Centralizing (Sigma, Tau)-Derivations on Prime Gamma-Rings

Salah M. Salih
Department of mathematics
College of Educations
Al-Mustansiriya University

Mazen O. Karim
Department of mathematics
College of Education
Al-Qadisiyah University

Abstract:

Let M be a Γ-ring and σ, τ be two endomorphisms of M.

In this paper, some result on the centralizing of (σ,τ)-derivations on a subset S of a prime Γ-ring M. Also we study the commutativity of M by using the concepts centralizing and commuting of a (σ,τ)-derivations of M.

If M is a prime Γ-ring of characteristic not equal 2 has a non-zero divisors and satisfying (∗). Suppose there exists a non-zero (σ,τ)-derivation d of M such that the mapping $x \longrightarrow [d(x\beta x),x]_\alpha$ is centralizing and $\sigma(x) \mp \tau(x) = 0$, $[\sigma(x),x]_\alpha = [\tau(x),x]_\alpha = 0$ for all $x \in M$ and $\alpha,\beta \in \Gamma$ then M is commutative.

Mathematics Subject Classification: 16A78 , 16W10 , 19U80.
Keywords: primeΓ-ring , centralizing, (σ,τ)-derivation.

1- Introduction:

The study of Γ-rings was introduced by Nobusawa [1] and further generalized by Barnes [2],M. Ashraf, A. Ali and S. Ali was study (σ,τ)-derivation on aprime near ring [3], In 2003,S.M.A.Zaidi ,M. Ashraf and S. Ali gave more properties of (σ,τ)-derivations on prime rings[4], afterward in 2008,M.A. Ozturk and Y. Ceven [5] defined (σ,τ)-derivation on gamma near rings, where σ, τ are endomorphisms .

In [6] S.M. Salih and A.M. Kamal in 2012 present the definition of (σ,τ)-derivations on a prime .

Note that Bresar[7] , Mayne [8] and J. Luh[9] have developed some remarkable results on prime rings with commuting and centralizing mappings. Y. Ceven[10] worked on Jordan left derivation on completely prime Γ-ring that make the Γ-ring commutative with an assumptions.

Barens in [2] defined the Γ-ring is a pair (M,Γ) of two additive abelian groups for which there exist a map from $M\times\Gamma\times M \longrightarrow M$, i.e. the image of (x,α,y) will be denoted by xαy, for all x, y ∈ M and α ∈ Γ and this map satisfying

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$

(ii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iii) $x\alpha(y + z) = x\alpha y + x\alpha z$

(iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$

holds for all x, y, z ∈ M and α, β ∈ Γ . Then M is called a Γ-ring.

Suppose that M is a Γ-ring. Then M is called a prime Γ-ring if $x\Gamma M\Gamma y = \{0\}$ implies x = 0 or y = 0, and M is called semi-prime Γ-ring if $x\Gamma M\Gamma x = \{0\}$ implies x = 0.forthermore M is said to be commutative Γ-ring if xαy= yαx hold for all x, y ∈ M and α ∈ Γ , moreover the set Z(M)={ x ∈ M | xαy= yαx , for all y ∈ M and α ∈ Γ} is called the center of the Γ-ring M[11].

A Γ-ring M is called 2-torsion free if 2x = 0 implies x = 0, for all x∈ M, [11].

For any x,y∈M and α∈Γ, the symbol $[x,y]_\alpha$ will be represent for the commutator xαy – yαx,.

We denote the following assumption by (*)

$x\alpha y\beta z = x\beta y\alpha z$ hold for all x, y , z ∈ M and α, β ∈ Γ

The above commutator satisfies the following

$[x\alpha y,z]_\beta = x\alpha[y,z]_\beta + [x,z]_\beta\alpha y$ and

$[x,y\alpha z]_\beta = y\alpha[x,z]_\beta + [x,y]_\beta\alpha z$

Suppose again that M is a Γ-ring, an additive mapping d: M ⟶ M is called a derivation if

$d(x\alpha y) = d(x)\alpha y + x\alpha d(y)$, and

it is called Jordan derivation if $d(x\alpha x) = d(x)\alpha x + x\alpha d(x)$

holds for all $x, y \in M$ and $\alpha \in \Gamma$.

In [12] the concept of (σ,τ)-derivations in rings defined as follow an additive mapping d: M ⟶ M is called (σ,τ)-derivation if

$d(x\alpha y) = d(x)\alpha\, \sigma(y) + \tau(x)\alpha d(y)$

and Jordan (σ,τ)-derivation if $d(x\alpha x) = d(x)\alpha\, \sigma(x) + \tau(x)\alpha d(x)$

holds for all $x, y \in M$ and $\alpha \in \Gamma$ where σ, τ are endomorphisms of M.

An additive mapping f of a prime Γ-ring M is called centralizing on a subset S of M if $[x, f(x)]_\alpha \in Z(M)$ for all $x \in S$ and $\alpha \in \Gamma$ and it called commuting on a subset S of M if $[x, f(x)]_\alpha = 0$ hold for all $x \in S$ and $\alpha \in \Gamma$, [11].

The objective of this paper is to study the centralization of the (σ,τ)-derivation on a subset S of a prime Γ-ring M and study the commutativity of M. We need the following lemma:

Lemma1.1:[13] let M be a prime Γ-ring. If $a\in Z(M)$ and $a\Gamma b\in Z(M)$ then either a=0 or $b\in Z(M)$.

2-Centralizing (σ,τ)-Derivations

The main purpose of this section is to study the centralization on asubset S of prime Γ-ring M.

Lemma2.1:

Let M be a prime Γ-ring of characteristic not equal 2 satisfying (∗) and let S be a Jordan subring of M, if d is a Jordan (σ,τ)-derivation of S such that $[x,\sigma(x)]_\alpha = [\tau(x),x]_\alpha = 0$, $\sigma(x) \mp \tau(x) = 0$ and $[x,d(x)]_\alpha \in Z(M)$ for all $x \in S$ and $\alpha \in \Gamma$.then $[x,d(x)]_\alpha=0$ for all $x \in S$ and $\alpha \in \Gamma$

Proof:

By assumption we have

$$[x + y,d(x + y)]_\alpha \in Z(M) \quad \text{….(1)}$$

for all $x, y \in S$ and $\alpha \in \Gamma$

therefore

$$[x + y,d(x + y)]_\alpha = [x,d(x)]_\alpha + [y,d(y)]_\alpha + [x,d(y)]_\alpha + [y,d(x)]_\alpha$$

since Z(M) is an additive subgroup of M and by assumption we have

$$[x,d(y)]_\alpha + [y,d(x)]_\alpha \in Z(M) \quad \text{…(2)}$$

for all $x, y \in S$ and $\alpha \in \Gamma$

In (2) replace y by $x\beta x$ for $\beta \in \Gamma$, we get

$$\begin{aligned}[x,d(x\beta x)]_\alpha + [x\beta x,d(x)]_\alpha &= [x,d(x)\beta\sigma(x) + \tau(x)\beta d(x)]_\alpha +[x\beta x,d(x)]_\alpha \\ &= [x,d(x)\beta\sigma(x)]_\alpha + [x,\tau(x)\beta d(x)]_\alpha +[x\beta x,d(x)]_\alpha \\ &= [x,d(x)]_\alpha\beta\sigma(x) + \tau(x)\beta [x,d(x)]_\alpha + x\beta[x,d(x)]_\alpha \\ &\quad + [x,d(x)]_\alpha\beta x\end{aligned}$$

and since $[x,d(x)]_\alpha\in Z(M)$ then the above relation becomes

$$[x,d(x\beta x)]_\alpha + [x\beta x,d(x)]_\alpha = (\sigma(x) + \tau(x))\beta [x,d(x)]_\alpha +2 x\beta[x,d(x)]_\alpha$$

but $\sigma(x) + \tau(x)=0$ so that $[x,d(x\beta x)]_\alpha + [x\beta x,d(x)]_\alpha= 2 x\beta[x,d(x)]_\alpha \in Z(M)$

by lemma 1.1 we have either $[x,d(x)]_\alpha=0$ or $2x\in Z(M)$ and hence

$$0=[2x,d(x)]_\alpha=2[x,d(x)]_\alpha$$

and since char.M ≠ 2 so we have $[x,d(x)]_\alpha=0$ holds for all $x \in S$ and $\alpha \in \Gamma$.■

Lemma 2.2:

Let M be a prime Γ-ring satisfying (∗) and S be a right ideal of M if d is (σ,τ)-derivation of M such that $[x,\sigma(x)]_\alpha = [x,\tau(x)]_\alpha = 0$ and $[x,d(x)]_\alpha \in Z(M)$ for all x, y ∈ S and α ∈ Γ then $[x,d(x)]_\alpha = 0$ for all x ∈ S and α ∈ Γ.

Proof:

If char.M ≠ 2 then by lemma (2.1) we conclude that $[x,d(x)]_\alpha = 0$ for all x ∈ S and α ∈ Γ.

Now suppose that M is of characteristic equal 2.

Let x, y ∈ S and d be an additive mapping then we have

$[[x,y]_\beta,d(x)]_\alpha = [x\beta y - y\beta x,d(x)]_\alpha$

$= [x\beta y,d(x)]_\alpha - [y\beta x,d(x)]_\alpha$

since char.M = 2 then we have

$[[x,y]_\beta,d(x)]_\alpha = [x\beta y,d(x)]_\alpha + [y\beta x,d(x)]_\alpha$

$= x\beta[y,d(x)]_\alpha + [x,d(x)]_\alpha\beta y + \beta y[x,d(x)]_\alpha + [y,d(x)]_\alpha\beta x$

$= x\beta[y,d(x)]_\alpha + [y,d(x)]_\alpha\beta x + 2y\beta[x,d(x)]_\alpha$

and since char.M = 2 the above relation becomes

$$[[x,y]_\beta,d(x)]_\alpha = x\beta[y,d(x)]_\alpha + [y,d(x)]_\alpha\beta x \quad \ldots(1)$$

we intend to prove that

$$[[x,y]_\beta,d(x)]_\alpha + [x\beta x,d(y)]_\alpha = 0 \quad \ldots(2)$$

from (1) we can write (2) as the following

$[[x,y]_\beta,d(x)]_\alpha + [x\beta x,d(y)]_\alpha = x\beta[y,d(x)]_\alpha + [y,d(x)]_\alpha\beta x + [x\beta x,d(y)]_\alpha$

$= x\beta[y,d(x)]_\alpha + [y,d(x)]_\alpha\beta x + x\beta[x,d(y)]_\alpha + [x,d(y)]_\alpha\beta x$

so that and since char.M = 2 we have

$$[[x,y]_\beta,d(x)]_\alpha + [x\beta x,d(y)]_\alpha = 0$$

in (2) let z=d(x) so we get

$$[[x,y]_\beta,z]_\alpha + [x\beta x,d(y)]_\alpha = 0 \quad \ldots.(3)$$

if we put y=x in (3) then

$$[[x,y]_\beta,z]_\alpha = 0 \quad \ldots.(4)$$

now for all x ∈ S and μ ∈ Γ, let y = xμz.

hence from (3) we have

$0 = [[x,x\mu z]_\beta,z]_\alpha + [x\beta x,d(x\mu z)]_\alpha$

$= [x\mu[x,z]_\beta + [x,x]_\alpha\mu z,z]_\alpha + [x\beta x,d(x\mu z)]_\alpha$

$= x\mu[[x,z]_\beta,z] + [x,z]_\alpha\mu[x,z]_\beta + [x\beta x,d(x\mu z)]_\alpha$

but $[x,z]_\alpha \in Z(M)$ which implies that

$0 = [x,z]_\beta\mu[x,z]_\alpha + [x\beta x,d(x\mu z)]_\alpha$

hence

$[x,z]_\beta\mu[x,z]_\alpha = - [x\beta x,d(x\mu z)]_\alpha$

$= [x\beta x,d(x\mu z)]_\alpha \quad \ldots(5)$

now from (5) we can conclude that

$[x,z]_\beta\mu[x,z]_\alpha = - [x\beta x,d(x\mu z)]_\alpha$

$= [x\beta x,d(x)\mu\sigma(z) + \tau(x)\mu d(z)]_\alpha$

$= [x\beta x,d(x)\mu d(z) + [x\beta x,\tau(x)\mu d(z)]_\alpha$

$= [x\beta x,z]_\alpha\mu\sigma(z) + z\mu[x\beta x,\sigma(z)]_\alpha + [x\beta x,\tau(x)\mu d(z)]_\alpha$

$= x\beta[x,z]_\alpha\mu\sigma(z) + [x,z]_\alpha\beta x\mu\sigma(z) + z\mu x\beta[x,\sigma(z)]_\alpha +$
$z\mu[x,\sigma(z)]_\alpha\beta x + \tau(x)\mu[x\beta x,d(z)]_\alpha + [x\beta x,\tau(x)]_\alpha\mu d(z)$

$= x\beta[x,z]_\alpha\mu\sigma(z) + [x,z]_\alpha\beta z\mu\sigma(z) + z\mu x\beta[x,\sigma(z)]_\alpha +$
$z\mu[x,\sigma(z)]_\alpha\beta x + \tau(x)\mu x\beta[x,d(z)]_\alpha + \tau(x)\mu[x,d(z)]_\alpha\beta x +$

$$x\beta[x,\tau(x)]_\alpha\mu d(z) + [x,\tau(x)]_\alpha\beta x\mu d(z)$$

so that

$[x,z]_\beta\mu[x,z]_\alpha = 2x\beta\sigma(z)\mu[x,z]_\alpha + \tau(x)\mu x\beta[x,d(z)]_\alpha + \tau(x)\mu[x,d(z)]_\alpha\beta x$ …(6)

in (6) replace x by z we get

$0 = 2x\beta\sigma(z)\mu[x,d(z)]_\alpha + 2\tau(x)\mu x\beta[x,d(z)]_\alpha$

$= 2(x\beta\sigma(z) + \tau(x)\beta x)\mu[x,d(x)]_\alpha$

Since M is a prime ring, we get either $[x,d(x)]_\alpha = 0$ or $2x\beta\sigma(z) + 2\tau(x)\beta x = 0$

If $2x\beta\sigma(z) + 2\tau(x)\beta x = 0$ then $2x\beta\sigma(z) = -2\tau(x)\beta x$ and since M has no zero divisors and σ, τ are non-zero maps then $x = 0$ which is a contradiction since x is an arbitrary element of S and S is a non-zero ideal so that $[x,d(x)]_\alpha = 0$ for all $x \in S$, and $\alpha \in \Gamma$.

Lemma 2.3:

Let M be a prime Γ-ring and S be a non-zero ideal of M if d is a non-zero (σ,τ)-derivation of M such that$[x,\sigma(x)]_\alpha= [x,\tau(x)]_\alpha=0$ and $[x,d(x)]_\alpha \in Z(M)$ for all $x \in S$, and $\alpha \in \Gamma$ then M is commutative.

Proof:

By lemma 2.2 we have $[x,d(x)]_\alpha = 0\ \forall\ x \in S, \forall\ \alpha \in \Gamma$

therefore

$0 = [x + y,d(x + y)]_\alpha$

$= [x,d(x)]_\alpha + [x,d(y)]_\alpha + [y,d(x)]_\alpha + [y,d(y)]_\alpha$

so that

$0 = [y,d(x)]_\alpha + [x,d(y)]_\alpha\ \forall\ x, y \in S, \forall\ \alpha \in \Gamma$ …(1)

since S is an ideal replace y by $x\beta y \in U$, so

$0 = [x\beta y,d(x)]_\alpha + [x,d(x\beta y)]_\alpha$

$= x\beta[y,d(x)]_\alpha + [x,d(x)]_\alpha\beta y + [x,d(x)\beta\sigma(y) + \tau(x)\beta d(y)]_\alpha$

$= x\beta[y,d(x)]_\alpha + [x,d(x)]_\alpha\beta y + [x,d(x)\beta\sigma(y)]_\alpha + [x,\tau(x)\beta d(y)]_\alpha$

$= x\beta[y,d(x)]_\alpha + [x,d(x)]_\alpha\beta y + d(x)\beta[x,\sigma(y)]_\alpha + [x,d(y)]_\alpha\beta\sigma(y) +$
$\tau(x)\beta[x,d(y)]_\alpha + [x,\tau(x)]_\alpha\beta d(y)$

So that

$0 = d(x)\beta[x,\sigma(y)]_\alpha + \tau(x)\beta[x,d(y)]_\alpha + [x,\tau(x)]_\alpha\beta d(y) + x\beta[y,d(x)]_\alpha$

in the above relation put x instead of $\tau(x)$.

hence, we get

$0 = d(x)\beta[x,\sigma(y)]_\alpha\ \forall\ x, y \in S, \forall\ \alpha, \beta \in \Gamma$ …(2)

in (2) for all $a \in M$, replace $\sigma(y)$ by $\sigma(y)\mu a$, so

$0 = d(x)\beta[x,\sigma(y)\mu a]_\alpha$

$= d(x)\beta\sigma(y)\mu[x,a]_\alpha + d(x)\beta[x,\sigma(y)]\mu a$

from (2) the above relation becomes

$0 = d(x)\beta\sigma(y)\mu[x,a]_\alpha, \forall\ x, y \in S, \forall\ \alpha, \beta, \mu \in \Gamma$ …(3)

from (3) we can conclude that

$d(x)\Gamma M\Gamma[x,a]_\alpha = 0$

now for all $m \in M$ and $\beta \in \Gamma$ we get

$d(x)\Gamma M\Gamma[x\beta\ m , a]_\alpha = 0$

$0 = d(x)\Gamma M\Gamma U\Gamma[\ x\beta m,a]_\alpha$

$= d(x)\Gamma M\Gamma\ x\beta\ [m,a]_\alpha+ d(x)\Gamma M\Gamma[x,a]_\alpha\ \beta m$

hence

$d(x)\Gamma M\Gamma x\beta[m,a]_\alpha = 0$ for all $m, a \in M$.

since M is prime Γ-ring and d is a non-zero (σ,τ)-derivation of M and since x is any arbitrary element of S then we have

$[m,a]_\alpha = 0$ for all $m, a \in M, \alpha \in \Gamma$

$\therefore$ M is commutative

3-The Main Results

In this section we present the main results of this paper.

Theorem 3.1:

Let M be a prime Γ-ring of characteristic not equal 2 which has no zero divisors and satisfying (∗). Suppose there exists a non-zero (σ,τ)-derivation $d:M \longrightarrow M$ such that the mapping $x \longrightarrow [d(x\beta x),x]_\alpha$ is commuting on M, $[x,\sigma(x)]_\alpha = [x,\tau(x)]_\alpha = 0$ and $[\sigma(x),\tau(y)]_\alpha = 0$ holds for all $x,y \in M$, $\alpha \in \Gamma$ then M is commutative.

Proof:

By assumption we have

$[[d(x\beta x),x]_\alpha,x]_\alpha = 0$ …(1)

for all $x \in M$ and $\alpha, \beta \in \Gamma$.

let us introduce a mapping $B(\cdot,\cdot): M\times M \longrightarrow M$ by

$B(x,y) = [d(x),\sigma(y)]_\alpha + [\tau(x),d(y)]_\alpha + [d(x),\sigma(x)]_\alpha + [\tau(y),d(x)]_\alpha$

for all $x, y \in M$ and $\alpha \in \Gamma$.

It is clear that $B(\cdot,\cdot)$ is symmetric ($B(x,y) = B(y,x)$) and bi-additive.

a simple calculation show that

$B(x\beta y,z) = [d(x\beta y),\sigma(z)]_\alpha + [\tau(x\beta y),d(z)]_\alpha + [d(z),\sigma(x\beta y)]_\alpha + [\tau(z),d(x\beta y)]_\alpha$

from the definition of the mapping $B(\cdot,\cdot)$ and by the assumption we have

$B(x\beta y,z) = B(x,z)\beta\sigma(y) + \tau(x)\beta B(y,z) + d(x)\beta[\sigma(y),\sigma(z)]_\alpha + [\tau(z),\tau(x)]_\alpha\beta d(y)$ …(2)

now we introduce a non-zero mapping $f: M \longrightarrow M$ by $f(x) = B(x,x)$.

so we have

$f(x) = 2\{[d(x),\sigma(x)]_\alpha + [\tau(x),d(x)]_\alpha\}$ …(3)

for all $x \in M$ and $\alpha \in \Gamma$.

It is obviously, that mapping f satisfies the relation

$f(x,y) = f(x) + f(y) + 2B(x,y)$ for all $x,y \in M$, and $\alpha \in \Gamma$ …(4)

so the relation (1) becomes

$[f(x),x]_\alpha = 0$ for all $x \in M$ and $\alpha \in \Gamma$ …(5)

the linearizing of (5) gives

$0 = [f(x + y),x + y]_\alpha$

$= [f(x),y]_\alpha + [f(y),x]_\alpha + 2[B(x,y),x]_\alpha + 2[B(x,y),y]_\alpha$ …(6)

for all $x, y \in M$ and $\alpha \in \Gamma$

the substitution $-x$ for x in the above elation get

$0 = [f(x),y]_\alpha - [f(y),x]_\alpha + 2[B(x,y),x]_\alpha - 2[B(x,y),y]_\alpha$ …(7)

from (6) and (7) we obtain

$2[f(x),y]_\alpha + 4[B(x,y),x]_\alpha = 0$

but char.M $\neq 2$ so we get

$[f(x),y]_\alpha + 2[B(x,y),x]_\alpha = 0$ …(8)

in (8) replace y by $x\beta y$ then

$0 = [f(x),x\beta y]_\alpha + 2[B(x,x\beta y), x]_\alpha$

$= x\beta[f(x),y]_\alpha + [f(x),x]_\alpha + 2[B(x,x)\beta\sigma(y) + \tau(x)\beta B(y,x) + d(x)\beta[\sigma(y),\sigma(x)]_\alpha + [\tau(x),\tau(x)]_\alpha\beta d(y),x]_\alpha$

$= x\beta[f(x),y]_\alpha + [f(x),x]_\alpha\beta y + 2[B(x,x)\mu\sigma(y),x]_\alpha + 2[\tau(x)\beta B(y,x),x]_\alpha + 2[d(x)\beta[\sigma(y),\sigma(x)]_\alpha,x]_\alpha + 2[[\tau(x),\tau(x)]_\alpha\beta d(y),x]_\alpha$

so that

$0 = x\beta[f(x),y]_\alpha+2f(x)\beta[\sigma(y),x]_\alpha + 2\tau(x)\beta B(y,x),x]_\alpha+2[d(x),x]_\alpha\beta[\sigma(y),\sigma(x)]_\alpha$
$2d(x)\beta[[\sigma(y),\sigma(x)]_\alpha,x]_\alpha$ …(9)

In the above relation replace $\tau(x)$ by x, we get

$0 = 2f(x)\beta[\sigma(y),x]_\alpha + 2[d(x),x]_\alpha\beta[\sigma(y),\sigma(x)]_\alpha + 2d(x)\beta[[\sigma(y),\sigma(x)]_\alpha,x]_\alpha$ …(10)

now replace $\sigma(x)$ by $\tau(x)$ in (10)

$0 = 2f(x)\beta[\sigma(y),x]_\alpha$ …(11)

put $\sigma(y) = z$ so (11) becomes

$0 = 2f(x)\beta[z,x]_\alpha$

Since char.M $\neq 2$, so

$0 = f(x)\beta[z,x]_\alpha$ …(12)

Since M is a ring has no zero divisor and since f is a non-zero mapping so we get

$0 = [z,x]_\alpha$, for all x, z $\in$ M and .

So M is commutative. ■

Theorem 3.2:

Let M be a prime Γ-ring has no-zero divisors of characteristic not equal 2 and satisfying (∗). Suppose that there exists a non-zero (σ,τ)-derivation $d:M \longrightarrow M$ such that the mapping $x \longrightarrow [d(x\beta x),x]_\alpha$ is centralizing and $[\sigma(x),x]_\alpha = [\tau(x),x]_\alpha = 0,\sigma(x) \mp \tau(x) = 0$for all x $\in$ M then M is commutative.

Proof:

Let $B(x,y) = [d(x),\sigma(y)]_\alpha + [\tau(x),d(y)]_\alpha + [d(x),\sigma(x)]_\alpha + [\tau(y),d(x)]_\alpha$

and let

$f(x) = B(x,x)$
$= 2\{[d(x),\sigma(x)]_\alpha + [\tau(x),d(x)]_\alpha\}$

since the map $x \longrightarrow [d(x)\beta\sigma(y) + \tau(x)\beta d(y),x]_\alpha$ is centralizing on M then we have

$[f(x),x]_\alpha \in Z(M)$ …(1)

by the same steps of theorem 3.1 we can proof that

$[f(x),y]_\alpha + 2[B(x,y),x]_\alpha \in Z(M)$ …(2)

in (2) put $x\beta x$ instead of y to get

$[f(x),x\beta x]_\alpha + 2[B(x,x\beta x),x]_\alpha \in Z(M)$

now from step (2) in theorem 3.1 we have

$[f(x),x\beta x]_\alpha + 2[B(x,x\beta x),x]_\alpha$
$= x\beta[f(x),x]_\alpha+[f(x),x]_\alpha\beta x +2[B(x,x)\beta\sigma(x)+\tau(x)\beta B(x,x)+d(x)\beta[\sigma(x),\sigma(x)]_\alpha +$
$[\tau(x),\tau(x)]_\alpha\beta d(x),x]_\alpha$
$= 2x\beta[f(x),x]_\alpha+ 2B(x,x)\beta[\sigma(x),x]_\alpha + 2[B(x,x),x]_\alpha\beta\sigma(x) + 2\tau(x)\beta[B(x,x),x] +$
$2[\tau(x),x]\beta B(x,x)$
$= 2x\beta[f(x),x]_\alpha+ 2f(x)\beta[\sigma(x),x]_\alpha + 2[f(x),x]_\alpha\beta\sigma(x) + 2\tau(x)\beta[f(x),x] +$
$2[\tau(x),x]\beta f(x)$
$= 2x\beta[f(x),x]_\alpha+ 2(\sigma(x) + \tau(x))\beta[f(x),x]_\alpha + 2[f(x),x]_\alpha\beta[\sigma(x),x]_\alpha +$
$[\tau(x),x]_\alpha\beta f(x)$

By assumption we have $[\sigma(x),x]_\alpha = [\tau(x),x]_\alpha=0$ and $\sigma(x) \mp \tau(x) = 0$, for all x $\in$ M, and $\alpha\in\Gamma$.

so that

$[f(x),x\beta x]_\alpha+ 2[B(x,x\beta x),x]_\alpha = 2x\beta[f(x),x]_\alpha \in Z(M)$

now for all $y \in M$ we have

$0 = [2x\beta[f(x),x],y]_\alpha$
so
$0 = 2[x\beta[f(x),x]_\alpha,y]_\alpha$
but char.M $\neq 2$ so $0 = [x\beta[f(x),x]_\alpha,y]_\alpha$
which leads to
$0 = x\beta[[f(x),x]_\alpha,y]_\alpha + [x,y]_\alpha\beta[f(x),x]_\alpha$
which implies that
$0 = [x,y]_\alpha\beta[f(x),x]_\alpha$
since M has no zero divisor so either $[x,y]_\alpha = 0$ or $[f(x),x]_\alpha = 0$
if $[x,y]_\alpha = 0$ for all x, y $\in$ M, and $\alpha \in \Gamma$ then M is commutative.
or if $[f(x),x]_\alpha = 0$ then by the same steps of theorem 3.1 we have that M is commutative.

References:

[1] Nobusawa N., On The Generalizations of the Ring Theory, Osaka J. Math., 1,(1964), 81-89.
[2] Barnes W.E., On the Γ-Rings of Nobusawa, Pacific J. Math., 18(3), (1966), 411-422.
[3] **Ashraf** M., Ali A. and Ali S., (σ,τ)-Derivations on Prime Near Rings, Ach. Math. (Brno), 42(3) ,(2004), 281-286 .
[4] Zaidi S.M.A.,Ashraf M. and Ali S., On Jordan Ideals and Left (θ,θ)-Derivations in Prime Rings , IJMMS, Vol.37 ,(2003), 1957-1964 .
[5] Ozturk M. A. and Ceven Y. , On (σ,τ)- gamma Derivations in Gamma Near Rings ,Advance in Algebra ,ISSN0973 -6964, Vol. 1, No.1, (2008) ,1-10 .
[6] Saleh S.M. and Kamal A.M. , (σ,τ)- Derivations on Prime Γ-Rings ,M.Sc. thesis , Al-Mustansiriya Unv., 2012.
[7] Bresar M., Centralizing Mappings and derivations in prime rings,Jornal of Algebra ,156,(1993),385-394.
[8] Mayne J., Centralizing Automorphisms of Prime Rings, Canada Math.Bull, 19,(1976) , 113-115.
[9] Luh L., On the Theory Of Simple Rings, Michigan Math. J.,19, (1969), ,65-75 .
[10] Ceven y. , Jordan Left Derivations On Completely Prime Gamma Rings ,G.U.Fen-Edebiyat Fakultesi ,Fen-Bilimleri Derigisi ,cilt23,sayi2,(2002) .
[11] Hoque M. F. and Paul A.C.Prime Gamma Rings With Centralizing And Commuting Generalized Derivations , International Jornal Of Algebra , Vol 7(13), (2013) , 645 -651 .
[12] ArgaG N., Kaya A. and Kisir A., (σ,τ)-Derivations on Prime Rings, Math.J. Okayama Univ., 29 (1987), 173-177.
[13] Motasher S. K. , Γ-Centralizing Mappings on Prime and Semi Prime Γ-Rings , M.Sc. Thesis , Unv. of Baghdad ,2011 .