Approximate Solution of Nonlinear fractional Integro-Differential Equations By He's Homotopy Perturbation Method And The Modification of He's Variational Iteration Method

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Abstract

In this paper, we compare the modification of He's variational iteration method (MVIM), and He's homotopy perturbation method (HPM), in order to obtain the approximate solution of nonlinear fractional integro-differential equations of Volterra and Fredholm integro-differential equations, we present some examples to find out accuracy of the methods.

keywords: Fractional integro-differential equations, Caputo derivative, modification of He's variational iteration method, homotopy perturbation method.

1. Introduction

The fractional integro-differential equations is a special kind of equations collecting integro-differential equations and calculus,([17],[18]). In recent years, there has been a growing mathematical formulations of physical phenomena, such as nonlinear fractional analysis and their applications in the theory of Engineering, Mechanics, Physics, Chemical kinetics, Astronomy, Biology Economics, potential theory and Electrostatics contain integro-differential equations, ([3],[6],[15],[16]).

The variational iteration method was first proposed by He, ([1],[2],[13]), and was been worked out over a number of years by many authors. This method has been shown to effectively, easily and accurately solve a large class of nonlinear problems.In this paper our propose the reliable modification of He's VIM (MVIM), that was introduced by Gharbani,[1], for solving the nonlinear fractional integro-differential equations by constructing an initial trial-function without unknown parameters so that one iteration leads to exact solution.

The other propose of this paper we study He's perturbation method,[7], for approximating the solution of nonlinear fractional integro-differential equations.We well consider fractional order integro-differential equations of the form:

$$D^\alpha y(x) = g(x) + \lambda \int_0^x k(x,t)\, F(y(t))\, dt, \tag{1.1}$$

and

$$D^\alpha y(x) = g(x) + \lambda \int_0^1 k(x,t)\, F(y(t))\, dt, \tag{1.2}$$

for $x,t \in [0,1]$, λ is a numerical parameter, where the function $g(x)$, $k(x,t)$ are known and $y(x)$ is the unknown function, D^α is Caputo's fractional derivative and α is a parameter describing the order of the fractional derivative and $F(y(x)) = f(y(t))^q, q > 1$, is a nonlinear continuous function.

2. Basic definitions

In this section we present some basic definitions and properties of the fractional calculus theory, which are used in this paper,([5],[9],[14]).

Definition: 2.1

A real function $y(x)$, $x > 0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p > \mu$, such that $y(x) = x^p y_1(x)$, where $y_1(x) \in C[0,1)$. Clearly $C_\mu \subset C_\beta$ if $\beta \leq \mu$.

Definition: 2.2

A function $y(x)$, $x > 0$, is said to be in the space C_μ^m, $m \in N \cup \{0\}$, if $y^{(m)} \in C_\mu$.

Definition: 2.3

The left sided Riemann-Liouville fractional integral operator $\alpha \geq 0$, of a function, $y \in C_\mu$, $\mu \geq -1$, is defined as:

$$I^\alpha y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} y(t)\, dt, \quad \alpha > 0, \quad x > 0. \tag{2.1}$$

$$I^0 y(x) = y(x) \quad ,(I^0 = I \ \ idendity \ \ operator). \tag{2.2}$$

Definition: 2.4

Let $y \in C_{-1}^m$, $m \in N \cup 0$, then the Caputo's fractional derivative of $y(x)$ is defined as.

$$D^\alpha y(x) = \begin{cases} J^{m-\alpha} y^m(x), & m-1 < \alpha \leq m \ , \ m \in \mathbf{N} \ , \\[2mm] \frac{D^m y(x)}{Dx^m} \ , & \alpha = m. \end{cases} \tag{2.3}$$

Hence, we have the following properties:

$$I^\alpha y(x) I^\beta y(x) = I^{\alpha+\beta} y(x), \quad for \ all \ \ \alpha, \beta \geq 0, \ y \in C_\mu, \ \mu > 0. \tag{2.4}$$

$$I^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha} , \tag{2.5}$$

for $x > 0$, $\alpha \geq 0$, $\gamma > -1$.

$$I^\alpha D^\alpha y(x) = y(x) - \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} , \quad x > 0. \tag{2.6}$$

And Caputo fractional differentiation is a linear operation, similar to inter order differentiation.
$D^\alpha[\lambda y(x) + \mu g(x)] = \lambda D^\alpha y(x) + \mu D^\alpha g(x)$, where λ and μ are constants.

3. Numerical solution of nonlinear Volterra and Fredholm fractional integro-differential equations

In this section the Modified of He's Variational iteration Method and He's Homotopy Perturbation method are applied for solving nonlinear fractional integro-differential equations.

The Modified of He's Variational iteration Method

In the first we will propose the reliable modification of the (VIM),[1], for solving nonlinear fractional integro-differential equations withe initial conditions by constructing an initial trial-function without unknown parameters.

Here,we consider the following fractional functional equation,

$$Ly(x) + Ry(x) + Ny(x) = g(x)\,, \tag{3.1}$$

where $L_x = D^\alpha$, R is a linear differential operator, N represents the nonlinear terms, and g is the source term. By using (2.6) applying the inverse L_x^{-1} to both sides of (3.1) we obtain:

$$y(x) = f(x) - L_x^{-1}[Ry(x)] - L_x^{-1}[Ny(x)]\,, \tag{3.2}$$

where $L_x^{-1} = I^\alpha$, and $L_x^{-1}[g(x)] = f(x)$. Can be applied in the above equations (1.1) and (1.2) and using the basic character of He s method is construction of a correction fractional for (3.1) which reads,

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(t)[Ly_N(t) + R\tilde{y}_N(t) + N\tilde{y}_n(t) - g(t)]\,dt\,, \tag{3.3}$$

and

$$y_{n+1}(x) = y_n(x) + \int_0^1 \lambda(t)[Ly_N(t) + R\tilde{y}_N(t) + N\tilde{y}_n(t) - g(t)]\,dt\,, \tag{3.4}$$

where λ is a Lagrange multiplier which can be identified optimally via Variational theory,[13], y_n is the nth approximate solution of (1.1) or (1.2), and $\tilde{y}_n$ denotes a restricted Variation, $\delta\tilde{y}_n = 0$, to solve equations (3.3) and (3.4) use a Lagrange multiplier resulting from the integration by parts.

Then the successive approximations $y_n(x)$, $n \geq 0$ of the solution $y(x)$ can obtain by using a Lagrange multiplier and by using any selective function $y_0(x)$,([10],[11],[12]). The exact solution may be obtain by using,

$$\lim_{x \to \infty} y_n(x) = y(x)\,. \tag{3.5}$$

As a result, we have the following Variational iteration formula for (3.2),

$$\begin{cases} y_0(x) \ is \ an \ arbitrary \ initial \ guess\,, \\ \\ y_{n+1}(x) = f(x) - L_x^{-1}[Ry_n(x)] - L_x^{-1}[Ny_n(x)], \end{cases} \tag{3.6}$$

The MVIM, that was introduced by Ghorbai et al,[1], can be established based on the assumption that the function $f(x)$ of the iterative relation (3.6) can be divided into two parts, namely $f_0(x)$ and $f_1(x)$, then we set,

$$f(x) = f_0(x) + f_1(x)\,. \tag{3.7}$$

According to the assumption (3.7) and by the relationship (3.6), we construct the following Variational iteration formula,

$$\begin{cases} y_0(x) = f_0(x)\,, \\ y_1(x) = f(x) - L_x^{-1}[Rf_0(x)] - L_x^{-1}[Nf_0(x)]\,, \\ y_{n+1}(x) = f(x) - L_x^{-1}[Ry_n(x)] - L_x^{-1}[Ny_n(x)]\,. \end{cases} \tag{3.8}$$

He's Homotopy perturbation method

The basic consider of homotopy perturbation method illustrated by consider the following nonlinear functional equation,([4],[8]).

$$A(u) = y(x),\tag{3.9}$$

with the following boundary conditions, $(U, \frac{\partial u}{\partial n}) = 0$, $x \in \Gamma$, where A is a general functional operator, U is a boundary operator, $y(x)$ is a known analytic function, and Γ is the boundary of domain Ω, the operator A can be decomposed into two parts L and N, where L is linear and N is a nonlinear operator, equation (3.9) can by rewritten as the following:

$$L(U) + N(U) - y(x) = 0.\tag{3.10}$$

We construct a homotopy $V(x, p) : \Omega \times [0, 1] \longrightarrow R$, which satisfies:

$$H(V, p) = (1 - p)[L(V) - L(U_0)] + p[A(V) - y(x)] = 0,\tag{3.11}$$

where $p \in [0, 1]$, $x \in \Omega$,

or

$$H(V, p) = L(V) - L(U_0) + pL(U_0) + p[N(V) - y(x)] = 0,\tag{3.12}$$

where y_0 is an initial approximation for the solution of equation (3.9). In this method, we use the homotopy parameter p to expand:

$$V = V_0 + pV_1 + p^2 V_2 + \dots.\tag{3.13}$$

The approximate will be obtained by taking the limit as p tends to 1,

$$U = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \dots.\tag{3.14}$$

To illustrate for equations (1.1) or (1.2) substituting (3.11) into (1.1) or (1.2) we obtain:

$$D^\alpha y_i(x) = p\left(g_i(x) + \lambda \int_0^x k(x, t)\, F(y(t))\, dt,\right.\tag{3.15}$$

or

$$D^\alpha y_i(x) = p\left(g_i(x) + \lambda \int_0^1 k(x, t)\, F(y(t))\, dt\right..\tag{3.16}$$

We expand the solution of equations (1.1) or (1.2) in the following form:

$$y_i(x) = \sum_{i=0}^\infty p^i y_i(x) = y_0(x) + p y_1(x) + p^2 y_2(x) + \dots.\tag{3.17}$$

Substituting (3.17) into (3.15) or (3.16) and collecting the terms with the same powers of p, we obtain a series of equations of the form:

$p^0 \; : \; D^\alpha y_0(x) = 0,$

$p^1 \; : \; D^\alpha y_1(x) = g(x) + \lambda \int_0^x k(x, t)\, F(y(t))\, dt,$

$p^2 \; : \; D^\alpha y_2(x) = \lambda \int_0^x k(x, t)\, F(y(t))\, dt,$

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or

$$p^0 \ : \ D^\alpha y_0(x) = 0,$$

$$p^1 \ : \ D^\alpha y_1(x) = g(x) + \lambda \int_0^1 k(x,t)\, F(y(t))\, dt,$$

$$p^2 \ : \ D^\alpha y_2(x) = \lambda \int_0^1 k(x,t)\, F(y(t))\, dt,$$

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that these equations can be easily solved by applying the operator I^α the inverse of the operator D^α, according to equation (2.6), that is by setting $p = 1$, in equations (3.15) or (3.16) we can entirely determine the (HPM) series solutions,([4],[8]).

$$y(x) = \sum_{i=0}^\infty y_i(x)\,. \tag{3.18}$$

4.Numeical Examples

In this section we present some numerical examples of nonlinear fractional integro-differential equations by the modification of He's Variational iteration method and He's Homotopy perturbation method.

Example 4.1

Consider the following nonlinear fractional integro-differential equation:

$$D^{0.9}y(x) = g(x) + \int_0^x (x-t)^2 [y(t)]^3\, dt\,, \tag{4.1}$$

where $g(x) = \frac{1}{2}\frac{\sqrt{\pi}}{x^{2/5}\Gamma(2/5)} - \frac{16}{315}x^{2/5}$, with the initial condition $y(0) = 0$, and exact solution $y(x) = \sqrt{x}$.

The solution according to(MVIM)

$$D^{0.9}y(x) = \frac{1}{2}\frac{\sqrt{\pi}}{x^{2/5}\Gamma(2/5)} - \frac{16}{315}x^{2/5} + \int_0^x (x-t)^2[y(t)]^3\, dt\,. \tag{4.1}$$

We take the operator $I^{9/10}$ on both sides of equation (4.1) we obtain:

$$y(x) = y(0) - \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + I^{9/10}\Big(\frac{1}{2}\frac{\sqrt{\pi}}{x^{2/5}\Gamma(2/5)} - \frac{16}{315}x^{2/5} + \int_0^x (x-t)^2[y(t)]^3\Big)\, dt\,. \tag{4.2}$$

According to the original VIM (3.3) and corresponding the recursive scheme (3.6), we obtain:

$$f(x) = f_0(x) + f_1(x) = I^{9/10}\Big(\frac{1}{2}\frac{\sqrt{\pi}}{x^{2/5}\Gamma(2/5)} - \frac{16}{315}\Big)\,, \tag{4.3}$$

$$f(x) = \sqrt{x} - 0.01103948449x^{27/5}\,, \tag{4.4}$$

by assuming

$$f(x) = \sqrt{x}\,, \quad f_1(x) = -0.01103948449x^{27/5}\,, \tag{4.5}$$

with starting of the initial approximation, $y(x) = f_0(x) = \sqrt{x}$, we obtain,

$$y_0(x) = \sqrt{x}\,, \tag{4.6}$$

$$y_1(x) = \sqrt{x} - 0.01103948449x^{27/5} + L_x^{-}1 \, , \tag{4.7}$$

$$y_1(x) = \sqrt{x} - 0.01103948449x^{27/5} + I^{9/10}(\int_0^x (x-t)^2[\sqrt{t}]^3 \, dt) \, ,$$

$$y_1(x) = \sqrt{x} \, .$$

$$y_{n+1}(x) = \sqrt{x} - 0.01103948449x^{27/5} + L_x^{-1}[y_n(x)] = \sqrt{x} \quad , n \geq 1 \tag{4.8}$$

in similarly view equation (4.7) it is obtained $y(x) = \sqrt{x}$, where it is the exact solution of equation (4.1).

The solution according to(HPM)

$$D^{9/10}y(x) = \frac{1}{2}\frac{\sqrt{\pi}}{x^{2/5}\Gamma(2/5)} - \frac{16}{315}x^{2/5} + \int_0^x (x-t)^2[y(t)]^3 \, dt \, . \tag{4.1}$$

According to (3.16) we construct the following homotopy:

$$D^{9/10}y(x) = p(g(x) + \int_0^x (x-t)^2[y(t)]^3 \, dt) \, , \tag{4.9}$$

substituting (3.13) into (4.9) we obtain:

$p^0 \ : \ D^{9/10}y_0(x) = 0,$

$p^1 \ : \ D^{9/10}y_1(x) = g(x) + \int_0^x (x-t)^2[(y_0(t)]^3 \, dt$

$P^2 \ : \ D^{9/10}y_2(x) = \int_0^x (x-t)^2[3(y_0(t))^2 y_1(t)] \, dt$

$p^3 \ : \ D^{9/10}y_3(x) = \int_0^x (x-t)^2[3(y_0(t))^2 y_2(t) + 3y_0(x)(y_1(t))^2] \, dt$

$p^4 \ : \ D^{9/10}y_4(x) = \int_0^x (x-t)^2[3(y_0(t))^2 y_3(t) + 6y_0(t) + y_1(t) + y_2(t) + (y-3(t))^2] \, dt,$

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by applying the operators $I^{9/10}$ to the above sets we obtain:

$y_0(x) = 0$

$y_1(x) = \frac{1}{2}\frac{\sqrt{\pi}}{x^{2/5}\Gamma(2/5)} - \frac{16}{315}x^{2/5} + \int_0^x (x-t)^2[y_0(t)]^3 \, dt \, ,$

$y_1(x) = \sqrt{x} - 0.01103948449x^{27/5}$

$y_2(x) = 0, \quad y_3(x) = 0, \ldots$

$$y(x) = \sum_{i=0}^{\infty} y_i(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + \ldots \ .$$

Therefore the approximate solution of (4.1),

$y(x) = \sqrt{x} - 0.01103948449x^{27/5}.$

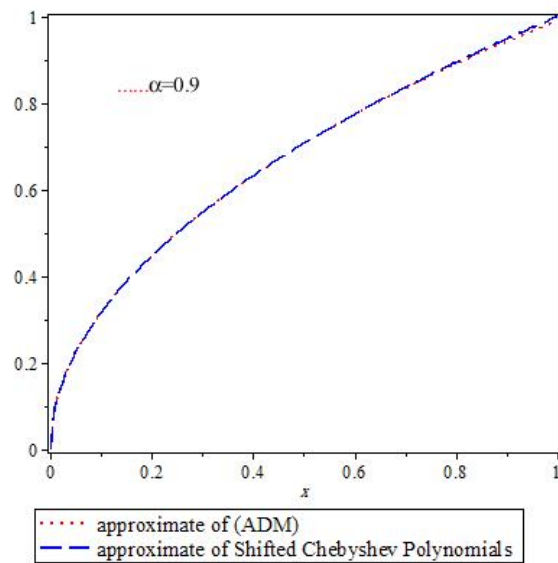Table 1, Figure 1 and 2 shown the numerical result of example 4.1 .
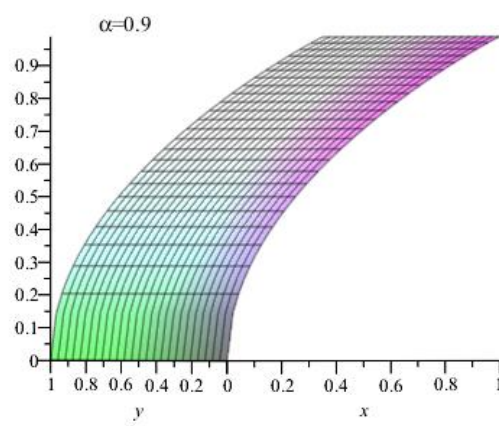
Figure 1: Numerical result of example 4.1

Figure 2: Approximate solution of example 4.1

x	$exact = (MVIM)$	$Approximant\,by\,(HPM)$	$Error\,of\,(HPM)$
0.1	0.3162277660	0.3162277221	4.39×10^{-8}
0.2	0.4472135955	0.4472117398	0.0000018557
0.3	0.5477225575	0.5477059844	0.0000165731
0.4	0.6324555320	0.6323771759	0.0000783561
0.5	0.7071067812	0.7068453323	0.0002614489
0.6	0.7745966692	0.7738968827	0.0006997865
0.7	0.8366600265	0.8350513148	0.0016087117
0.8	0.8944271910	0.8911186637	0.0033085273
0.9	0.9486832981	0.9424336100	0.0062496881
1	1.0	0.9889605155	0.0110394845

Table 1. Indicate the amount of error in Example 4.1

Example 4.2

Consider the following nonlinear fractional integro-differential equation:

$$D^{1/3}y(x) = \frac{9}{5}\frac{x^{5/3}}{\Gamma(2/3)} - \frac{7}{40}x + \frac{1}{4}\int_0^1 x(1-t)[y(t)]^2\,dt, \tag{4.10}$$

with the initial condition $y(0) = 1$ and exact solution $y(x) = 1 + x^2$.

The solution according to(MVIM)

$$D^{1/3}y(x) = \frac{9}{5}\frac{x^{5/3}}{\Gamma(2/3)} - \frac{7}{40}x + \frac{1}{4}\int_0^1 x(1-t)[y(t)]^2\,dt,$$

we take the operator $I^{1/3}$ on both sides of equation (4.10) we obtain:

$$y(x) = y(0) - \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + I^{1/3}\left(\frac{9}{5}\frac{x^{5/3}}{\Gamma(2/3)} - \frac{7}{40}x + \frac{1}{4}\int_0^1 x(1-t)[y(t)]^2\,dt\right). \tag{4.11}$$

According to the original VIM (3.3) and corresponding the recursive scheme (3.6) we obtain:

$$f(x) = f_0(x) + f_1(x) = 1 + I^{1/3}\left(\frac{9}{5}\frac{x^{5/3}}{\Gamma(2/3)} - \frac{7}{40}x\right), \tag{4.12}$$

$$f(x) = 1 + x^2 - 0.1469798559x^{4/3},$$

by assuming,

$$f_0(x) = 1 + x^2,\ \ f_1(x) = -0.1469798559x^{4/3},$$

with starting the initial approximation, $y_0(x) = f_0(x) = 1 + x^2$, we obtain:

$$y_0(x) = 1 + x^2,$$

$$y_1(x) = 1 + x^2 - 0.1469798559x^{4/3} + L_x^{-1}[f_0(x)],$$

$$y_1(x) = 1 + x^2 - 0.1469798559x^{4/3} + I^{4/3}\left(\int_0^1 x(1-t)[y(t)]^2\,dt\right),$$

$$y_1(x) = 1 + x^2,$$

$$y_{n+1}(x) = 1 + x^2 - 0.1469798559x^{4/3} + L_x^{-1}[y_n(x)] = 1 + x^2 \ \ ,n \geq 1, \tag{4.13}$$

in similarly view equation (4.14) it is obtained, $y(x) = 1 + x^2$, where it is the exact solution of (4.10).

The solution according to(HPM)

$$D^{1/3}y(x) = \frac{9}{5}\frac{x^{5/3}}{\Gamma(2/3)} - \frac{7}{40}x + \frac{1}{4}\int_0^1 x(1-t)[y(t)]^2\,dt, \tag{4.10}$$

According to (3.16) we construct the following homotopy:

$$D^{1/3} = p(g(x) + \frac{1}{4}\int_0^1 x(1-t)[y(t)]^2\,dt). \tag{4.14}$$

Substituting (3.13) into (4.14),

$p^0 \;:\; D^{1/3}y_0(x) = 0,$

$p^1 \;:\; D^{1/3}y_1(x) = g(x) + \frac{1}{4}\int_0^1 x(1-t)[y_0(t)]^2\,dt,$

$p^2 \;:\; D^{1/3}y_2(x) = \frac{1}{4}\int_0^1 x(1-t)[2y_0(t)y_1(t)]\,dt,$

$p^3 \;:\; D^{1/3}y_3(x) = \frac{1}{4}\int_0^1 x(1-t)[2y_0(t)y_2(t) + (y_1(t))^2]\,dt,$

$p^4 \;:\; D^{1/3}y_3(x) = \frac{1}{4}\int_0^1 x(1-t)[2y_0(t)y_3(t) + 2y_2(t)y_1(t)]\,dt,$

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by applying the operators $I^{1/3}$ to the above sets we obtain:

$y_0(x) = 1,$

$y_1(x) = x^2 - 0.04199424454x^{4/3},$

$y_2(x) = 0.03272782533x^{4/3},$

$y_3(x) = 0.008024681284x^{4/3},$

$y_{(}x) = 0.0009942270254x^{4/3},$

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therefor the approximations to the solution of equation (4.10) will be determined as:

$$y(x) = \sum_{i=0}^{\infty} y_i(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) + \dots \;,$$

Table 2, Figure 3 and 4 shown the numerical result of example 4.2 .

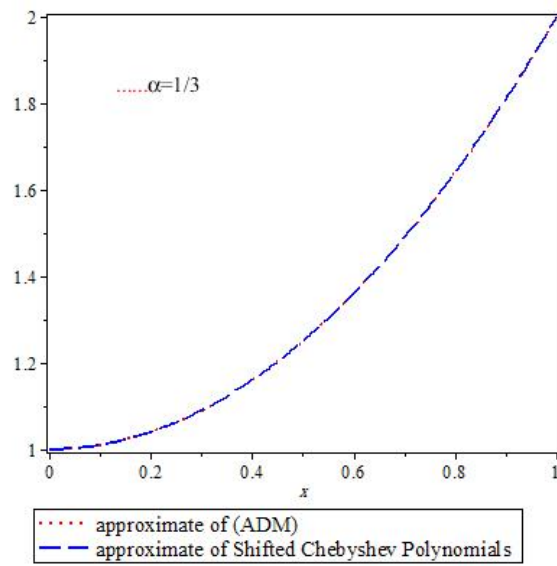$$y(x) \cong 1 + x^2 - 0.00024751090x^{4/3}.$$
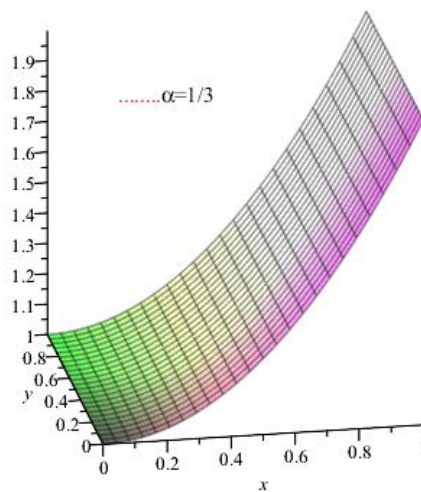
Figure 3: Numerical result of example 4.2

Figure 4: Approximate solution of example 4.2

x	$exact = (MVIM)$	$Approximant\,by(HPM)$	$Error\,of(HPM)$
0	1	1.0	0.
0.1	1.01	1.009988512	0.000011488
0.2	1.04	1.039971051	0.000028949
0.3	1.19	1.089950292	0.000049708
0.4	1.16	1.159927053	0.000072947
0.5	1.25	1.249901775	0.000098225
0.6	1.36	1.359874745	0.000125255
0.7	1.49	1.489846164	0.000153836
0.8	1.64	1.639816185	0.000183815
0.9	1.81	1.809784928	0.000215072
1	2.00	1.999752489	0.000247511

Table 2. Indicate the amount of error in Example 4.2

Example 4.3

Consider the following nonlinear fractional integro-differential equation:

$$D^{2/3}y(x) = y(0) - \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + \frac{81}{28}\frac{x^{7/3}\sqrt{3}\Gamma(2/3)}{\pi} - \frac{\sqrt{x}}{8} + \int_0^1 \sqrt{x}\,t\,[y(t)]^2\,dt\,, \qquad (4.15)$$

with the initial condition $y(0) = 0$ and exact solution $y(x) = x^3$.

The solution according to(MVIM)

Now we take the operator $I^{2/3}$ on both sides of equation (4.14) we obtain:

$$y(x) = I^{2/3}(\frac{81}{28}\frac{x^{7/3}\sqrt{3}\Gamma(2/3)}{\pi} - \frac{\sqrt{x}}{8} + \int_0^1 \sqrt{x}\,t\,[y(t)]^2\,dt), \qquad (4.16)$$

according to the original VIM (3.3) and corresponding the recursive scheme (3.6) we obtain:

$$f(x) = f_0(x) + f_1(x) = I^{2/3}(\tfrac{81}{28}\tfrac{x^{7/3}\sqrt{3}\Gamma(2/3)}{\pi} - \tfrac{\sqrt{x}}{8}), \qquad (4.17)$$

$f(x) = x^3 - 0.1023508743x^{7/6}$,

by assuming

$$f_0(x) = x^3, \quad f_1(x) = -0.1023508743x^{7/6}, \qquad (4.18)$$

with starting of the initial approximation, $y_0(x) = f_0(x) = x^3$, we obtain:

$y_0(x) = x^3$,

$$y_1(x) = x^3 - 0.1023508743x^{7/6} + L_x^{-1}[f_0(x)], \qquad (4.19)$$

$y_1(x) = x^3 - 0.1023508743x^{7/6} + I^{2/3}(\int_0^1 \sqrt{x}t(t^3))^2\,dt)$,

$$y_1(x) = x^3$$

$$y_{n+1}(x) = 0.1023508743x^{7/6} + L_x^{-1}[y_n(x)] = x^3, \quad n \geq 1, \tag{4.20}$$

in similarly view equation (4.18) it is obtained, $y(x) = x^3$, where it is the exact solution of equation (4.10).

The solution according to(HPM)

$$D^{2/3}y(x) = \frac{81}{28}\frac{x^{7/3}\sqrt{3}\Gamma(2/3)}{\pi} - \frac{\sqrt{x}}{8} + \int_0^1 \sqrt{x}\,t\,[y(t)]^2\,dt, \tag{4.14}$$

According to (3.16) we construct the following homotopy:

$$D^{2/3}y(x) = p(g(x) + \int_0^1 \sqrt{x}t[y(t)]^2\,dt). \tag{4.21}$$

Substituting (3.13) into (4.21),

$$p^0 \; : \; D^{1/3}y_0(x) = 0,$$

$$p^1 \; : \; D^{1/3}y_1(x) = g(x) + \int_0^1 \sqrt{x}\,t\,[y_0(t)]^2\,dt,$$

$$p^2 \; : \; D^{1/3}y_2(x) = \int_0^1 \sqrt{x}\,t\,[2y_0(t)y_1(t)]\,dt,$$

$$p^3 \; : \; D^{1/3}y_3(x) = \int_0^1 \sqrt{x}\,t\,[2y_0(t)y_2(t) + (y_1(t))^2]\,dt,$$

$$p^4 \; : \; D^{1/3}y_3(x) = \int_0^1 \sqrt{x}\,t\,[2y_0(t)y_3(t) + 2y_2(t)y_1(t)]\,dt,$$

.
.
.

by applying the operators $I^{2/3}$ to the above sets we obtain:

$$y_0(x) = 0,$$

$$y_1(x) = x^3 - 0.1023508743x^{7/6},$$

$$y_2(x) = 0,$$

$$y_3(x) = 0.07715011686x^{7/6},$$

$$y_4(x) = 0,$$

.
.
.

therefor the approximations to the solution of equation (4.14) will be determined as:

$$y(x) = \sum_{i=0}^{\infty} y_i(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) + \dots \;,$$

$$y(x) = \cong x^3 - 0.02520075744x^{7/6}.$$

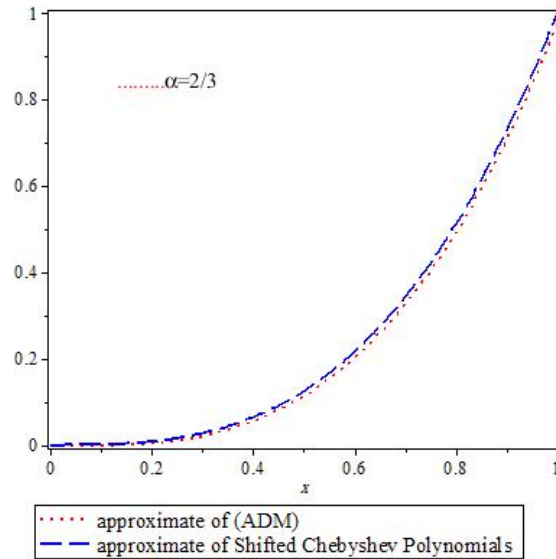Table 3, Figure 5 and 6 shown the numerical result of example 4.3 .
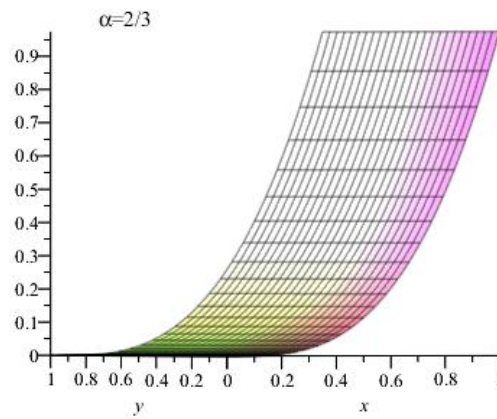

Figure 5: Numerical result of example 4.3

Figure 6: Approximate solution of example 4.3

x	$exact = (MVIM)$	$Approximant\,by(HPM)$	$Error\,of(HPM)$
0.1	0.001	-0.000716907618	0.001716907618
0.2	0.008	0.004145672716	0.003854327284
0.3	0.027	0.02081430658	0.00618569342
0.4	0.064	0.05534732781	0.00865267219
0.5	0.125	0.1137743388	0.0112256612
0.6	0.216	0.2021135878	0.0138864122
0.7	0.343	0.3263775651	0.0166224349
0.8	0.512	0.4925754077	0.0194245923
0.9	0.729	0.7067141165	0.0222858835
1	1.000	0.9747992426	0.0252007574

Table 3. Indicate the amount of error in Example 4.3

5. Conclusion

From the above result we find that the modification of He s variational iteration method (MVIM), is better than He s homotopy perturbation method (HPM), the results obtain by using Maple 16 .

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