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Application of Homotopy Perturbation Method to Nonlinear System of PDEs

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Abstract: In this study, Homotopy Perturbation Method (HPM) is used to obtain the analytically exact solution of linear and nonlinear systems of partial differential equations (PDEs). The efficiency and accuracy of HPM are demonstrated through several test examples. HPM yields solutions in convergent series forms with easily computable terms. Generally, the closed form of the exact solution is obtained without any noise terms.
Keywords: Homotopy Perturbation Method, Linear and Nonlinear system of PDEs.

1. Introduction

The nonlinear phenomena played a very significant role in the field of applied Mathematics and mathematical Physics. It is known that phenomena coming from applied physics and engineering for example propagation of waves and shallow water waves can be modeled by systems of linear and nonlinear PDEs. To findaccurate, a reliable and efficient method to solve nonlinear system of PDEs is the interest of most of the researchers now a day. Adomian Decomposition Method (ADM) is used to tackle the systems of PDEs which become more difficult when finding the Adomian Polynomials [1-3,8]. Variational Iteration Method (VIM) is also helpful for the approximate solutions of the systems of PDEs [4-5,11].

In recent years, a lot of attention has been given to study the homotopy perturbation method (HPM) by different researchers such asto solve the nonlinear system of PDEs [7-12]. HPM deforms a difficult problem into a set of problems which are easy for solving without any need to transform nonlinear terms.

In this article, HPM is applied to some linear and nonlinear system of PDEs to find the exact solutions of the systems. Here HPM is also applied to a two dimensional system of PDEs proving the accuracy, efficiency and reliability of the proposed method.

2. Basic Idea of Homotopy Perturbation Method (HPM)

Consider system of nonlinear differential equations,

$$L_j(u,v) + N_j(u,v) = f_j(q) \qquad q \in \Omega \ , \tag{1}$$

with the boundary conditions of,

$$B_1(u, \frac{\partial u}{\partial n}) = 0, \ B_2(v, \frac{\partial v}{\partial n}) = 0,$$

where L_j are linear operators and N_j are nonlinear operators. The He's homotopy perturbation technique defines the homotopy,

$$U(q, p) : \Omega \times [0,1] \to R,$$

and

$$V(q, p) : \Omega \times [0,1] \to R.$$

which satisfies

$$H_j(U,V,p) = (1-p)[L_j(U,V) - L_j(v_0,u_0)] + p[L_j(U,V) + N_j(U,V) - f_j(q) = 0 \tag{2}$$

Where $p \in [0,1]$ is an imbedding parameter, u_0, v_0 are initial approximations which satisfy the boundary conditions.

The basic assumption is that the solution of Eq. (2) can be expressed as a power series in p,

$$U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2 U_2 + \ldots$$

$$V = \sum_{n=0}^{\infty} p^n V_n = V_0 + pV_1 + p^2 V_2 + \ldots$$

The approximate solution of Eq. (1) can be obtained as,

$$u = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n U_n = U_0 + U_1 + U_2 + \ldots$$

$$v = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n V_n = V_0 + V_1 + V_2 + \ldots$$

3. Numerical Applications

To demonstrate the effectiveness of the method, we consider the following problems with given initial condition.

Example 3.1 Consider the following linear System of PDEs

$$\begin{aligned} u_x - v_y &= 0, \\ v_x + u_y &= 0, \end{aligned} \tag{3}$$

with the initial conditions of,

$$u(0, y) = \cos y,$$

$$v(0, y) = \sin y .$$

Taking inverse operator, we have

$$\begin{aligned} u(x, y) &= \cos y + L^{-1}(v_y) \\ v(x, y) &= \sin y + L^{-1}(u) . \end{aligned} \tag{4}$$

Let solution of the Eq. (3) be,

$$u(x, y) = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots$$

$$v(x, y) = \sum_{n=0}^{\infty} p^n v_n = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots$$

Putting in Eq. (4),

$$u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots = \cos y + pL^{-1}[\frac{\partial}{\partial y}(v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots)]$$

$$v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots = \sin y - pL^{-1}[\frac{\partial}{\partial y}(u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots)] .$$

Equating powers of "p",

p^0, $\quad u_0(x, y) = \cos y$

$\quad v_0(x, y) = \sin y$

p^1, $\quad u_1(x, y) = L^{-1}(\frac{\partial v_0}{\partial y})$

$$v_1(x, y) = -L^{-1}(\frac{\partial u_0}{\partial y})$$

$$u_1(x, y) = x.\cos y$$

$$v_1(x, y) = x.\sin y$$

p^2, $\quad u_2(x,y) = L^{-1}(\frac{\partial v_1}{\partial y})$

$$v_2(x,y) = -L^{-1}(\frac{\partial u_1}{\partial y})$$

$$u_2(x,y) = \frac{x^2}{2!}(\cos y)$$

$$v_2(x,y) = \frac{x^2}{2!}(\sin y)$$

p^3, $\quad u_3(x,y) = L^{-1}(\frac{\partial v_2}{\partial y})$

$$v_3(x,y) = -L^{-1}(\frac{\partial u_2}{\partial y})$$

$$u_3(x,y) = \frac{x^3}{3!}(\cos y)$$

$$v_3(x,y) = \frac{x^3}{3!}(\sin y).$$

So, the solution of the system of PDEs (3) is given by,

$$u(x,y) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n = u_0 + u_1 + u_2 + u_3 + \dots$$

$$v(x,y) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n v_n = v_0 + v_1 + v_2 + v_3 + \dots$$

Consequently, we have

$$u(x,y) = \cos y + x.\cos y + \frac{x^2}{2!}.\cos y + \frac{x^3}{3!}.\cos y + \dots$$

$$u(x,y) = \cos y[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots]$$

$$u(x,y) = e^x \cos y$$

$$v(x,y) = \sin y + x.\sin y + \frac{x^2}{2!}.\sin y + \frac{x^3}{3!}.\sin y + \dots$$

$$v(x,y) = \sin y[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots]$$

$$v(x,y) = e^x \sin y.$$

Example 3.2 Consider another example of linear system of PDEs

$$u_t - v_x + u + v = 0$$

$$v_t - u_x + u + v = 0, \qquad (5)$$

with the initial conditions of,

$$u(x,0) = \sinh x$$

$$v(x,0) = \cosh x.$$

Taking inverse operator, we have

$$u(x,t) = \sinh x + L^{-1}(v_x - u - v)$$

$$v(x,t) = \cosh x + L^{-1}(u_x - u - v) \qquad (6)$$

Let solution of the Eq. (6) be,

$$u(x,t)=\sum_{n=0}^{\infty}p^n u_n = u_0 + pu_1 + p^2u_2 + p^3u_3 + ...$$

$$v(x,t)=\sum_{n=0}^{\infty}p^n v_n = v_0 + pv_1 + p^2v_2 + p^3v_3 + ...$$

Putting these values in Eq. (6),

$$u_0 + pu_1 + p^2u_2 + p^3u_3 + ... = \sinh x + pL^{-1}[\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...) - (u_0 + pu_1 + p^2u_2 + p^3u_3 + ...) - (v_0 + pv_1 + p^2v_2 + p^3v_3 + ...)]$$

$$v_0 + pv_1 + p^2v_2 + p^3v_3 + ... = \cosh x + pL^{-1}[\frac{\partial}{\partial x}(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...) - (v_0 + pv_1 + p^2v_2 + p^3v_3 + ...) - (u_0 + pu_1 + p^2u_2 + p^3u_3 + ...)]$$

Equating powers of "p",

p^0, $\quad u_0(x,t) = \sinh x$

$v_0(x,t) = \cosh x$

p^1, $\quad u_1(x,t) = L^{-1}(\frac{\partial v_0}{\partial x} - u_0 - v_0)$

$v_1(x,t) = L^{-1}(\frac{\partial u_0}{\partial x} - u_0 - v_0)$

$u_1(x,t) = -t(\cosh x)$

$v_1(x,t) = -t(\sinh x)$

p^2, $\quad u_2(x,t) = L^{-1}(\frac{\partial v_1}{\partial x} - u_1 - v_1)$

$v_2(x,t) = L^{-1}(\frac{\partial u_1}{\partial x} - u_1 - v_1)$

$u_2(x,t) = \frac{t^2}{2!}\sinh x$

$v_2(x,t) = \frac{t^2}{2!}\cosh x$

p^3, $\quad u_3(x,t) = L^{-1}(\frac{\partial v_2}{\partial x} - u_2 - v_2)$

$v_3(x,t) = L^{-1}(\frac{\partial u_2}{\partial x} - u_2 - v_2)$

$u_3(x,t) = -\frac{t^3}{3!}\cosh x$

$v_3(x,t) = -\frac{t^3}{3!}\sinh x$

p^4, $\quad u_4(x,t) = L^{-1}(\frac{\partial v_3}{\partial x} - u_3 - v_3)$

$v_4(x,t) = L^{-1}(\frac{\partial u_3}{\partial x} - u_3 - v_3)$

$$u_4(x,t) = \frac{t^4}{4!}\sinh x$$

$$v_4(x,t) = \frac{t^4}{4!}\cosh x .$$

So the solution of the system of Eq. (5) is,

$$u(x,t) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n u_n = u_0 + u_1 + u_2 + u_3 + \ldots$$

$$v(x,t) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n v_n = v_0 + v_1 + v_2 + v_3 + \ldots$$

$$u(x,t) = \sinh x - t(\cosh x) + \frac{t^2}{2!}(\sinh x) - \frac{t^3}{3!}(\cosh x) + \frac{t^4}{4!}(\sinh x) + \ldots$$

$$u(x,t) = \sinh(x-t)$$

and,

$$v(x,t) = \cosh x - t(\sinh x) + \frac{t^2}{2!}(\cosh x) - \frac{t^3}{3!}(\sinh x) + \frac{t^4}{4!}(\cosh x) + \ldots$$

$$v(x,t) = \cosh(x+t) .$$

Example3.3 Now consider the nonlinear system of PDEs

$$\begin{aligned} v_t &= vv_x + wv_y \\ w_t &= ww_x + vw_y , \end{aligned} \tag{7}$$

with the initial conditions of,

$$v(x,y,0) = w(x,y,0) = x + y .$$

Taking inverse operator, we have

$$\begin{aligned} v(x,y,t) &= x + y + L^{-1}(vv_x + wv_y) \\ w(x,y,t) &= x + y + L^{-1}(ww_x + vw_y) . \end{aligned} \tag{8}$$

Let solution of the Eq. (8) be,

$$v(x,y,t) = \sum_{n=0}^{\infty} p^n v_n = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots$$

$$w(x,y,t) = \sum_{n=0}^{\infty} p^n w_n = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \ldots$$

Putting these values in Eq. (8),

$$\begin{aligned} v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots = x + y + pL^{-1}[(v_0 + pv_1 + p^2 v_2 + p^3 v_3 +)\frac{\partial}{\partial x}(v_0 + pv_1 + \\ p^2 v_2 + p^3 v_3 + \ldots) + (w_0 + pw_1 + p^2 w_2 + p^3 w_3 +) \\ \frac{\partial}{\partial y}(v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots)] \end{aligned}$$

$$\begin{aligned} w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \ldots = x + y + pL^{-1}[(w_0 + pw_1 + p^2 w_2 + p^3 w_3 +)\frac{\partial}{\partial x}(w_0 + pw_1 + \\ p^2 w_2 + p^3 w_3 + \ldots) + (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \\ \frac{\partial}{\partial y}(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \ldots)] \end{aligned}$$

Equating powers of “p”,

$p^0,\quad v_0(x,y,t) = x + y$

$\quad\quad w_0(x,y,t) = x + y$

$p^1,\quad v_1(x,y,t) = L^{-1}[v_0 \frac{\partial v_0}{\partial x} + w_0 \frac{\partial v_0}{\partial y}]$

$\quad\quad w_1(x,y,t) = L^{-1}[w_0 \frac{\partial w_0}{\partial x} + v_0 \frac{\partial w_0}{\partial y}]$

$\quad\quad v_1(x,y,t) = 2(x+y)t$

$\quad\quad w_1(x,y,t) = 2(x+y)t$

$p^2,\quad v_2(x,y,t) = L^{-1}[(v_0 \frac{\partial v_1}{\partial x} + v_1 \frac{\partial v_0}{\partial x}) + (w_0 \frac{\partial v_1}{\partial y} + w_1 \frac{\partial v_0}{\partial y})]$

$\quad\quad w_2(x,y,t) = L^{-1}[(w_0 \frac{\partial w_1}{\partial x} + w_1 \frac{\partial w_0}{\partial x}) + (v_0 \frac{\partial w_1}{\partial y} + v_1 \frac{\partial w_0}{\partial y})]$

$\quad\quad v_2(x,y,t) = 4(x+y)t^2$

$\quad\quad w_2(x,y,t) = 4(x+y)t^2$.

So solution of the Eq. (7) is given by,

$$v(x,y,t) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n v_n = v_0 + v_1 + v_2 + v_3 + \ldots$$

$$w(x,y,t) = \lim_{p\to 1} \sum_{n=0}^{\infty} p^n w_n = w_0 + w_1 + w_2 + w_3 + \ldots$$

Putting values, we have,

$$v(x,y,t) = (x+y) + 2(x+y)t + 4(x+y)t^2 + \ldots$$
$$v(x,y,t) = (x+y)[1 + 2t + 4t^2 + \ldots]$$
$$w(x,y,t) = (x+y) + 2(x+y)t + 4(x+y)t^2 + \ldots$$
$$w(x,y,t) = (x+y)[1 + 2t + 4t^2 + \ldots].$$

Example 3.4 Consider another system of nonlinear PDEs

$$\begin{aligned} &u_t + v_x w_y - v_y w_x = -u \\ &v_t + w_x u_y + u_x w_y = v \\ &w_t + u_x v_y + u_y v_x = w, \end{aligned} \tag{9}$$

with the initial condition of,

$$u(x,y,0) = e^{x+y}, \qquad v(x,y,0) = e^{x-y}, \qquad w(x,y,0) = e^{y-x} .$$

Taking inverse operator, we have

$$\begin{aligned} &u(x,y,0) = e^{x+y} - L^{-1}(u - v_x w_y + v_y w_x) \\ &v(x,y,t) = e^{x-y} + L^{-1}(v - w_x u_y - u_x w_y) \\ &w(x,y,t) = e^{y-x} + L^{-1}(w - u_x v_y - u_y v_x). \end{aligned} \tag{10}$$

Let solution of the Eq. (10) be,

$$u(x,y,t)=\sum_{n=0}^{\infty}p^n u_n = u_0+pu_1+p^2u_2+p^3u_3+\ldots$$

$$v(x,y,t)=\sum_{n=0}^{\infty}p^n v_n = v_0+pv_1+p^2v_2+p^3v_3+\ldots$$

$$w(x,y,t)=\sum_{n=0}^{\infty}p^n w_n = w_0+pw_1+p^2w_2+p^3w_3+\ldots$$

Putting these values in Eq. (10),

$$\begin{aligned}u_0+pu_1+p^2u_2+p^3u_3+\ldots = e^{x+y} - L^{-1}[(u_0+pu_1+p^2u_2+p^3u_3+\ldots)-\frac{\partial}{\partial x}(v_0+pv_1+\\ p^2v_2+p^3v_3+\ldots)\frac{\partial}{\partial y}(w_0+pw_1+p^2w_2+p^3w_3+\ldots)+\frac{\partial}{\partial y}(v_0\\ +pv_1+p^2v_2+p^3v_3+\ldots)\frac{\partial}{\partial x}(w_0+pw_1+p^2w_2+p^3w_3+\ldots)]\end{aligned}$$

$$\begin{aligned}v_0+pv_1+p^2v_2+p^3v_3+\ldots = e^{x-y} + L^{-1}[(v_0+pv_1+p^2v_2+p^3v_3+\ldots)-\frac{\partial}{\partial x}(w_0+pw_1+\\ p^2w_2+p^3w_3+\ldots)\frac{\partial}{\partial y}(u_0+pu_1+p^2u_2+p^3u_3+\ldots)-\frac{\partial}{\partial x}(u_0\\ +pu_1+p^2u_2+p^3u_3+\ldots)\frac{\partial}{\partial y}(w_0+pw_1+p^2w_2+p^3w_3+\ldots)]\end{aligned}$$

$$\begin{aligned}w_0+pw_1+p^2w_2+p^3w_3+\ldots = e^{y-x} + L^{-1}[(w_0+pw_1+p^2w_2+p^3w_3+\ldots)-\frac{\partial}{\partial x}(u_0+pu_1+\\ p^2u_2+p^3u_3+\ldots)\frac{\partial}{\partial y}(v_0+pv_1+p^2v_2+p^3v_3+\ldots)-\frac{\partial}{\partial y}(u_0\\ +pu_1+p^2u_2+p^3u_3+\ldots)\frac{\partial}{\partial x}(v_0+pv_1+p^2v_2+p^3v_3+\ldots)].\end{aligned}$$

Equating powers of "p",

p^0,
$$u_0(x,y,t)=e^{x+y}$$
$$v_0(x,y,t)=e^{x-y}$$
0
$$w_0(x,y,t)=e^{y-x}$$

p^1,
$$u_1(x,y,t)=-u_0-\left[\frac{\partial v_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+\left[\frac{\partial w_0}{\partial x}\frac{\partial v_0}{\partial y}\right]$$
$$v_1(x,y,t)=v_0-\left[\frac{\partial w_0}{\partial x}\frac{\partial u_0}{\partial y}\right]-\left[\frac{\partial u_0}{\partial x}\frac{\partial w_0}{\partial y}\right]$$
$$w_1(x,y,t)=w_0-\left[\frac{\partial u_0}{\partial x}\frac{\partial v_0}{\partial y}\right]-\left[\frac{\partial u_0}{\partial y}\frac{\partial v_0}{\partial x}\right]$$
$$u_1(x,y,t)=-te^{x+y}$$
$$v_1(x,y,t)=te^{x-y}$$
$$w_1(x,y,t)=te^{y-x}$$

$$p^2, \quad u_2(x,y,t) = -u_1 - \left[\frac{\partial v_0}{\partial x}\frac{\partial w_1}{\partial y} + \frac{\partial v_1}{\partial x}\frac{\partial w_0}{\partial y}\right] + \left[\frac{\partial v_1}{\partial y}\frac{\partial w_0}{\partial x} + \frac{\partial w_1}{\partial x}\frac{\partial v_0}{\partial y}\right]$$

$$v_2(x,y,t) = v_1 - \left[\frac{\partial w_0}{\partial x}\frac{\partial u_1}{\partial y} + \frac{\partial w_1}{\partial x}\frac{\partial u_0}{\partial y}\right] - \left[\frac{\partial w_1}{\partial y}\frac{\partial u_0}{\partial x} + \frac{\partial u_1}{\partial x}\frac{\partial w_0}{\partial y}\right]$$

$$w_2(x,y,t) = w_1 - \left[\frac{\partial u_0}{\partial x}\frac{\partial v_1}{\partial y} + \frac{\partial u_1}{\partial x}\frac{\partial v_0}{\partial y}\right] - \left[\frac{\partial u_0}{\partial y}\frac{\partial v_1}{\partial x} + \frac{\partial u_1}{\partial y}\frac{\partial v_0}{\partial x}\right]$$

$$u_2(x,y,t) = \frac{t^2}{2!}e^{x+y}$$

$$v_2(x,y,t) = \frac{t^2}{2!}e^{x-y}$$

$$w_2(x,y,t) = \frac{t^2}{2!}e^{y-x}$$

$$p^3, \quad u_3(x,y,t) = -u_2 - \left[\frac{\partial v_0}{\partial x}\frac{\partial w_2}{\partial y} + \frac{\partial v_1}{\partial x}\frac{\partial w_1}{\partial y} + \frac{\partial v_2}{\partial x}\frac{\partial w_0}{\partial y}\right] + \left[\frac{\partial v_0}{\partial y}\frac{\partial w_2}{\partial x} + \frac{\partial v_1}{\partial y}\frac{\partial w_1}{\partial x} + \frac{\partial v_2}{\partial y}\frac{\partial w_0}{\partial x}\right]$$

$$v_3(x,y,t) = v_2 - \left[\frac{\partial w_0}{\partial x}\frac{\partial u_2}{\partial y} + \frac{\partial w_1}{\partial x}\frac{\partial u_1}{\partial y} + \frac{\partial w_2}{\partial x}\frac{\partial u_0}{\partial y}\right] - \left[\frac{\partial u_0}{\partial x}\frac{\partial w_2}{\partial y} + \frac{\partial u_1}{\partial x}\frac{\partial w_1}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial w_0}{\partial y}\right]$$

$$w_3(x,y,t) = w_2 - \left[\frac{\partial u_0}{\partial x}\frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial v_0}{\partial y}\right] - \left[\frac{\partial u_0}{\partial y}\frac{\partial v_2}{\partial x} + \frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y}\frac{\partial v_0}{\partial x}\right]$$

$$u_3(x,y,t) = -\frac{t^3}{3!}e^{x+y}$$

$$v_3(x,y,t) = \frac{t^3}{3!}e^{x-y}$$

$$w_3(x,y,t) = \frac{t^3}{3!}e^{y-x}\ .$$

So the solution of the system of PDEs in (9) is given as,

$$u(x,y,t) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n u_n = u_0 + u_1 + u_2 + u_3 + \ldots$$

$$u(x,y,t) = e^{x+y} - te^{x+y} + \frac{t^2}{2!}e^{x+y} - \frac{t^3}{3!}e^{x+y} + \ldots$$

$$u(x,y,t) = e^{x+y}\left[1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots\right]$$

$$u(x,y,t) = e^{x+y}.e^{-t} = e^{x+y-t}$$

$$v(x,y,t) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n v_n = v_0 + v_1 + v_2 + v_3 + \ldots$$

$$v(x,y,t) = e^{x-y} + te^{x-y} + \frac{t^2}{2!}e^{x-y} + \frac{t^3}{3!}e^{x-y} + \ldots$$

$$v(x,y,t) = e^{x-y}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right]$$

$$v(x,y,t) = e^{x-y}.e^{t} = e^{x-y+t}$$

$$w(x,y,t) = \lim_{p\to 1}\sum_{n=0}^{\infty} p^n w_n = w_0 + w_1 + w_2 + w_3 + \ldots$$

$$w(x,y,t) = e^{y-x} + te^{y-x} + \frac{t^2}{2!}e^{y-x} + \frac{t^3}{3!}e^{y-x} + \ldots$$

$$w(x,y,t) = e^{y-x}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\right]$$

$$w(x,y,t) = e^{y-x}.e^{t} = e^{y-x+t}\ .$$

Hence the solution of the system of PDEs given in (9) is,

$$u(x,y,t) = e^{x+y-t}$$

$$v(x,y,t) = e^{x-y+t}$$

$$w(x,y,t) = e^{y-x+t}\ .$$

Example 3.5 Consider two dimensional system of nonlinear PDEs

$$u_t = u_{xx} + u_{yy} + 2u(u_x + u_y) - (uv)_x - (uv)_y$$

$$v_t = v_{xx} + v_{yy} + 2v(v_x + v_y) - (uv)_x - (uv)_y\ , \tag{11}$$

with the initial conditions of,

$$u(x,y,0) = v(x,y,0) = \cos(x+y)\,.$$

Taking inverse operator, we gat

$$L^{-1}[L(u)] = L^{-1}\left[u_{xx} + u_{yy} + 2u(u_x + u_y) - (uv)_x - (uv)_y\right]$$

$$L^{-1}[L(u)] = L^{-1}\left[v_{xx} + v_{yy} + 2v(v_x + v_y) - (uv)_x - (uv)_y\right]$$

$$u(x,y,t) = \cos(x+y) + L^{-1}\left[u_{xx} + u_{yy} + 2u(u_x + u_y) - (uv)_x - (uv)_y\right] \tag{12}$$

$$v(x,y,t) = \cos(x+y) + L^{-1}\left[v_{xx} + v_{yy} + 2v(v_x + v_y) - (uv)_x - (uv)_y\right].$$

Let solution of the Eq. (12) be,

$$u(x,y,t)=\sum_{n=0}^{\infty}p^n u_n = u_0 + pu_1 + p^2u_2 + p^3u_3 + ...$$
$$v(x,y,t)=\sum_{n=0}^{\infty}p^n v_n = v_0 + pv_1 + p^2v_2 + p^3v_3 + ...$$

Putting in Eq. (12)

$$u_0 + pu_1 + p^2u_2 + p^3u_3 + ... = \cos(x+y) + pL^{-1}[\frac{\partial^2}{\partial x^2}(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...)$$
$$\frac{\partial^2}{\partial y^2}(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...) + 2(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...)$$
$$\{\frac{\partial}{\partial x}(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...) + \frac{\partial}{\partial y}(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...)\}$$
$$\frac{\partial}{\partial x}\{(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...)(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...)\}$$
$$\frac{\partial}{\partial y}\{(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...)(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...)\}]$$

$$v_0 + pv_1 + p^2v_2 + p^3v_3 + ... = \cos(x+y) + pL^{-1}[\frac{\partial^2}{\partial x^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...)$$
$$\frac{\partial^2}{\partial y^2}(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...) + 2(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...)$$
$$\{\frac{\partial}{\partial x}(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...) + \frac{\partial}{\partial y}(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...)\}$$
$$\frac{\partial}{\partial x}\{(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...)(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...)\}$$
$$\frac{\partial}{\partial y}\{(u_0 + pu_1 + p^2u_2 + p^3u_3 + ...)(v_0 + pv_1 + p^2v_2 + p^3v_3 + ...)\}].$$

Equating powers of "p",

p^0, $\quad u_0(x,y,t)=\cos(x+y)$

$$v_0(x,y,t)=\cos(x+y)$$

p^1, $\quad u_1(x,y,t)=L^{-1}[\frac{\partial^2 u_0}{\partial x^2}+\frac{\partial^2 u_0}{\partial y^2}+2u_0\{\frac{\partial u_0}{\partial x}+\frac{\partial u_0}{\partial y}\}-\frac{\partial}{\partial x}(u_0v_0)-\frac{\partial}{\partial y}(u_0v_0)]$

$$v_1(x,y,t)=L^{-1}[\frac{\partial^2 v_0}{\partial x^2}+\frac{\partial^2 v_0}{\partial y^2}+2v_0\{\frac{\partial v_0}{\partial x}+\frac{\partial v_0}{\partial y}\}-\frac{\partial}{\partial x}(u_0v_0)-\frac{\partial}{\partial y}(u_0v_0)]$$
$$u_1(x,y,t)=-2t\cos(x+y)$$
$$v_1(x,y,t)=-2t\cos(x+y)$$

$p^2,\quad u_2(x,y,t)=L^{-1}[\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}+2u_0(\frac{\partial u_1}{\partial x}+\frac{\partial u_1}{\partial y})+2u_1(\frac{\partial u_0}{\partial x}+\frac{\partial u_0}{\partial y})-\frac{\partial}{\partial x}(u_0v_1+u_1v_0)$
$$-\frac{\partial}{\partial y}(u_0v_1+u_1v_0)]$$

$$v_2(x,y,t)=L^{-1}[\frac{\partial^2 v_1}{\partial x^2}+\frac{\partial^2 v_1}{\partial y^2}+2v_0(\frac{\partial v_1}{\partial x}+\frac{\partial v_1}{\partial y})+2v_1(\frac{\partial v_0}{\partial x}+\frac{\partial v_0}{\partial y})-\frac{\partial}{\partial x}(u_0v_1+u_1v_0)$$
$$-\frac{\partial}{\partial y}(u_0v_1+u_1v_0)]$$

$$u_2(x,y,t)=\frac{(2t)^2}{2!}\cos(x+y)$$

$$v_2(x,y,t)=\frac{(2t)^2}{2!}\cos(x+y)$$

$p^3,\quad u_3(x,y,t)=L^{-1}[\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}+2u_0(\frac{\partial u_2}{\partial x}+\frac{\partial u_2}{\partial y})+2u_2(\frac{\partial u_0}{\partial x}+\frac{\partial u_0}{\partial y})-\frac{\partial}{\partial x}(u_0v_2+u_2v_0)$
$$-\frac{\partial}{\partial y}(u_0v_2+u_2v_0)]$$

$$v_3(x,y,t)=L^{-1}[\frac{\partial^2 v_2}{\partial x^2}+\frac{\partial^2 v_2}{\partial y^2}+2v_0(\frac{\partial v_2}{\partial x}+\frac{\partial v_2}{\partial y})+2v_2(\frac{\partial v_0}{\partial x}+\frac{\partial v_0}{\partial y})-\frac{\partial}{\partial x}(u_0v_2+u_2v_0)$$
$$-\frac{\partial}{\partial y}(u_0v_2+u_2v_0)$$

$$u_3(x,y,t)=-\frac{(2t)^3}{3!}\cos(x+y)$$

$$v_3(x,y,t)=-\frac{(2t)^3}{3!}\cos(x+y)$$

So the solution of the Eq. (11) is given by,

$$u(x,y,t)=\lim_{p\to 1}\sum_{n=0}^{\infty}p^n u_n=u_0+u_1+u_2+u_3+\ldots$$

$$v(x,y,t)=\lim_{p\to 1}\sum_{n=0}^{\infty}p^n v_n=v_0+v_1+v_2+v_3+\ldots$$

Consequently, we have

$$u(x,y,t)=\cos(x+y)-2t\cos(x+y)+\frac{(2t)^2}{2!}\cos(x+y)-\frac{(2t)^3}{3!}\cos(x+y)+\ldots$$

$$u(x,y,t)=\cos(x+y)\left[1-(2t)+\frac{(2t)^2}{2!}-\frac{(2t)^3}{3!}+\ldots.\right]$$

$$=\cos(x+y)e^{-2t}$$

$$v(x,y,t)=\cos(x+y)-2t\cos(x+y)+\frac{(2t)^2}{2!}\cos(x+y)-\frac{(2t)^3}{3!}\cos(x+y)+\ldots$$

$$=\cos(x+y)\left[1-(2t)+\frac{(2t)^2}{2!}-\frac{(2t)^3}{3!}+\ldots.\right]$$

$$=\cos(x+y)e^{-2t}$$

Hence the solution of the system of Eq. (11) is given by,

$$u(x,y,t)=v(x,y,t)=e^{-2t}\cos(x+y).$$

4. Conclusion

In this paper, the Homotopy Perturbation Method (HPM) was employed successfully for solving linear and nonlinear system of partial differential equations. The HPM is applied without determining the Adomian's polynomials, unrealistic assumptions, and transformation formulas. The results given here provide further evidence of the usefulness of Homotopy Perturbation Method. The HPM is clearly very efficient and powerful technique to find exact solutions of the nonlinear systems of PDEs and can be extended to other mathematical problems.

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