph *G* with $\Delta(G) < p - 1$, $\gamma(G) \leq \gamma_t(G) \leq \gamma_c(G)$.

The following figure illustrates the formation of a block subdivision graph *BS*(*G*) of a graph *G*.



G *S*(*G*) *BS*(*G*)

$p(G) = 7, q(G) = 8, n(G) = 4, \alpha_0(G) = 3, \beta_0(G) = 4, \alpha_1(G) = 4,$

$\beta_1(G) = 3, \gamma(G) = 1, \gamma_{re}(G) = 3, \gamma_t(G) = 2, \gamma_c(G) = 1, s(G) = 1,$

$$\gamma'(G) = 2, \Delta(G) = 6, \chi(G) = 3, \gamma_c[BS(G)] = 2, \gamma_c[BS(\bar{G})] = 2$$

In this paper, many bonds on $\gamma_c[BS(G)]$ were obtained in terms of vertices, blocks and other parameters of *G*. Also, we obtain some results on $\gamma_c[BS(G)]$ with other domination parameters of *G*. Finally, Nordhaus-Gaddam type results were established.

Results

Initially we present the exact value of connected domination number of a block subdivision graph of a non separable graph *G*.

Theorem 1: For any non separable graph *G*, $\gamma_c[BS(G)] = 1$.

The following result gives an upper bound on $\gamma_c[BS(G)]$ in terms of vertices *p* of *G*.

Theorem 2: For any connected (p, q) graph *G*, $\gamma_c[BS(G)] < 2p - 2$.

Proof: We prove the result in the following two cases.

Case (i): Suppose *G* is a tree then $q = p - 1$. And $|E(G)| = q \Longrightarrow |E[S(G)]| = 2q \Longrightarrow |V[BS(G)]| = 2q = 2(p-1) = 2p - 2$. Let $D \subseteq V[BS(G)]$ is a connected dominating set in $BS(G)$ such that $\gamma_c[BS(G)] = |D|$. Since total number of vertices in $BS(G)$ are $2p - 2$, from the definition of connected dominating set in $BS(G)$, $\gamma_c[BS(G)] = |D| < 2p - 2$.

Case (ii): Suppose *G* is not a tree and at least one block contains maximum number of vertices. Then clearly, $\gamma_c[BS(G)] = |D| < 2p - 2$.

From the above two cases we have, $\gamma_c[BS(G)] < 2p - 2$.

The following upper bound is a relationship between $\gamma_c[BS(G)]$, number of vertices *p* of *G* and number of cut vertices *s* of *G*.

Theorem 3: For any connected (p, q) graph *G*, $\gamma_c[BS(G)] < p(G) + s(G)$ where *s*(*G*) is number of cut vertices of *G*.

Proof: If *G* has no cut vertices then *G* is non separable. By Theorem 1, $\gamma_c[BS(G)] = 1 < p(G) + s(G)$. For any separable graph *G* we consider the following two cases.

Case(i): Let *G* be a tree. Since *s* is number of cut vertices of *G*, $C \subseteq V(G) \Longrightarrow |C| = s(G)$. Suppose $C' \subseteq V[BS(G)]$ be the set of cut vertices of $BS(G)$ such that $|C'| \geq |C| = s(G)$. Let $D \subseteq V[BS(G)]$ is a dominating set in $BS(G)$ such that $\gamma[BS(G)] = |D|$. Now $F = \{u_i \in N(D)\} - \{v_j\}$ where $\{u_i\}$ is the set of elements in neighbourhood of *D* and $\{v_j\}$ is a set of end vertices of $BS(G)$ such that $\langle D \cup F \rangle$ forms a connected dominating set of $BS(G)$. Hence, $\gamma_c[BS(G)] = |\langle D \cup F\rangle| = |D| + |F| = \gamma[BS(G)] + |F|$. In [11] we have, $\gamma[BS(G)] < p$. Clearly, $\gamma_c[BS(G)] < p(G) + s(G)$.

Case(ii): Suppose *G* is not a tree and at least one block contains maximum number of vertices. Then, clearly $\gamma_c[BS(G)] < p(G) + s(G)$.

From above, we get $\gamma_c[BS(G)] < p(G) + s(G)$.

We thus have a result, due to Ore [12].

Theorem A [12]: If *G* is a (p, q) graph with no isolated vertices, then $\gamma(G) \leq \frac{p}{2}$.

In the following Theorem we obtain the relation between $\gamma_c[BS(G)], \gamma(G)$ and *p* of *G*.

Theorem 4: For any connected (p, q) graph *G*, $\gamma_c[BS(G)] + \gamma(G) < \frac{5p}{2} - 2$.

Proof: From Theorem 2 and Theorem A, $\gamma_c[BS(G)] + \gamma(G) < 2p - 2 + \frac{p}{2} = \frac{5p}{2} - 2$. Hence, $\gamma_c[BS(G)] + \gamma(G) < \frac{5p}{2} - 2$.

We have a following result due to Harary [6].

Theorem B [6, P.95]: For any nontrivial (*p*, *q*) connected graph *G*, $\alpha_0(G) + \beta_0(G) = p = \alpha_1(G) + \beta_1(G)$.

The following Theorem is due to V.R.Kulli [9].

Theorem C [9, P.19]: For any graph *G*, $\gamma(G) \leq \beta_0(G)$.

In the following Corollary we develop the relation between $\gamma_c[BS(G)], \gamma(G), \alpha_0(G)$ and $\beta_0(G)$.

Corollary 1: For any connected (p, q) graph G, $\gamma_c[BS(G)] + \gamma(G) < 2\alpha_0(G) + 3\beta_0(G) - 2$.

Proof: From Theorem 2, Theorem B and Theorem C,

$$\gamma_c[BS(G)] + \gamma(G) < (2p - 2) + \beta_0(G) = [2(\alpha_0(G) + \beta_0(G)) - 2] + \beta_0(G) = 2\alpha_0(G) + 3\beta_0(G) - 2$$

. Hence, $\gamma_c[BS(G)] + \gamma(G) < 2\alpha_0(G) + 3\beta_0(G) - 2$.

T.W.Haynes et al. [7] establish the following result.

Theorem D [7, P.165]: For any connected graph G, $\gamma_c(G) \leq 2\beta_1(G)$.

In the following Corollary we develop the relation between $\gamma_c[BS(G)], \gamma_c(G), \alpha_1(G)$ and $\beta_1(G)$.

Corollary 2: For any connected (p, q) graph G, $\gamma_c[BS(G)] + \gamma_c(G) < 2\alpha_1(G) + 4\beta_1(G) - 2$.

Proof: From Theorem 2, Theorem B and Theorem D,

$$\gamma_c[BS(G)] + \gamma_c(G) < (2p - 2) + 2\beta_1(G) = [2(\alpha_1(G) + \beta_1(G)) - 2] + 2\beta_1(G) = 2\alpha_1(G) + 4\beta_1(G) - 2$$

. Hence, $\gamma_c[BS(G)] + \gamma_c(G) < 2\alpha_1(G) + 4\beta_1(G) - 2$.

The following Theorem establishes an upper bound on $\gamma_c[BS(G)]$.

Theorem 5: For any connected (p, q) graph G, $\gamma_c[BS(G)] < p + q - 1$.

Proof: Let G be a connected graph with p vertices and q edges. Since for any connected graph G, $p - 1 \leq q$, by Theorem 2, $\gamma_c[BS(G)] < 2p - 2 = p + (p - 1) - 1 \leq p + q - 1$. Hence, $\gamma_c[BS(G)] < p + q - 1$.

The following Theorem relates connected domination number of a block subdivision graph $BS(G)$ and number of blocks *n* of *G*.

Theorem 6: For any connected (p,q) graph *G*, $\gamma_c[BS(G)] \leq 2n(G)-1$ where $n(G)$ is number of blocks of *G*. Equality holds for any non separable graph *G*.

Proof: We consider the following two cases.

Case (i): For an equality, suppose *G* is a non separable graph. Then by Theorem 1, $\gamma_c[BS(G)] = 1$ and $n(G) = 1$. Therefore, $\gamma_c[BS(G)] = 1 = 2(1) - 1 = 2n(G) - 1$. Hence, $\gamma_c[BS(G)] = 2n(G) - 1$.

Case (ii): Suppose *G* is a separable graph. Then *G* contains at most $p-1$ blocks in it. From Theorem 2, $\gamma_c[BS(G)] < 2p - 2 = 2(p-1) = 2n(G)$. Hence, $\gamma_c[BS(G)] < 2n$. Since $n(G) \geq 2$, clearly $\gamma_c[BS(G)] \leq 2n(G) - 1$.

From the above two cases, we have $\gamma_c[BS(G)] \leq 2n(G) - 1$.

A relationship between the connected domination number of $BS(G)$, *p* of *G* and number of blocks *n* of *G* is given in the following result.

Theorem 7: For any connected (p,q) graph *G*, $\gamma_c[BS(G)] < p(G) + n(G) - 1$.

Proof: From Theorem 2 and Theorem 6, we get

$$\gamma_c[BS(G)] + \gamma_c[BS(G)] < (2p-2) + 2n - 1 \Longrightarrow 2\gamma_c[BS(G)] < 2p(G) + 2n(G) - 3 \Longrightarrow$$
$$\gamma_c[BS(G)] < p(G) + n(G) - 1.5 < p(G) + n(G) - 1$$

. Hence the proof.

The following upper bound was given by S.T.Hedetniemi and R.C.Laskar [8].

Theorem E[8]: For any connected (p,q) graph *G*, $\gamma_c(G) \le p - \Delta(G)$.

Now we obtain the following result.

Theorem 8: For any connected (p,q) graph *G*, $\gamma_c[BS(G)] + \gamma_c(G) < 3p - \Delta(G) - 2$.

Proof: From Theorem 2 and Theorem E, the result follows.

The following Theorem is due to F.Haray [6].

Theorem F [6, P.128]: For any graph *G*, the chromatic number is at most one greater than the maximum degree, $\chi(G) \le 1 + \Delta(G)$.

We establish the following upper bound.

Theorem 9: For any connected (p,q) graph *G*, $\gamma_c[BS(G)] + \chi(G) \le 2n(G) + \Delta(G)$. Equality holds if *G* is isomorphic to K_m.

Proof: From Theorem 6 and Theorem F, $\gamma_c[BS(G)] \le 2n(G) - 1$ and $\chi(G) \le 1 + \Delta(G)$, $\gamma_c[BS(G)] + \chi(G) \le 2n(G) - 1 + 1 + \Delta(G) = 2n(G) + \Delta(G)$.

For the equality, if *G* is isomorphic to K_m then $\gamma_c[BS(G)] = 1, \chi(G) = m, n(G) = 1$ and $\Delta(G) = m - 1$. Hence $\gamma_c[BS(G)] + \chi(G) = 1 + m = 2(1) + (m-1) = 2n(G) + \Delta(G)$.

The following Theorem is due to E.Sampathkumar and H.B.Walikar[13].

Theorem G[13]: If *G* is a connected (p,q) graph with $p \ge 3$ vertices, $\gamma_c(G) \le p - 2$.

The following result provides another upper bound for $\gamma_c[BS(G)]$ and $\gamma_c(G)$.

Theorem 10: If *G* is a connected graph with $p \geq 3$ vertices, $\gamma_c[BS(G)] + \gamma_c(G) < 3p - 4$.

Proof: From Theorem 2 and Theorem G, the result follows.

The following upper bound was given by V.R.Kulli[9].

Theorem H[9, P.44]: If *G* is connected (p, q) graph and $\Delta(G) < p - 1$, then $\gamma_t(G) \leq p - \Delta(G)$.

We obtain the following result.

Theorem 11: If *G* is a connected (*p*, *q*) graph and $\Delta(G) < p - 1$, $\gamma_c[BS(G)] + \gamma_t(G) < 3p - \Delta(G) - 2$.

Proof: Suppose *G* is a connected (*p*, *q*) graph and $\Delta(G) < p - 1$. From Theorem 2 and Theorem H, $\gamma_c[BS(G)] < 2p - 2$ and $\gamma_t(G) \leq p - \Delta(G)$,

$\gamma_c[BS(G)] + \gamma_t(G) < (2p - 2) + (p - \Delta(G)) = 3p - \Delta(G) - 2$. Hence the proof.

The following Theorem is due to S.Arumugam et al. [1].

Theorem I[1]: For any (*p*, *q*) graph *G*, $\gamma'(G) \leq \left\lfloor \frac{p}{2} \right\rfloor$. The equality is obtained for $G = K_p$.

Now we establish the following upper bound.

Theorem 12: For any (*p*, *q*) graph *G*, $\gamma_c[BS(G)] + \gamma'(G) \leq 2n(G) - 1 + \left\lfloor \frac{p}{2} \right\rfloor$. Equality is obtained for $G = C_4, C_5, W_4, K_p, K_{m,n}$.

Proof: From Theorem 6 and Theorem I, the result follows. For the equality if $G = C_4$, $\gamma_c[BS(G)] = 1, \gamma'(G) = 2, n(G) = 1, p = 4$ then $\gamma_c[BS(G)] + \gamma'(G) = 1+2 = 3 = 2(1) - 1 + \left\lfloor \frac{4}{2} \right\rfloor = 2n(G) - 1 + \left\lfloor \frac{p}{2} \right\rfloor$.

Next the following upper bound was established.

Theorem 13: For any (*p*, *q*) tree *T*, $\gamma_c[BS(T)] + \gamma_{re}(T) < \left\lfloor \frac{5p}{2} \right\rfloor + m(T)$, where *m* is the number of end vertices of *T*.

Proof: Suppose (*p*, *q*) be any tree *T*, then $q = p - 1$. Let $H = \{u_1, u_2, \ldots, u_k\} \subseteq V(T)$ be the set of vertices of $\deg(u_i) \geq 2, 1 \leq i \leq k$. If $I = \{v_1, v_2, \ldots, v_m\} \subseteq V(T)$ be the set of all end vertices in *T*, then $I \cup H_1$ where $H_1 \subseteq H$, forms a minimal restrained dominating set of *T*. Then $\gamma_{re}(T) = |I \cup H_1| = |I| + |H_1| \leq m(T) + \left\lfloor \frac{p}{2} \right\rfloor$. Since $V[BS(T)] = E[S(T)] = 2q = 2(p - 1) = 2p - 2$ and $D \subseteq V[BS(T)]$ be the connected dominating set such that $\gamma_c[BS(T)] = |D| < 2p - 2 < 2p$. From the above, clearly $\gamma_c[BS(T)] + \gamma_{re}(T) = |I \cup H_1| \cup |D| < 2p + m(T) + \left\lfloor \frac{p}{2} \right\rfloor = \left\lfloor \frac{5p}{2} \right\rfloor + m(T)$. Hence, $\gamma_c[BS(T)] + \gamma_{re}(T) < \left\lfloor \frac{5p}{2} \right\rfloor + m(T)$.

Bonds on the sum and product of the connected domination number of a block subdivision graph $BS(G)$ and its complement $BS(\bar{G})$ were given under.

Nordhaus – Gaddam type results:

Theorem 14: For any (*p*, *q*) graph *G* such that both *G* and $\bar{G}$ are connected. Then

$$\gamma_c[BS(G)] + \gamma_c[BS(\bar{G})] \leq 2q$$

$$\gamma_c[BS(G)].\gamma_c[BS(\bar{G})] \leq q^2.$$

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