Compact Composition Operators between Hardy Spaces

Elhadi Elnour Elniel

Department of Mathematics, College of Science and Arts, Al Mandag, Al Baha University, Saudi Arabia

elhadielniel_2003@hotmail.com

Abstract

We characterise composition operators between Hardy spaces. Certain growth conditions for generalized Nevanlinna counting functions of the inducing map are shown to be necessary and sufficient for such operators to be bounded or compact. Under a mind condition we show that a composition operators C_φ is compact on the Hardy spaces of the open unit ball in C^n.

Keywords: Hardy Spaces, composition operators, boundedness, Compactness, Nevanlinna counting functions.

1. Introduction

In this paper we follow the same literature and methods of Elhadi Elniel, M.E. Hassan [1], and Wayne Smith [2] with a little change. Let D be the open unit disk in the complex plane and denote Lebesgue measure on D by dA, normalized so that $A(D) = 1$. The Hardy space H^p is the space of functions f that are analytic on D and satisfy

$$\|f\|_{H^p}^p = \lim_{r \to 1-} \frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{i\theta}) \right|^p d\theta < \infty,$$

and the Bergman space A^p consists of those analytic functions such that

$$\|f\|_{A^p}^p = \int_D |f|^p \, dA < \infty.$$

Let $\varphi : D \to D$ be an analytic self-map of D. It is a well-known consequence of Littlewood's subordination principle [2], [3] that φ induces through composition a bounded linear operator on the classic hardy and Bergman spaces (see for example [4], [5], [6], [7] or [8]. That is, if we define C_φ by $C_\varphi(f) = fo\varphi$, then $C_\varphi : H^p \to H^q$ and $C_\varphi : A^p \to A^p$ are bounded operators. Such operators are called composition operators.

Theorem1.1. Let $0 < p \leq q$ and suppose φ is an analytic self-map of D. Then

 a) $C_\varphi : A^p \to A^q$ is bounded if and only if $N_{\varphi,2}(w) = O([\log(1/|w|)]^{2q/p})$ $(|w| \to 1)$;

 b) $C_\varphi : A^p \to H^q$ is bounded if and only if $N_{\varphi,1}(w) = O([\log(1/|w|)]^{2q/p})$ $(|w| \to 1)$;

 c) $C_\varphi : H^p \to A^q$ is bounded if and only if $N_{\varphi,2}(w) = O([\log(1/|w|)]^{q/p})$ $(|w| \to 1)$;

Theorem1.2. Let $0 < p \leq q$ and suppose φ is an analytic self-map of D. Then $C_\varphi : H^p \to H^q$ is bounded if and only if

$$N_{\varphi,1}(w) = O([\log(1/|w|)]^{q/p}),$$

and moreover C_φ is compact if and only if

$$N_{\varphi,1}(w) = O(\log(1/|w|)^{q/p})$$

where $|w| \to 1$.

Corollary1.3. Let $\eta \geq 2$ and suppose φ is an analytic self-map of D. Then the following are equivalent.

 a) There exists $p > 0$ such that $C_\varphi : H^p \to H^{\eta q}$ is bounded;

 b) $C_\varphi : H^p \to H^{\eta p}$ is bounded for all $p > 0$;

 c) There exists $p > 0$ such that $C_\varphi : A^p \to H^{\eta p/2}$ is bounded;

 d) $C_\varphi : A^p \to H^{\eta p/2}$ is bounded for all $p > 0$.

Moreover, these four statements remain equipment when "bounded" is replaced by "compact"

Corollary1.4. Let $\eta \geq 1$ and suppose φ is an analytic self-map of D. Then the following are equivalent.

    a) There exists p> 0 such that $C_\varphi : A^p \to H^{\eta p}$ is bounded;

    b) $C_\varphi : A^p \to A^{\eta p}$ is bounded for all $p > 0$;

    c) There exists p> 0 such that $C_\varphi: H^p \to A^{2\eta p}$ is bounded;

    d) $C_\varphi : H^p \to A^{2\eta p}$ is bounded for all p> 0.

Moreover, these four statements remain equipment when "bounded" is replaced by "compact".

Corollary1.5. Let $\eta \geq 1$ and suppose φ is an analytic self-map of D.

If $C_\varphi : H^p \to H^{\eta p}$ is bounded for some (and hence all) p> 0, then $C_\varphi : A^p \to A^{\eta p}$ is bounded for all p> 0. Moreover, this remains true when "bounded" is replaced by "compact".

2. Preliminaries

Definition2.1. We introduce a family of weighted Bergman type spaces that allows us to handle the classical Bergman and Hardy spaces in a unified manner. For $\alpha > -1$ define the measure dA_α on D by $dA_\alpha(w) = [\log(1/|w|)]^\alpha dA(w)$. For $0<p<\infty$ and $\alpha > -1$ we define the weighted Bergman space A_α^p to be those functions f analytic on D and satisfying.

$$\|f\|_{A_\alpha^p}^p = \int_D |f(w)|^p dA_\alpha(w) < \infty.$$

In this definition, the measure dA_α can be replaced by the measure $(1 - |w|)^\alpha dA(w)$, as in [4], [9], [10], and [11]. This result in the same space of functions and an equivalent norm, since $(1 - |w|)^\alpha$ and $[\log(1/|w|)]^\alpha$ are comparable for

$\frac{1}{2} \leq |w| < 1$, and the singularity of dA_α at the origin is integrable.

Definition2.2. Let dA (z) be the area measure on D normalized so that area of D is 1. For each $\alpha \in (-1,\infty)$, we set $dA_\alpha(z) = (\alpha+1)(1 - |z|^2)^\alpha dA(z)$, $z \in D$. Then dA_α is a probability measure on D. For $0 <p<\infty$ the weighted Bergman space A_α^p is defined as

$$A_\alpha^p = \left\{ f \in H(D) : \|f\|_{A_\alpha^p} = \left(\int_D |f(z)|^p dA_\alpha(z) \right)^{1/p} < \infty \right\}.$$

Note that $\|f\|_{A_\alpha^p}$ is a true norm only if $1 \leq p < \infty$ and in this case A_α^p is a Branch space.

Definition2.3. For any $\alpha > 0$, the space $A^{-\alpha}$ consists of Analytic functions f in D such that

$$\|f\|_{A^{-\alpha}} = sup\{(1 - |z|^\alpha)^\alpha |f(z)| : z \in D\} < \infty.$$

Each $A^{-\alpha}$ is a non-separable Branch space with the norm defined above and contains all bounded analytic functions on D. The closure in $A^{-\alpha}$ of the set of polynomials will be denoted by $A_0^{-\alpha}$, which is a separable Banach space and consists of exactly those functions f in $A^{-\alpha}$ with

$$\lim_{z \to 1-} (1 - |z|^2)^\alpha |f(z)| = 0$$

For general background on weighted Berman spaces A_α^p and Bergman type spaces, $A^{-\alpha}$ and $A_0^{-\alpha}$, one may consult [12] [13], and [14] and the references therein.

Lemma2.4. If $0<p<\infty$, then $\|f\|_{A_\alpha^p}^p \approx |f(0)|^p + \int_D |f(w)|^{p-2}|f'(w)|^2 dA_{\alpha+2}(w)$.

Here the symbol "$\approx$" means that the left hand side is bounded above and below by constant multiples of the right hand side, where the constants are positive and independent of f.

Proposition2.5. Let φ be an analytic self-map of D and let f be analytic on D. Then, for $\alpha \geq$ -1,

$$\|f \circ \varphi\|_{A_\alpha^p}^p \approx |f(\varphi(0))|^p + \int_D |f|^{p-2}|f'|^2 N_{\varphi,\alpha+2} \, dA.$$

Lemma2.6. Let $0<p<\infty$ and $\alpha \geq$ -1. If $f \in A_\alpha^p$ and $w \in D$, then $|f(w)| \leq C\|f\|_{A_\alpha^p}(1-|w|)^{-(\alpha+2)/p}$.

Lemma2.7. Suppose $p>0$ and $\alpha>$ -1. Then the following conditions are equivalent for any positive Borel measure μ on B_n.

(i) μ is a Carleson measure for A_α^p, that is, there exists a constant $C > 0$ such that

$$\int_{B_n} |f|^p d\mu \leq C \int_{B_n} |f(z)|^p dv_\alpha \quad \text{for all } f \in A_\alpha^p$$

(ii)   For some (or each) R >0 there exists a constant $C>0$ (depending on R and α but independent of a) such that $\mu(D(a,R)) \leq Cv_\alpha(D(a,R))$ For all $a \in B_n$ where $D(a,R)$ is the Bergman metric ball at a with radius R.

Proof. See [12] for example.

Corollary 2.8. Suppose $p>0$, $q>0$, and $\alpha>$ -1. Then C_α is compact on A_α^p if and only if C_φ is compact on A_α^p We need two more technical lemmas. The first of which is called Schur's test and concerns the boundedness of integral operators on L^p spaces. Thus we consider a measure space (X,μ) and an integral operator

$$T f(x) = \int_X H(x,y)f(y) \, d\mu(y), \tag{1}$$

where H is a nonnegative measurable function on $X \times X$.

Lemma 2.9. Suppose there exist a positive measurable function h on X such that $\int_X H(x,y)h(y)d\mu(y) \leq Ch(x)$ for almost all x and $\int_X H(x,y)h(x)d\mu(x) \leq Ch(y)$ for almost all y, where C is a positive constant. Then the integral operator T defined in (1) is bounded on $L^2(X,d\mu)$. Moreover, the norm of T on $L^2(X,d\mu)$ is less than or equal to the constant C.

Lemma 2.10. Suppose $\alpha>$ -1 and $t>0$. Then there exists a constant $C>0$ such that

$$\int_{B_n} \frac{dv_\alpha(w)}{|1-\langle z,w\rangle|^{n+1+\alpha+t}} \leq \frac{C}{(1-|z|^2)^t}$$

for all $z \in B_n$.

Proof. See [2].

Theorem 2.11. Suppose $p>0$, $\alpha>$ -1 and $t>0$. Then the composition operator C_α is bounded on A_α^p if and only if

$$\underset{a\in B_n}{Sup}(1-|a|^2)^t \int_{B_n} \frac{dv_\alpha(z)}{|1-\langle\alpha,\varphi(z)\rangle|^{n+1+\alpha+t}} < \infty \tag{2}$$

Proof. It follows from Lemma2.10 that the boundedness of C_φ on A_α^p implies condition (2).Next we assume that condition (2) holds. Then by the change of variables formula there exists a constant $C>0$ such that

$$(1-|a|^2)^t \int_{B_n} \frac{d\mu_{\varphi,\alpha}(z)}{|1-\langle a,z\rangle|^{n+1+\alpha+t}} \leq C, \quad \text{for all } a \in B_n.$$

For any fixed positive radius R we have

$$(1-|a|^2)^t \int_{D(a,R)} \frac{d\mu_{\varphi,\alpha}(z)}{|1-\langle a,z\rangle|^{n+1+\alpha+t}} \leq C, \text{ for all } a \in B_n.$$

It is well known that $|1-\langle a,z\rangle| \sim 1-|a|^2$, for $z \in D(a,R)$, and it is also well known that

$$(1-|a|^2)^{n+1+\alpha} \sim v_\alpha(D(a,R));$$

See [12]. It follows that there exists another positive constant C (independent of a) such that

$$\mu_{\varphi,\alpha}(D(a,R)) \leq C v_{\alpha}(D(a,R)) \text{ for all } a \in B_n.$$

By Lemma 2.7, the measure $\mu_{\varphi,\alpha}$ is Carleson for A_{α}^{p} and so the composition operator C_{φ} is bounded on A_{α}^{p} .

3. Necessary and Sufficient Conditions for Bounded

In this section we justify Necessary and sufficient conditions for $C_{\varphi} : H^p \rightarrow H^q$ to be bounded (see [2] and [14]).For any holomorphic $\varphi : B_n \rightarrow B_n$ we can define a positive Borel measure $\mu_{\varphi,}$ on B_n as follows. Given a Borel set E in B_n, we set

$$\mu_{\varphi,}(E) = v(\varphi^{-1}(E)) = C_{\varphi} \int_{\varphi^{-1}(E)} (1 - |(z)|^2) dv(z) \quad .$$

Obviously, $\mu_{\varphi,}$ is the pullback measure of dv under the map φ . Therefore, we have the following change of variables formula:

$$\int_{B_n} f(\varphi)dv = \int_{B_n} f d\mu_{\varphi} ,$$

where f is either nonnegative or belongs to $L^1(B_n, d\mu_{\varphi,})$.

3.1 A sufficient condition for $C_{\varphi} : H^p \rightarrow H^q$ to be bounded.

Theorem3.1.1 Let $0 < p \leq q$ suppose φ is an analytic self-map of D satisfing.

$$N_2(w) = O([\log(1/|w|)]^{2q/p}) \qquad (|w| \rightarrow 1).$$

Then $C_{\varphi} : H^p \rightarrow H^q$ is bounded.

Proof. By the Closed Graph Theorem, it suffices to show that $C_{\varphi}(f) \in H^q$ if $f \in H^p$. Let $f \in H^p$. Then, by Proposition 2.5,

$$\|f \circ \varphi\|_{H^q}^q \approx |f(\varphi(0))|^q + \int_D |f|^{p-2}|f'|^2 N_2 \, dA$$

and it is clear that $C_{\varphi}(f) \in H^q$ if and only if there is $r \in (0,1)$ such that

$$\int_{D \setminus rD} |f|^{q-2}|f'|^2 N_2 \, dA < \infty.$$

By our hypothesis on the growth of N_2 there is a constant $K < \infty$ and $r_0 \in (0,1)$ such that

$$N_2(w) \leq K(\log(1/|w|))^{2q/p} \quad ,w \in D \setminus r_0 D$$

Using this and the growth estimate for $|f|$ from Lemma 2.6,

$$\int_{D \setminus r_0 D} |f|^{q-2}|f'|^2 N_2 \, dA \leq$$

$$CK\|f\|_{H^p}^{q-p} \int_{D \setminus r_0 D} |f(w)|^{p-2}|f'(w)^2|(1-|w|)^{2(p-q)/p} dA_{2q/p}(w).$$

We may assume $r_0 \geq \frac{1}{2}$, so that $\log(1/|w|) \leq 2(1 - |w|)$. Using this estimate in our upper bound above, we see that

$$\int_{D \setminus r_0 D} |f|^{q-2}|f'|^2 N_2 dA \leq C\|f\|_{H^p}^{q-p} \int_{D \setminus r_0 D} |f(w)|^{p-2}|f'(w)|^2 dA_2(w) \leq C\|f\|_{H^p}^q$$

which completes the proof.

Theorem3.1.2. Let $0 < q \leq p$ and suppose φ is an analytic self-map of D satisfying $N_2(w) = O([\log(1/|w|)]^\eta)(|w| \rightarrow 1)$, for some

$$\eta > \frac{q+p}{p}$$

Then $C_{\varphi} : H^p \rightarrow H^q$ is bounded.

Proof. Let $f \in H^p$. As in the proof of Theorem 3.1.1, it suffices to show that $C_{\varphi}(f) \in H^q$ and for this it suffices to show there is $r \in (0,1)$ such that

$$\int_{D\backslash rD} |f|^{q-2}|f'|^2 N_2 dA < \infty.$$

By our hypothesis on the growth of N_2, there is a constant $K<\infty$ and $r_0\in(0,1)$ such that $N_2(w) \leq K(\log(1/|w|))^\eta$, $w\in D\backslash r_0 D$ where

$$\eta > \frac{q+p}{p}$$

Now $\eta>1$. Thus, by Lemma 2.4, it suffices to show that

$$\int_{D\backslash r_0 D} |f(w)|^q (1-|w|)^{\eta-2} dA(w) < \infty.$$

By Hölder's inequality, this integral is bounded by

$$\left(\int_D [|f(w)|^q (1-|w|)]^{p/q}\right)^{q/p} \left(\int_D [(1-|w|)^{\eta-2}]^{\frac{p}{p-q}} dA(w)\right)^{(p-q)/p}$$

The first factor is bounded by $\|f\|_{H^p}^q$ and so is finite, while the second factor is finite because the assumed lower bound for η is equivalent to the exponent in the integral being strictly greater than -1. Thus the proof is complete.

3.2 A necessary condition for $C_\varphi: H^p \to H^q$ to be bounded.

Lemma3.2.1. Let ψ be an analytic self-map of D and let $\gamma \geq 1$. If $\psi(0)$ 0 and $0<r<|\psi(0)|$, then

$$N_{\psi,\gamma}(0) \leq \frac{1}{r^2} \int_{rD} N_{\psi,\gamma} dA.$$

The next lemma shows how the counting functions transform under composition. The case $\gamma=1$ of this lemma can be found in [6].

Lemma3.2.2. Let ψ be an analytic self-map of D let $\alpha\in D$ and let $\sigma_a(w) \leq \frac{a-w}{1-\bar{a}w}$

be the Möbius self-map of D that interchanges 0 and a. Then
$$(N_{\psi,\gamma})\circ\sigma_a = N_{\sigma_a\circ\psi,\gamma}.$$

Theorem3.2.3. Suppose that φ is an analytic self-map of D that induces a bounded composition operator $C_\varphi; H^p \to H^q$. Then $N_2(w) = O((\log(1/|w|))^{2q/p})$ $(|w|\to 1)$.

Proof: By Proposition 2.5, there is a constant C_1 such that

$$C_1\|C_\varphi(k_a)\|_{H^q}^q \geq \int_D |k_a(w)|^{q-2}|k'_a(w)|^2 N_2(w) dA(w)$$
$$= \frac{16}{p^2}|a|^2 (1-|a|^2)(1-|a|^2)^{2q/p}\int_D \frac{N_2(w)}{|1-\bar{a}w|^{2+4q/p}} dA(w)$$
$$= \frac{16}{p^2}|a|^2 (1-|a|^2)^{2q/p-2}\int_D \frac{N_2(w)}{|1-\bar{a}w|^{4q/p-2}}|\sigma'_a(w)|^2 dA(w)$$
$$= \frac{16}{p^2}|a|^2 (1-|a|^2)^{2q/p-2}\int_D \frac{N_2(\sigma_a(z))}{|1-\bar{a}\sigma_a(z)|^{4q/p-2}} dA(z)$$

Here $\sigma_a = \sigma_a^{-1}$ is the Möbius self-map of D that interchanges 0 and a, as in Lemma 3.2.2 and the change of variable

$z = \sigma_a(w)$ was made in the last line. Now,

$$\frac{1}{|1-\bar{a}\sigma_a(z)|} = \frac{|1-\bar{a}z|}{1-|a|^2} \geq \frac{1}{2}\frac{1}{1-|a|^2}, \ |z| \leq \frac{1}{2}$$

and so

$$C_1\|C_\varphi(k_a)\|_{H^q}^q \geq \frac{4^{2-2q/p}4|a|^2}{p^2(1-|a|^2)^{2q/p}}\int_{\frac{1}{2}D} N_2(\sigma_a(z)) dA(z).$$

We now apply first Lemma 3.2.2, then Lemma 3.2.1 and then Lemma 3.2.2 again to see, provided that $\sigma_a \circ \varphi(0) > \frac{1}{2}$, the integral in the line above is at least

$$\frac{1}{4}N_2, \sigma_a \circ \varphi(0) = \frac{1}{4}N_2 \quad (a).$$

Since $\sigma_a \circ \varphi(0) > \frac{1}{2}$ if $|a|$ is sufficiently close to 1, this provides the estimate that

$$N_2(a) \leq C1 \|C\varphi(ka)\|_{H^q}^q \frac{4^{2q/p-1}p^2}{4|a|^2}(1-|a|^2)2q/p \tag{3}$$

for all such a. Since $\|k_a\|_{A_{(\alpha+1)^2}^p} \approx 1$ and $\log(1/|a|)$ is comparable to $(1-|a|^2)$ for $\frac{1}{2} < |a| < 1$, the assumption that C_φ is bounded provides the asserted bound for N_2 and the proof is complete.

As noted at the beginning of this section, the necessary condition in Theorem 3.2.3 for C_φ to be pounded agrees with the sufficient condition from Theorem 3.1.1 that holds for $q \geq p$, and so the following corollary results.

Corollary3.2.4 Let $0 < p \leq q$ and let φ be an analytic self-map of D. Then $C_\varphi: \to H^q$ is bounded if and only if

$$N_2(w) = O((\log(1/|w|))^{2q/p}) \qquad (|w| \to 1).$$

4. Compactness of Hardy Spaces

A bounded linear operator T from a Branch space X to a Branch space Y is said to be compact provided the closure of $T(B)$ is a compact subset of Y, where B is the unit ball of X. Equivalently, T is compact if and only if some subsequence of $\{T(x_n)\}$ converges in Y, whenever $\{x_n\}$ is a bounded sequence in X. Our goal here is to characterize those analytic functions $\varphi : D \to D$ that induce bounded composition operators from one Hardy to another such spaces (see [2], [15], and [16]).We also show the conditions that a composition operators $C\varphi$ is compact on Hardy spaces of the open unit ball in C^n (see [17] and [18].

Theorem4.1. Let $0 < p \leq q$ and let φ be an analytic self-map of D. Then $C_\varphi; H^p \to H^q$ is compact if and only if

$$N_2(w) = O((\log(1/|w|))^{2q/p}) \qquad (|w| \to 1).$$

Proof. We first show that C_φ is compact, assuming that N_2 satisfying the given growth condition. Let $\|f_n\|_{H^p} \leq 1$ for $n \geq 1$. We must show that $\{f_n \circ \varphi\}$ has a subsequence that converges in H^q. By Lemma 2.6, the f_n are uniformly bounded on compact subsets of D, and hence $\{f_n\}$ is a normal family there. Thus there is a subsequence, which for simplicity we continue to denote by $\{f_n\}$, that converges uniformly on compact subsets of D to an analytic function f. Now, for each $r \in (0,1)$,

$$\int_{rD}|f|^{p-2}|f'_n|^2 dA = \lim_{n\to\infty}\int_{rD}|f_n|^{p-2}|f'_n|^2 dA \leq \lim_{n\to\infty} \sup C\|f_n\|_{H^p} \leq C,$$

by Lemma 2.4. It follows that $f \in H^p$, and so $f \circ \varphi \in H^q$ by Theorem 3.1.1

To complete the proof, it suffices to show $\|f_n \circ \varphi - f \circ \varphi\|_{H^q} \to 0$ as $n \to \infty$.To establish this, note that from Proposition 2.5 it suffices to show that

$$|f_n(\varphi(0) - f(\varphi(0))|^p + \int_{rD}|f_n - f|^{q-2}|f'_n - f'|^2 N_2 dA + \int_{D\backslash rD}|f_n - f|^{q-2}|f'_n - f|^2 N_2 dA \tag{4}$$

can be made arbitrarily small by choosing n large. For any fixed $r \in (0,1)$, the uniform convergence of f_n to f on compact subsets of D shows that the first two terms in the display above converge to 0 as $n \to \infty$. Thus it suffices to show that the third term in (4) tends to zero, uniformly of n, as $r \to 0$. To this end, let $\varepsilon > 0$, and note that by hypothesis we can choose $r \in (0,1)$ so that

$$\int_{D\backslash rD}|f_n - f|^{q-2}|f'_n - f|^2 N_2 dA \leq \varepsilon \int_{D\backslash rD}|f_n - f|^{q-2}|f'_n - f|^2 dA_{2q/p}$$

We are now in exactly the same situation that occurred at the end of the proof of Theorem 3.1.1 without providing all the details from that proof, the bound for $|f_n - f|$ from Lemma 2.6 leads to the estimate

$$\int_{D\backslash rD}|f_n - f|^{q-2}|f'_n - f'|^2 N_2 dA \leq \varepsilon C\|f_n - f\|_{H^q}^q \leq \varepsilon C(\|f\|_{H^q} + 1)^q.$$

Since $\varepsilon > 0$ was arbitrary, C_φ is compact and the first part of the proof is complete.

We now finish the proof by assuming that C_φ is compact and proving that N_2 satisfies the stated growth conditions. For $a \in D$, let K_a be as defined in $\xi\, 3$,

$$ka(w) = \frac{(1 - |a|^2)^{2/p}}{(1 - \bar{a}w)^{4/p}}$$

and recall that $\|k_a\|_{H^p} \approx 1$. Let $\{a_n\} \subset D$ satisfy $|a_n| \to 1$ as $n \to \infty$. From the definition of k_a, it is clear that this implies $k_{a_n}(z)$ converges uniformly to 0 on compact subsets of D as $n \to \infty$. Hence the zero element of H^q is the only possible limit point of $\{k_{a_n}(z) \circ \varphi\}$. The compactness of C_φ therefore yields that $\lim_{|a| \to 1} \|C_\varphi(k_a)\|_{H^q} = 0$

The required growth condition for N_2 is an immediate consequence of this and (3), the bound for N_2 that was derived in the proof of theorem 3.2.3, and the proof is complete.

Theorem4.2. Let $0 < q < p$ and suppose φ is an analytic self-map of D satisfying the conditions of Theorem 3.1.1 Then $C_\varphi : H^p \to H^q$ is compact.

Theorem4.3. Suppose $p > 0$. If C_φ is bounded on H^q for some $q > 0$, then C_φ is compact on H^p if and only if

$$\lim_{|z| \to 1^-} \left\{ \frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right\} = 0 \tag{5}$$

Proof. According to Corollary 2.8, we may assume that $p=2$. The normalized reproducing kernels of H^2 are given by

$$k_z(\theta) = \frac{(1 - |re^{i\theta}|^2)^{(n+1)/2}}{|1 - (re^{i\theta}, z)|^{n+1}}.$$

Each k_z is a unit vector in H^2 and it is clear that

$$\lim_{|z| \to 1^-} k_z(re^{i\theta}) = 0, \quad w \in Bn$$

Furthermore, the coverage is uniform when θ is restricted to any compact subset of B_n. A standard computation shows that

$$\frac{1}{2\pi} \int_{B_n} |C_\varphi^* k_z|^2 d\theta = \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{n+1}$$

So the compactness of C_φ on H^2 (which is the same as the compactness of C_φ^* on H^2) implies condition (2). We proceed to show that condition (5) implies the compactness of C_φ on H^2, provided that C_φ is bounded on H^q. An easy computation shows that the operator

$$C_\varphi C_\varphi^* : H^2 \to H^2$$

admits the following integral representation:

$$C_\varphi C_\varphi^* f(z) = \frac{1}{2\pi} \int_{B_n} \frac{f(re^{i\theta}) d\theta}{(1 - \langle \varphi(z), \varphi(re^{i\theta}) \rangle)^{n+1}}, \quad f \in H^2 \tag{6}$$

We will actually prove the compactness of $C_\varphi C_\varphi^*$ on H^2, which is equivalent to the compactness of C_φ on H^2. In fact, our arguments will prove the compactness of the following integral operator on $L^2(B_n, d\theta)$:

$$Tf(z) = \frac{1}{2\pi} \int_{B_n} \frac{f(re^{i\theta}) d\theta}{(1 - \langle \varphi(z), \varphi(re^{i\theta}) \rangle)^{n+1}} \tag{7}$$

For any $r \in (0,1)$ let X_r denote the characteristic function of the set $\{z \in \mathbb{C}^n : r < |z| < 1\}$. Consider the following integral operator on $L^2(B_n, d\theta)$:

$$T_r f(z) = \frac{1}{2\pi} \int_{B_n} H_r\left(z, re^{i\theta}\right) f\left(re^{i\theta}\right) d\theta, \tag{8}$$

where

$$H_r\left(z, re^{i\theta}\right) = \frac{\chi_r(z)\chi_r(re^{i\theta})}{(1 - \langle \varphi(z), \varphi(re^{i\theta})\rangle)^{n+1}}$$

is a nonnegative integral kernel. We are going to estimate the norm of T_r on $L^2(B_n, d\theta)$ in terms of the quantity.

$$M_r = \operatorname*{Sup}_{r<|z|<1} \frac{1 - |z|^2}{1 - |\varphi(z)|^2}$$

We do this with the help of Schur's test. Let $\sigma > 0$, and consider the function $h(z) = (1 - |z|^2)^{-\sigma}$, $z \in B_n$ we have

$$\int_{B_n} H_r\left(z, re^{i\theta}\right) h\left(re^{i\theta}\right) d\theta \leq C \int_{B_n} \frac{\chi_r(z)\chi_r\left(re^{i\theta}\right) d\theta}{(1 - \langle \varphi(z), \varphi(re^{i\theta})\rangle)^{n+\sigma+1}}$$

$$\leq C \int_{B_n} \frac{\chi_r(z) d\theta}{(1 - \langle \varphi(z), \varphi(re^{i\theta})\rangle)^{n+\sigma+1}}$$

By the boundedness of C_φ on H^q, there exists a constant $C_1 > 0$, independent of r and z, such that

$$\int_{B_n} H_r\left(z, re^{i\theta}\right) h\left(re^{i\theta}\right) d\theta \leq C_1 \chi_r(z) \int_{B_n} \frac{d\theta}{|1 - \langle \varphi(z), re^{i\theta}\rangle|^{n+\sigma+1}}$$

We apply Lemma 2.10 to find another positive constant C_2, independent of r and z, such that

$$\frac{1}{2\pi} \int_{B_n} H_r\left(z, re^{i\theta}\right) h\left(re^{i\theta}\right) d\theta \leq \frac{C_2 \chi_r(z)}{(1 - |\varphi(z)|^2)^\sigma}$$

$$= C_2 \chi_r(z) \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2}\right)^\sigma h(z) \leq C_2 M_r^\sigma h(z)$$

For all $z \in B_n$, by the symmetry of $H_r(z, re^{i\theta})$, we also have $\frac{1}{2\pi} \int_{B_n} H_r\left(z, re^{i\theta}\right) h(z) d\theta \leq C_2 M_r^\sigma h(re^{i\theta})$ for all

$re^{i\theta} \in B_n$. It follows from Lemma 2.9 that the operator T_r is bounded on $L^2(B_n, d\theta)$ and the norm of T_r on $L^2(B_n, d\theta)$ does not exceed the constant $C_2 M_r^\sigma$.

Now fix some $r \in (0,1)$ and fix a bounded sequence $\{f_k\}$ in H^2 that converges to 0 uniformly on every compact subset of B_n. In particular, $\{f_k\}$ converges uniformly to 0 on $|z| \leq r$. We use (6) to write

$$C_\varphi C_\varphi^* f_k(z) = F_k(z) + G_k(z), \quad z \in B_n,$$

where

$$F_k(z) = \frac{1}{2\pi} \int_{|e^{i\theta}| \leq 1} \frac{f_k\left(re^{i\theta}\right) d\theta}{(1 - \langle \varphi(z), \varphi(re^{i\theta})\rangle)^{n+1}},$$

and

$$G_k(z) = \frac{1}{2\pi} \int_{B_n} \frac{\chi_r\left(re^{i\theta}\right) f_k(re^{i\theta}) d\theta}{(1 - \langle \varphi(z), \varphi(re^{i\theta})\rangle)^{n+1}}$$

Since $\{f_k(re^{i\theta})\}$ converges to 0 uniformly for $|e^{i\theta}| \leq 1$, we have

$$\lim_{k \to \infty} \frac{1}{2\pi} \int_{B_n} |f_k((z)|^2 d\theta = 0$$

For any fixed $z \in B_n$, the weak convergence of $\{f_k\}$ to 0 in $L^2(B_n, d\theta)$ implies that $G_k(z) \to 0$ as $k \to \infty$. In fact, by splitting the ball into $|z| \leq \delta$ and $\delta < |z| < 1$, it is easy to show that

$$\lim_{k \to \infty} G_k(z) = 0$$

Uniformly for z in any compact subset of B_n. It follows from the definition of T_r that

$$\int_{B_n} |G_k|^2 d\theta \le \int_{|z| \le r} |G_k|^2 d\theta + \int_{B_n} |T_r(|f_k|)|^2 d\theta.$$

Since $\{f_k\}$ is bounded in $L^2(B_n, d\theta)$, and since the norm of the operator T_r on $L^2(B_n, d\theta)$ does not exceed $C_2 M_r^\sigma$ we can find a constant $C_3 > 0$, independent of r and k, such that

$$\frac{1}{2\pi} \int_{B_n} |T_r(|f_k|)|^2 d\theta \le C_3 M_r^{2\sigma}$$

for all k. Combining this with $\lim_{k \to \infty} \frac{1}{2\pi} \int_{|z| \le r} |G_k|^2 d\theta = 0$. we obtain

$$\lim_{k \to \infty} \operatorname{Sup} \frac{1}{2\pi} \int_{B_n} |G_k|^2 d\theta \le C_3 M_r^{2\sigma}$$

This along with the estimates for F_k in the previous paragraph gives

$$\lim_{k \to \infty} \operatorname{Sup} \frac{1}{2\pi} \int_{B_n} |C_\varphi C_\varphi^* f_k|^2 d\theta \le C_3 M_r^{2\sigma}$$

Since r is arbitrary and $M_r \to 0$ as $r \to 1-$ (which is equivalent to the condition in (5), we conclude that

$$\lim_{k \to \infty} \frac{1}{2\pi} \int_{B_n} |C_\varphi C_\varphi^* f_k|^2 d\theta = 0$$

So C_φ is compact on H^2, and the proof of the theorem is complete.

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