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A Note an asymptotic Expansion of Spericity Test

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Abstract

This research is devoted to study the moment of the likelihood ratio criterion for sphericity test for one- way multivariate repeated measurements analysis of variance model. Also we obtain the asymptotic expansion and limiting distribution of the test statistic. As practical research, a study has been taken to diagnostics and isolation for kinds of becteria which complain with tissue cultivation for the dates and the study of frustrate affection for three kinds of extractor plant , which are called Rhus coriaria and cinnamomum zeylanicum , the excretes of adhesive for the Bswellia Sp plant and by using four kinds of solvent and two different condense. An experimental has been made for getting measurement for the best reacting extractor plant with the solvent by using different affection on frustrate core .

Key words: One-Way Multivariate Repeated Measures Model, Sphericity test, Asymptotic Expansion, MANOVA.

1. **Introduction**

Repeated measurements analysis is widely used in many fields , for example, in the health and life sciences, epidemiology, biomedical, agricultural, industrial, psychological, educational research and so on. Repeated measurements is a term used to describe data in which the response variable for each experimental unite is observed on multiple occasions and possibly under different experimental conditions [9]. Repeated measures designs involving two or more independent groups are among the most common experimental designs in a variety of research settings. Various statistical procedures have been suggested for analyzing data from split-plot designs when parametric model assumptions are violated[6]. Repeated measurements analysis of variance, often referred to as randomized block and split-plot designs [5] and [8].

The focus of this paper is to study the moment of the likelihood ratio criterion for sphericity test for one- way multivariate repeated measurements analysis of variance model. Also we obtain the asymptotic expansion and limiting distribution of the test statistic. The practical side of this paper is about tissue agriculture of Date palm trees. The date palm Phonenix dactyliferal is regarded as the most important fruit tree in Arab and Islamic Worlds. The purpose of our study , having specified and separated three kinds of bacteria, is to examine the effect of the transactions of distance inhibitory of bacteria. The results of application are obtained by MATLAB (R 2012) program .

2- **One –Way MRM Mode**

There are a variety of possibilities for the between- units factors in a one-way design. In a randomized one-way MRM experiment, the experimental units are randomized to one between-units factor (Groups with q levels), one within-units factor(Time with p levels) and random effect to experimental unit i within treatment group j, we define the following linear model and parameterization for the one-way multivariate repeated measurements design with one between- units factor :-

$$\mathbf{y}_{\mathbf{ijk}} = \boldsymbol{\mu} + \boldsymbol{\tau}_{\mathbf{j}} + \boldsymbol{\delta}_{\mathbf{i(j)}} + \boldsymbol{\gamma}_{\mathbf{k}} + (\boldsymbol{\tau\gamma})_{\mathbf{jk}} + \mathbf{e}_{\mathbf{ijk}} \qquad (2.1)$$

where

$i = 1, \cdots, n_j$ is an index for experimental unit within group j ,

$j = 1, \cdots, q$ is an index for levels of the between-units factor (Group),

$k = 1, \cdots, p$ is an index for levels of the within-units factor (Time),

$Y_{ijk} = \left[Y_{ijk1}, \cdots, Y_{ijkr}\right]'$ is the response measurement at time k for unit i within group j,

$\mu = \left[\mu_1, \cdots, \mu_r\right]'$ is the overall mean,

$\tau_j = \left[\tau_{j1}, \cdots, \tau_{jr}\right]'$ is the added effect for treatment group j,

$\delta_{i(j)} = \left[\delta_{i(j)1}, \cdots, \delta_{i(j)r}\right]'$ is the random effect due to experimental unit i within treatment group j,

$\gamma_k = \left[\gamma_{k1}, \cdots, \gamma_{kr}\right]'$ is the added effect for time k,

$(\tau\gamma)_{jk} = \left[(\tau\gamma)_{jk1}, \cdots, (\tau\gamma)_{jkr}\right]'$ is the added effect for the group j $\times$ time k interaction, and

$e_{ijk} = \left[e_{ijk1}, \cdots, e_{ijkr}\right]'$ is the random error on time k for unit i within groupj.

For the parameterization to be of full rank, we imposed the following set of conditions $\sum_{j=1}^{q} \tau_j = 0$, $\sum_{k=1}^{p} \gamma_k = 0$, $\sum_{j=1}^{q} (\tau\gamma)_{jk} = 0$ for $k = 1, \cdots, p$ $\sum_{k=1}^{p} (\tau\gamma)_{jk} = 0$ for $j = 1, \cdots, q$;
(2.2)

And we assumed that the e_{ijk}'s ,$\delta_{i(j)}$'s are independent with

$e_{ijk} = \left[e_{ijk1}, \cdots, e_{ijkr}\right]' \sim \text{i.i.d.}\, N_r(0, \Sigma_e)$ and

$$\delta_{i(j)} = \left[\delta_{i(j)1}, \cdots, \delta_{i(j)r}\right]' \sim \text{i.i.d.}\, N_r(0, \Sigma_\delta) \qquad (2.3)$$

where Σ_e, Σ_δ are $r \times r$ positive definite matrices. Let $Y_{ij} = \left[Y_{ij1}, Y_{ij2}, \cdots, Y_{ijp}\right]'$, that is

$$Y_{ij} = \begin{bmatrix} Y_{ij11} & Y_{ij21}\cdots & Y_{ijp1} \\ Y_{ij12} & Y_{ij22}\cdots & Y_{ijp2} \\ \vdots & \vdots\ddots & \vdots \\ Y_{ij1r} & Y_{ij2r}\cdots & Y_{ijpr} \end{bmatrix} \qquad (2.4)$$

The variance- covariance matrix of $\vec{Y}_{ij}$ is denoted as $\sum$, where$\vec{Y}_{ij} = \mathrm{Vec}(Y_{ij})$.The $\mathrm{Vec}(\cdot)$ operator creates a column vector from a matrixY_{ij} by simply stacking the column vectors of Y_{ij}below one another . The variance- covariance matrix $\sum$ of the model (2.1) satisfies the assumption of compound symmetry, i.e.

$$\Sigma = I_p \otimes \Sigma_e + J_P \otimes \Sigma_\delta = \begin{bmatrix} \Sigma_e + \Sigma_\delta & \Sigma_\delta & \cdots & \Sigma_\delta \\ \Sigma_\delta & \Sigma_e + \Sigma_\delta & \cdots & \Sigma_\delta \\ \vdots & \vdots & \ddots & \vdots \\ \Sigma_\delta & \Sigma_\delta & \cdots & \Sigma_e + \Sigma_\delta \end{bmatrix} (2.5)$$

WhereI_p denotes the $p \times p$ identity matrix, J_p denotes $p \times p$ matrix of one's and$\otimes$ is the Kronecker product operation of two matrices. obviously, we have that

$$e_{ij} = \left[e_{ij1}, \cdots, e_{ijp}\right]' \sim \text{i.i.d.}\, N_{p\times r}\left(0, I_p \otimes \Sigma_e\right) \qquad (2.6)$$

Let U^* be $p \times p$ orthogonal matrix. It is partitioned as follows:

$$U^* = \begin{pmatrix} p^{\frac{-1}{2}} j_p^{`} \\ U^{`} \end{pmatrix} \qquad (2.7)$$

where j_p denotes the $p \times 1$ vector of one's, U is $p \times (p-1)$ matrix . Because$U^*$is chosen to be orthogonal, we have that $U' j_p = 0$and$U'U = I_{p-1}$.

Let $Y_{ij}^* = Y_{ij} U^*$

$$\text{Where } Y_{ij}{}^* = \begin{bmatrix} Y_{ij11}^* & Y_{ij21}^*\cdots & Y_{ijp1}^* \\ Y_{ij12}^* & Y_{ij22}^*\cdots & Y_{ijp2}^* \\ \vdots & \vdots\ddots & \vdots \\ Y_{ij1r}^* & Y_{ij2r}^*\cdots & Y_{ijpr}^* \end{bmatrix} \qquad (2.8)$$

So $\mathrm{Cov}(\vec{Y}_{ij}^*) = \mathrm{Cov}\,(\overrightarrow{Y_{\iota J} U^*}) = \mathrm{Cov}\left(\left(U^{*'} \otimes I_r\right) \vec{Y}_{ij}\right)$

$$= \left(U^{*'} \otimes I_r\right) \Sigma (U^* \otimes I_r) \qquad (2.9)$$

For(2.5) we get $\mathrm{Cov}(\vec{Y}_{ij}^*) = \left(U^{*'} \otimes I_r\right)\left(\mathrm{I_p} \otimes \Sigma_e + \mathrm{J_p} \otimes \Sigma_\delta\right)(U^* \otimes I_r)$.

$$= \mathrm{I_P} \otimes \Sigma_e + U^{*'} \mathrm{J_P} U^* \otimes \Sigma_\delta \qquad (2.10)$$

That means $\mathrm{Y}_{ij1}^*, \ldots, \mathrm{Y}_{ijP}^*$ are independent of each other $\mathrm{Cov}\,(\mathrm{Y}_{ijk}^*) = \Sigma_e + P\Sigma_\delta$ and $\mathrm{Cov}\,(\mathrm{Y}_{ijk}^*) = \Sigma_e$, for each k=2,…,p

Now $Y_{ij1}^* = Y_{ij} P^{-1/2} J_P$, $[\,\mathrm{Y}_{ij2}^* \ldots \mathrm{Y}_{ijP}^*] = Y_{ij} U$

$$\mathrm{Cov}(\vec{Y}_{ij}^*) = \begin{bmatrix} \Sigma_e + p\Sigma_\delta & 0 & \cdots & 0 \\ 0 & \Sigma_e & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma_e \end{bmatrix} \qquad (2.11)$$

$$\mathrm{SO}\ \ Y_{ij1}^* = \begin{bmatrix} Y_{ij11}^* \\ Y_{ij12}^* \\ \vdots \\ Y_{ij1r}^* \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{p}} \sum_{k=1}^{p} Y_{ijk1} \\ \frac{1}{\sqrt{p}} \sum_{k=1}^{p} Y_{ijk2} \\ \vdots \\ \frac{1}{\sqrt{p}} \sum_{k=1}^{p} Y_{ijkr} \end{bmatrix},$$

3 The Hypothesis of the Sphericity Test

In this section, we focus on testing the null hypothesis that the variance matrix of $r \times p$ random matrix $Y_{ij} = (Y_{ij1}, \cdots, Y_{ijp})$, $Cov\left(Y_{ij1}^{'}, \cdots, Y_{ijp}^{'}\right)^{'} = \Sigma$, $i = 1 \cdots n_j$,

$j = 1, \cdots, q$, is of type H $\mathrm{H_0}: \Sigma = \mathrm{I_p} \otimes \Sigma_e + \mathrm{J_P} \otimes \Sigma_\delta$ .A generally speaking,the $pr \times pr$ matrix Σ is said to be of the type H it satisfies the following condition: $\Sigma = \mathrm{I}_p \otimes V_1 + \left(j_p \otimes \alpha^{'} + \alpha \otimes j_p^{'}\right) \otimes V_2$

$$= \begin{bmatrix} V_1 + 2\alpha_1 V_2 & (\alpha_1 + \alpha_2) V_2 & \cdots & (\alpha_1 + \alpha_p) V_2 \\ (\alpha_2 + \alpha_1) V_2 & V_1 + 2\alpha_2 V_2 & \cdots & (\alpha_2 + \alpha_p) V_2 \\ \vdots & \vdots & \ddots & \vdots \\ (\alpha_p + \alpha_1) V_2 & (\alpha_p + \alpha_2) V_2 & \cdots & V_1 + 2\alpha_p V_2 \end{bmatrix}, \qquad (3.1)$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)^{'}$ and each of V_1 and V_2 are diagonal matrices . This means that the null hypothesis be in the form $H_0: \Sigma = I_p \otimes \Sigma_e + J_p \otimes \Sigma_\delta$ (3.2) In fact, the Multivariate analysis of variance (MANOVA) discussed in section two of this paper is based on Type H of the covariance matrix of random matrix Y_{ij}.

$$\left(U^{*'}\otimes I_r\right)\Sigma(U^{*}\otimes I_r)=\left(U^{*'}\otimes I_r\right)\left(I_p\otimes\Sigma_e+J_p\otimes\Sigma_\delta\right)(U^{*}\otimes I_r)$$

$$\left(U^{*'}\otimes I_r\right)\Sigma(U^{*}\otimes I_r)=U'I_PU\otimes I_r\Sigma_eI_r+U'J_PU\otimes I_r\Sigma_\delta I_r$$

$$=U'U\otimes\Sigma_e+(U'j_P)(j_P'U)\otimes\Sigma_\delta=U'U\otimes\Sigma_e=I_{P-1}\otimes\Sigma_e, \qquad (3.3)$$

because $U'j_P=j_P'U=0$ .This shows that if Σ is of Type H, then any set of $(p-1)$ orthogonal contrasts of the observations of the form $Y_{ij}\,U$, $i=1,\cdots,n_j, j=1,\cdots,q$, has covariance matrix Σ^*, $\Sigma^*=I_{P-1}\otimes V$ (3.4)
such situation, each transformed $Y_{ij}\,U$, $i=1,\cdots,n_j$, $j=1\cdots q$, is said to have a spherical distribution. Thus, the problem of testing the null hypothesis (3.2) based on the observations $Y_{ij}\,U$, $i=1,\cdots,n_j$, $j=1,\cdots,q$, is transformed to the problem of testing the null hypothesis $H_0(3.4)$ .H_0: $\Sigma^*=I_{P-1}\otimes V$ (3.5)

based on $Y^*_{ij(2)}=Y_{ij}U$, wher $i=1,\cdots,n_j$, $j=1,\cdots,q$.

4- Likelihood Ratio Criterion for Sphericity Test

In canonical form the hypothesis H_0 is a combination of the hypotheses:

H_{01}: Σ^* is block diagonal or the components of Y_{ij} are independent, and H_{02}: the block diagonal elements of Σ^* are equal given Σ^* is block diagonal or the variances of the components of Y_{ijk} are equal given the components are independent. The likelihood ratio criterion λ for H_0 is the product of the criteria λ_1 and λ_2, where λ_1 is the likelihood ratio criterion for the hypothesis that Σ^* is block diagonal

$$\therefore\ \lambda_1=\frac{|\mathrm{A}|^{\frac{n}{2}}}{\prod_{l=1}^{p-1}|\mathrm{A}_{ll}|^{\frac{n}{2}}} \qquad (4.1)$$

Where $\bar{Y}_j^*=\frac{\sum_{i=1}^{n_j}\vec{Y}^*_{ij(2)}}{n_j}$, $\vec{Y}^*_{ij(2)}=\begin{pmatrix}Y^*_{ij2}\\Y^*_{ij3}\\\vdots\\Y^*_{ijp}\end{pmatrix}$, $Y^*_{ijk}=\begin{pmatrix}Y^*_{ijk1}\\Y^*_{ijk2}\\\vdots\\Y^*_{ijkr}\end{pmatrix}$,

$$A=\sum_{j=1}^{q}\sum_{i=1}^{n_j}\left(\vec{Y}^*_{ij(2)}-\bar{Y}_j^*\right)\left(\vec{Y}^*_{ij(2)}-\bar{Y}_j^*\right)' \qquad (4.2)$$

And λ_2 is the likelihood ratio criterion for the hypothesis that the block diagonal elements of Σ^* are equal given Σ^* is block diagonal.

$$\lambda_2 = \frac{\prod_{l=1}^{p-1} |\mathrm{A}_{ll}|^{\frac{n}{2}}}{|\mathrm{B}|^{\frac{n(p-1)}{2}}} \times (p-1)^{\frac{rn(p-1)}{2}} . \qquad (4.3)$$

Where $\mathrm{A}_{ll} = \sum_{j=1}^{q} \sum_{i=1}^{n_j} (\vec{Y}_{ij(2)}^{*} - \mu)(\vec{Y}_{ij(2)}^{*} - \mu)' \quad , l = 1, \dots . p-1.$ $B = \sum_{l=1}^{p-1} \mathrm{A}_{ll}$ hypothesis H_0 according to the Anderson is the product of two criteria, λ_1, λ_2 , then $\lambda = \lambda_1 \lambda_2$

$$\lambda = \frac{|A|^{\frac{n}{2}}}{\left(\frac{|\mathrm{B}|}{(p-1)^{r}}\right)^{\frac{n(p-1)}{2}}} . \qquad (4.4)$$

5-The Moment of the Criterion for Sphericity Test The distribution of λ cannot be easily obtained in an explicit form for a general n , but its moment is easily found when the hypothesis tested is true. So ,we identify the distribution of λ by finding its moment. As it was observed when Σ^* is block diagonal, the correlation coefficients R_{ij} are distributed of the variances

A_{ii} , $i = 1, \cdots, p-1$. So that we obtain the h^{th}moment of λ_1 and the h^{th} moment of λ_2 in the following propositions.

Proposition1: The h^{th} moment of λ_1 is given as :$E\left(\lambda_1^h\right) = \frac{\prod_{j=1}^{m} \Gamma(\frac{n(h+1)-q-j+1}{2}) \left(\prod_{j=1}^{r} \Gamma(\frac{n-q-j+1}{2})\right)^{p-1}}{\prod_{j=1}^{m} \Gamma(\frac{n-q-j+1}{2}) \left(\prod_{j=1}^{r} \Gamma(\frac{n(1+h)-q-j+1}{2})\right)^{p-1}}$ Where λ_1 is given in (4.1)

Proof

Since the density function of R is:$P_R(R) = \frac{\left[2^{\frac{r(n-q)}{2}} \pi^{\frac{r(r-1)}{4}} \prod_{j=1}^{p-1} \Gamma(\frac{n-q-j+1}{2})\right]^{p-1} |R|^{\frac{n-q-m-1}{2}}}{2^{\frac{m(n-q)}{2}} \pi^{\frac{m(m-1)}{4}} \prod_{j=1}^{p-1} \Gamma(\frac{n-q-j+1}{2})}$

$$E(|R|^h) = \frac{\left[2^{\frac{r(n-q)}{2}} \pi^{\frac{r(r-1)}{4}} \prod_{j=1}^{p-1} \Gamma(\frac{n-q-j+1}{2})\right]^{p-1}}{2^{\frac{m(n-q)}{2}} \pi^{\frac{m(m-1)}{4}} \prod_{j=1}^{p-1} \Gamma(\frac{n-q-j+1}{2})} \int_{R_{ij}, p-1 \geq i \geq j \geq 1} \cdots \int |R|^h \, |R|^{\frac{n-q-m-1}{2}} \, d(R)$$

$$E(|R|^h) = \frac{\left[2^{\frac{r(n-q)}{2}} \pi^{\frac{r(r-1)}{4}} \prod_{j=1}^{p-1} \Gamma(\frac{n-q-j+1}{2})\right]^{p-1}}{2^{\frac{m(n-q)}{2}} \pi^{\frac{m(m-1)}{4}} \prod_{j=1}^{p-1} \Gamma(\frac{n-q-j+1}{2})} \int_{R_{ij, p-1 \geq i \geq j \geq 1}} \dots \int |R|^{\frac{n-q+2h-m-1}{2}} \, d(R)$$

$$E(|R|^h) = \frac{\left[2^{\frac{r(n-q)}{2}}\pi^{\frac{r(r-1)}{4}}\prod_{j=1}^{p-1}\Gamma(\frac{n-q-j+1}{2})\right]^{p-1}}{2^{\frac{m(n-q)}{2}}\pi^{\frac{m(m-1)}{4}}\prod_{j=1}^{p-1}\Gamma(\frac{n-q-j+1}{2})\ 2^{\frac{m(n+2h-q)}{2}}\pi^{\frac{m(m-1)}{4}}\prod_{j=1}^{p-1}\Gamma(\frac{n+2h-q-j+1}{2})}$$

$$\int\cdots\int 2^{\frac{m(n+2h-q)}{2}}\pi^{\frac{m(m-1)}{4}}\prod_{j=1}^{p-1}\Gamma(\frac{n+2h-q-j+1}{2})|R|^{\frac{n-q+2h-m-1}{2}}d(R)$$

$$= \left[\frac{2^{\frac{r(n-q)}{2}}\pi^{\frac{r(r-1)}{4}}\prod_{j=1}^{p-1}\Gamma(\frac{n-q-j+1}{2})}{2^{\frac{r(n+2h-q)}{2}}\pi^{\frac{r(r-1)}{4}}\prod_{j=1}^{p-1}\Gamma(\frac{n+2h-q-j+1}{2})}\right]^{p-1} \times \frac{2^{\frac{m(n+2h-q)}{2}}\pi^{\frac{m(m-1)}{4}}\prod_{j=1}^{p-1}\Gamma(\frac{n+2h-q-j+1}{2})}{2^{\frac{m(n-q)}{2}}\pi^{\frac{m(m-1)}{4}}\prod_{j=1}^{p-1}\Gamma(\frac{n-q-j+1}{2})}$$

$$= \frac{\prod_{j=1}^{m}\Gamma(\frac{n-q+2h-j+1}{2})\left(\prod_{j=1}^{r}\Gamma(\frac{n-q-j+1}{2})\right)^{p-1}}{\prod_{j=1}^{m}\Gamma(\frac{n-q-j+1}{2})\left(\prod_{j=1}^{r}\Gamma(\frac{n-q+2h-j+1}{2})\right)^{p-1}} \quad \text{Since } \lambda_1 = \frac{|A|^{\frac{n}{2}}}{\prod_{l=1}^{p-1}|A_{ll}|^{\frac{n}{2}}} = |R|^{\frac{n}{2}},\ n = \sum_{j=1}^{q} n_j$$

$$\therefore\ E(\lambda_1^h) = E(|R|^{\frac{nh}{2}}). \text{ Hence } E(\lambda_1^h) = \frac{\prod_{j=1}^{m}\Gamma(\frac{n(1+h)-q-j+1}{2})\left(\prod_{j=1}^{r}\Gamma(\frac{n-q-j+1}{2})\right)^{p-1}}{\prod_{j=1}^{m}\Gamma(\frac{n-q-j+1}{2})\left(\prod_{j=1}^{r}\Gamma(\frac{n(1+h)-q-j+1}{2})\right)^{p-1}} \quad (5.1)$$

The proof of the proposition 1 is completed.

Proposition2

The h^{th} moment of λ_2 is given

as $E(\lambda_2^h) = \left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \left[\frac{\Gamma_r(\frac{n-q-nh}{2})}{\Gamma_r(\frac{n-q}{2})}\right]^{p-1} \left[\frac{\Gamma_r(\frac{(n-q)(p-1)}{2})}{\Gamma_r(\frac{(n-q+nh)(p-1)}{2})}\right]$

Where λ_2 is given in (4.3).

Proof

Since $\lambda_2 = \frac{\prod_{l=1}^{p-1}|A_{ll}|^{\frac{n}{2}}}{|B|^{\frac{n(p-1)}{2}}} \times (p-1)^{\frac{rn(p-1)}{2}}$, where $A_{11},\cdots,A_{p-1,p-1}$ are independent, $B = \sum_{l=1}^{p-1} A_{ll}$,$A_{ll}\sim W_r(n-q,V_i)$, $l = 1,\cdots,p-1$.

First, we consider $\Lambda_2 = \frac{\prod_{l=1}^{p-1}|A_{ll}|^{\frac{n}{2}}}{|B|^{\frac{n(p-1)}{2}}}$.

Now, let

$$\Lambda_{2,g} = \frac{|A_{1,1}+\cdots+A_{g-1,g-1}|^{\frac{n(g-1)}{2}}|A_{gg}|^{\frac{n}{2}}}{|A_{1,1}+\cdots+A_{g,g}|^{\frac{ng}{2}}},\ g = 2,\cdots,p-1 \quad (5.2)$$

Thus $\Lambda_2=\Lambda_{2,2},\cdots,\Lambda_{2,p-1}$. Since $A_{1,1},\cdots,A_{p-1,p-1}$ are independent, then by Lemma (10.4.1) of Anderson (1984) we have $\Lambda_{2,2}^{(1)}\cdots\Lambda_{2,p-1}^{(2)}, A_{1,1}+\cdots+A_{p-1,p-1}$ are independent, where $\Lambda_{2,g}^{(1)}=\left(A_{1,1}+\cdots+A_{g,g}\right)^{-\frac{1}{2}}\left(A_{1,1}+\cdots+A_{g-1,g-1}\right)\left(A_{1,1}+\cdots+A_{g,g}\right)^{-\frac{1}{2}}$ for $g=2,\cdots,p-1$. Then we can write (3.17) as follows:

$$\Lambda_{2,g}=\left|\frac{A_{1,1}+\cdots+A_{g-1,g-1}}{A_{1,1}+\cdots+A_{g,g}}\right|^{\frac{n(g-1)}{2}}\left|\frac{A_{g,g}}{A_{1,1}+\cdots+A_{g,g}}\right|^{\frac{n}{2}}=\left|\Lambda_{2,g}^{(1)}\right|^{\frac{n(g-1)}{2}}\left|I-\Lambda_{2,g}^{(1)}\right|^{\frac{n}{2}}$$

Hence $\Lambda_2=\Lambda_{2,2}\cdots\Lambda_{2,p-1}$, $A_{11}+\cdots+A_{p-1,p-1}$ are independent.

Also , by Lemma (10.4.1) of Anderson (1984) resulting from

$$\Lambda_{2,g}^{(1)}=\left(A_{1,1}+\cdots+A_{g,g}\right)^{-\frac{1}{2}}\left(A_{1,1}+\cdots+A_{g-1,g-1}\right)\left(A_{1,1}+\cdots+A_{g,g}\right)^{-\frac{1}{2}}$$

we have $\Lambda_{2,g}^{(1)}$ that has the multivariate Beta distribution with (g-1) (n-q) and (n-q) degrees of freedom,where

$$\Lambda_{2,g}^{(1)}\sim\frac{\Gamma_r\left(\frac{1}{2}g(n-q)\right)}{\Gamma_r\left(\frac{1}{2}(g-1)(n-q)\right)\Gamma_r\left(\frac{1}{2}(n-q)\right)}\left|\Lambda_{2,g}^{(1)}\right|^{\frac{1}{2}((g-1)(n-q-r-1))}\left|I-\Lambda_{2,g}^{(1)}\right|^{\frac{1}{2}(n-q-r-1)}$$

$\Gamma_p(t)=\Pi^{\frac{p(p-1)}{4}}\prod_{i=1}^{p}\Gamma\left(t-\frac{1}{2}(i-1)\right)$ is the multivariate Gamma function.

Then $E\left(\Lambda_{2,g}^{h}\right)=\int\left[\left|\Lambda_{2,g}^{(1)}\right|^{\frac{n(g-1)}{2}}\left|I-\Lambda_{2,g}^{(1)}\right|^{\frac{n}{2}}\right]^{h}\left[\frac{\Gamma_r\left(\frac{1}{2}g(n-q)\right)}{\Gamma_r\left(\frac{1}{2}(g-1)(n-q)\right)\Gamma_r\left(\frac{1}{2}(n-q)\right)}\right]$

$$\left[\left|\Lambda_{2,g}^{(1)}\right|^{\frac{1}{2}((g-1)(n-q-r-1))}\left|I-\Lambda_{2,g}^{(1)}\right|^{\frac{1}{2}(n-q-r-1)}\right]\ d\left(\Lambda_{2,g}^{(1)}\right)$$

$$=\frac{\Gamma_r\left(\frac{1}{2}g(n-q)\right)}{\Gamma_r\left(\frac{1}{2}(g-1)(n-q)\right)\Gamma_r\left(\frac{1}{2}(n-q)\right)}\int\left[\left|\Lambda_{2,g}^{(1)}\right|^{\frac{1}{2}((g-1)(nh+n-q-r-1))}\left|I-\Lambda_{2,g}^{(1)}\right|^{\frac{1}{2}(nh+n-q-r-1)}\right]\ d\left(\Lambda_{2,g}^{(1)}\right)$$

$$=\frac{\Gamma_r\left(\frac{1}{2}g(n-q)\right)\Gamma_r\left(\frac{1}{2}(g-1)(nh+n-q)\right)\Gamma_r\left(\frac{1}{2}(nh+n-q)\right)}{\Gamma_r\left(\frac{1}{2}(g-1)(n-q)\right)\Gamma_r\left(\frac{1}{2}(n-q)\right)\Gamma_r\left(\frac{1}{2}g(nh+n-q)\right)}$$. Hence

$$E\left(\Lambda_2^h\right)=E\left(\Lambda_{22}^h\,\Lambda_{23}^h\cdots\Lambda_{2,p-1}^h\right)=E\left(\Lambda_{22}^h\right)E\left(\Lambda_{23}^h\right)\cdots E\left(\Lambda_{2,P-1}^h\right)=\prod_{g=2}^{p-1}E\left(\Lambda_{2,g}^h\right)$$

Thus $E(\Lambda_2^h) = \prod_{g=2}^{p-1}[\frac{\Gamma_r\left(\frac{1}{2}g(n-q)\right)\Gamma_r\left(\frac{1}{2}(g-1)(nh+n-q)\right)\Gamma_r\left(\frac{1}{2}(nh+n-q)\right)}{\Gamma_r\left(\frac{1}{2}(g-1)(n-q)\right)\Gamma_r\left(\frac{1}{2}(n-q)\right)\Gamma_r\left(\frac{1}{2}g(nh+n-q)\right)}]$

Since $\lambda_2 = (p-1)^{\frac{rn(p-1)}{2}}\Lambda_2$. Then $E(\lambda_2^h) = \left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h E(\Lambda_2^h)$

$$= \left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \prod_{g=2}^{p-1}[\frac{\Gamma_r\left(\frac{1}{2}g(n-q)\right)}{\Gamma_r\left(\frac{1}{2}(g-1)(n-q)\right)\Gamma_r\left(\frac{1}{2}(n-q)\right)}] [\frac{\Gamma_r\left(\frac{1}{2}(g-1)(nh+n-q)\right)\Gamma_r\left(\frac{1}{2}(nh+n-q)\right)}{\Gamma_r\left(\frac{1}{2}g(nh+n-q)\right)}]$$

Hence $E(\lambda_2^h) = [(p-1)^{\frac{rn(p-1)}{2}}]^h \frac{[\Gamma_r\left(\frac{1}{2}(nh+n-q)\right)]^{p-1}\Gamma_r\left(\frac{1}{2}(p-1)(n-q)\right)}{[\Gamma_r\left(\frac{1}{2}(n-q)\right)]^{p-1}\Gamma_r\left(\frac{1}{2}(p-1)(n-q+nh)\right)}$

The proof of proposition 2 is completed.

Proposition 3

The h^{th} moment of λ is given as:

$$E(\lambda^h) = \left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \frac{\prod_{j=1}^m \Gamma(\frac{n-q+nh-j+1}{2})}{\prod_{j=1}^m \Gamma(\frac{n-q-j+1}{2})} \times \frac{\prod_{j=1}^r \Gamma(\frac{(p-1)(n-q)-j+1}{2})}{\prod_{j=1}^r \Gamma(\frac{(p-1)(n-q+nh)-j+1}{2})}$$

Where λ is given in (4.4).

Proof

By Lemma (10.3.1) which is given by Anderson (1984) λ is the product of λ_1 and λ_2 , and by proposition 1 above , λ_1 and λ_2 are independent, when H_0 is true, then

$$E(\lambda^h) = E(\lambda_1^h) E(\lambda_2^h)$$

$$= \left[\frac{\prod_{j=1}^m \Gamma(\frac{n(1+h)-q-j+1}{2})\left(\prod_{j=1}^r \Gamma(\frac{n-q-j+1}{2})\right)^{p-1}}{\prod_{j=1}^m \Gamma(\frac{n-q-j+1}{2})\left(\prod_{j=1}^r \Gamma(\frac{n(1+h)-q-j+1}{2})\right)^{p-1}}\right] \times$$

$$\left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \left[\frac{\Gamma_r(\frac{n-q+nh}{2})}{\Gamma_r(\frac{n-q}{2})}\right]^{p-1} \left[\frac{\Gamma_r(\frac{(n-q)(p-1)}{2})}{\Gamma_r(\frac{(n-q+nh)(p-1)}{2})}\right]$$

Hence $E(\lambda^h) = \left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \frac{\prod_{j=1}^m \Gamma(\frac{n-q+nh-j+1}{2})}{\prod_{j=1}^m \Gamma(\frac{n-q-j+1}{2})} \times \frac{\prod_{j=1}^r \Gamma(\frac{(p-1)(n-q)-j+1}{2})}{\prod_{j=1}^r \Gamma(\frac{(p-1)(n-q+nh)-j+1}{2})}$ (5.3)

The proof of the proposition 3 is completed.

6- Asymptotic Expansion of Sphericity Test

In multivariate analysis, the exact distribution of likelihood ratio tests is often too complicated to be of any practical use. An asymptotic expansion due to Box (1949) is rather simple to obtain the distribution function to any degree of accuracy. This approximation is applied to several testing situations (see Bilodeau and Brenner (1999). In at least one situation where the exact distribution is known, an evaluation of the approximation is carried out for small to moderate sample size. The method can be used whenever the likelihood ratio criterion λ has a moment of order h. The likelihood ratio test (LRT) of sphericity was given in (3.9) and its moment is derived in proposition 4. Now, it is simply a matter of rewriting things in the form (1) of Section (8.6.1) of Anderson (1984) to obtain the asymptotic expansion. Since

$$E(\lambda^h) = \left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \frac{\prod_{j=1}^{r}\Gamma(\frac{(p-1)(n-q-j+1)}{2})}{\prod_{j=1}^{m}\Gamma(\frac{n-q-j+1}{2})} \times \frac{\prod_{j=1}^{m}\Gamma(\frac{n-q+nh-j+1}{2})}{\prod_{j=1}^{r}\Gamma(\frac{(p-1)(n-q+nh)}{2})}$$

Then $E(\lambda^h) = \left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \frac{\Gamma_m(\frac{n+nh-q}{2})}{\Gamma_m(\frac{n-q}{2})} \times \frac{\Gamma_r(\frac{(p-1)(n-q)}{2})}{\Gamma_r(\frac{(p-1)(n+nh-q)}{2})}$

$$E(\lambda^h) = \frac{\prod_{j=1}^{r}\Gamma(\frac{(p-1)((n-q)-j+1)}{2})}{\prod_{j=1}^{m}\Gamma(\frac{n-q-j+1}{2})} \left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \frac{\prod_{j=1}^{m}\Gamma(\frac{(n-q+nh)-j+1}{2})}{\prod_{j=1}^{r}\Gamma(\frac{(p-1)((n-q+nh)-j+1)}{2})}$$

$$E(\lambda^h) = K\left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \frac{\prod_{j=1}^{m}\Gamma(\frac{(n-q+nh)-j+1}{2})}{\prod_{j=1}^{r}\Gamma(\frac{(p-1)((n-q+nh)-j+1)}{2})}$$

where K is just constant (not depending on h), so the form can be written as follows:

$$E(\lambda^h) = K\left[(p-1)^{\frac{rn(p-1)}{2}}\right]^h \frac{\prod_{j=1}^{m}\Gamma\,\frac{1}{2}n(1+h)+\frac{1}{2}(1-q-j)}{\prod_{j=1}^{r}\Gamma\,\frac{1}{2}n(p-1)(1+h)+\frac{1}{2}(1-(p-1)q-j)}$$

It turns out that this is $E(\lambda^h)$ is similar to the relationship (1) in (8.5.1) of Anderson such that

$a = m, k = \frac{1}{2}n\,, \xi_k = \frac{1}{2}(1-q-k), \;\; k = 1, \dots, m$

$b = r\,, y_{j=}\frac{1}{2}n(p-1), \eta_j = \frac{1}{2}n(1-(p-1)q-j), j = 1, \dots, r.$

$\sum_{k=1}^{m} x_k = \sum_{j=1}^{r} y_j$ is satisfied and x_k and y_j are terms behaving as $O(n)$.The asymptotic expansion with remainder $O(n^{-(l+1)})$ as in Theorem (8.6.1) of Anderson (1984) is now a simple matter of calculating with form (10) of section (8.6.1) of Anderson (1984)

$$f = -2[\sum_{k=1}^{a} \xi_k - \sum_{j=1}^{b} \eta_j - \frac{1}{2}(a-b)]$$

$$= -2[\sum_{k=1}^{m} \frac{1}{2}(1-q-k) - \sum_{j=1}^{r} \frac{1}{2} n(1-(p-1)q-j) - \frac{1}{2}(m-r)]$$

$$= -m(1-q) + \sum_{k=1}^{m} k + r - r(p-1)q - \sum_{j=1}^{r} j + m - r$$

$$= -m + mq + \sum_{k}^{m} k + r - mq - \sum_{j=1}^{r} j + m - r \ = \sum_{k=1}^{m} k - \sum_{j=1}^{r} j$$

$$f = \frac{1}{2}[m(m+1) - r(r+1)] \qquad (\mathbf{6.1})$$

Now, we want to choose ρ to annihilate of order (n^{-1}) , i.e. to make $w_1 = 0$. Recalling β_k and ϵ_j as in Anderson (1984) and $B_2(h) = h^2 - h + \frac{1}{6}$, where B_r is the Bernoulli polynomial of degree r , (see, Apostol (1982) , and De Bruijn (1981) for Bernoulli polynomial), we have

$\beta_k = \frac{1}{2} n\ (\rho - 1)$, $\epsilon_j = \frac{1}{2} n\ (p-1)(1-\rho)$, and

$$W_r = \frac{(-1)^{r+1}}{r(r+1)} \left[\sum_k \frac{B_{r+1}(\beta_k + \xi_k)}{(px_k)^r} - \sum_j \frac{B_{r+1}(\epsilon_j + \eta_j)}{(py_j)^r}\right]$$

If $r = 1 \ \therefore \ W_1 = \frac{1}{2}\left[\sum_k \frac{B_2(\beta_k + \xi_k)}{\frac{np}{2}} - \sum_j \frac{B_2(\epsilon_j + \eta_j)}{\frac{np(p-1)}{2}}\right]$

$$W_1 = \frac{1}{2\rho} \left\{\sum_{k=1}^{m} (\tfrac{n}{2})^{-1} ((\tfrac{1}{2} n(1-\rho) + (1-q-k))^2 - (\tfrac{1}{2} n\ (1-\rho) + \ (1-q-k)) + \tfrac{1}{6} - \sum_{j=1}^{r} \left(\tfrac{n(p-1)}{2}\right)^{-1} ((\tfrac{1}{2}\ n(p-1)(1-\rho) + \tfrac{1}{2}(1-(p-1)q-j))^2 - (\tfrac{1}{2} n(p-1)(1-\rho) + (1-(p-1)q-j) + \tfrac{1}{6}\right\}$$

$$W_1 = \left\{-(1-\rho)f + \sum_{k=1}^{m} (\tfrac{n}{2})^{-1} [\left(\tfrac{1}{2}(1-q-k)\right)^2 + \tfrac{1}{2}\ (1-q-k) + \tfrac{1}{6}\right] - \sum_{j=1}^{r} (\tfrac{n(p-1)}{2})^{-1}\ [\ (\tfrac{1}{2}\ (1-(p-1)q-j) + \tfrac{1}{6}]\}$$

$$W_1 = \frac{1}{2\rho} - (1-\rho)f - \left(\frac{n}{2}\right)^{-1} \left[\frac{1}{3} m - m(m+1)q - \frac{m(m+1)(2m+1)}{6} - \frac{1}{3}\ r(p-1)^{-1} + r(r+1)q + \frac{1}{6}\ r(r+1)(2r+1)(p-1)^{-1}\right]$$

Thus to make $w_1 = 0$, we require that:

$$\rho = 1 - f^{-1}\{-\tfrac{1}{2}n^{-1}\left[\tfrac{1}{3}m - m(m+1)q - \tfrac{1}{6}m(m+1)(2m+1) - \tfrac{1}{3}r(p-1)^{-1} + r(r+1)q + \tfrac{1}{6}r(r+1)(2r+1)\,(p-1)^{-1}\right]\} \qquad (6.2)$$

Hence, when the null hypothesis (3.2) is true, the distribution function of $-2\rho\log\lambda$ can be expanded for large n. That is, when the null hypothesis is true, the limiting distribution of $-2\rho\log\lambda$ is χ_f^2 for large n, where f and ρ are given in (6.1) and (6.2) respectively, and χ_f^2 denotes the chi-square distribution with f degrees of freedom.

7- The Experiment

The tissue agriculture is considered as modern technology to propagate many plants which belong to different plant families. The technology of the tissue agriculture has proved its efficiency in the propagation of the plants, which can be produced from the root of the same plant and the matching plants arising from their origins, in terms of genetic stability[9].The data of the experiment was taken from Date palm research center, Basrah university – which represent for isolation and identification of bacterial types that contaminated date palm tissue culture., and studied the inhibiting activities of three types of plant extracts on fruit of Rhus coriaria , bark of Cinnamomum zeylanicum and gummy extraction of Bswellia sp., using four types of solvent water , methyl alcohol , normal hexane and ethyl acetate , in two concentrations (0.5 , 1) % . The results of isolation and identification of bacteria appeared contanmination of callus tissue of date palm tissue culture by three genera of bacteria Stapylloccus aureus , Bacillus subtillu and proteus spp . According to the mathematical formula of the model study(2.1) and by applying the model to the experiment, in order to calculate the sphericity test aspect of test hypothesis $H_0 : \sum_{11} = \sum_{22} = \sum_{33} = \sum_{44}$ we find the variance-covariance matrix between observations ,in calculated likelihood ratio test the hypothesis according to the equation (3.2).We find $\rho = 0.0545$ according to the formula (6.1) . And we find f= 26 according to the formula (6.2) . It is clear from above that : -2 ρ $\log\lambda \sim x^2$ (26) where $x_{0.05}^2$ (26) =15.4 .When we compare the calculated x^2 value with tabulated value at 26 degree of freedom and at o.o5 level of significant . We find $x_{col.}^2 < x_{tab.}^2$ .So we accept the null hypothesis. All the block diagonal matrix are equal.

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